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THE COEFFICIENTS OF THE LAURENT EXPANSION OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper it is shown that the coefficient a_n with $n = 0, \pm 1, \pm 2, \ldots$ of the Laurent expansion $\ldots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots$ of a function $f(z)$ analytic in an annulus with center at 0 is the limit as $k \to \infty$ of the coefficient $a_{k,n}$ of the rational function $a_{k,-k}z^{-k} + \ldots + a_{k,0} + \ldots + a_{k,k}z^k$ which interpolates to $f(z)$ in the $2k + 1$ points equally spaced on a circle with center at 0 and located inside the annulus. Hence, $a_n = \lim\limits_{k \to \infty} a_{k,n}$. Consequently, a_n is evaluated as the limit of the quotient of two appropriate determinants.

Clearly, the $2k + 1$ coefficients of the rational function $q_k(z)$ given (in its Laurent expansion) as:

$$q_k(z) = a_{k,-k}z^{-k} + \ldots + a_{k,-1}z^{-1} + a_{k,0} + a_{k,1}z + \ldots + a_{k,k}z_k \tag{1}$$

over, say, the field of complex numbers are uniquely determined by $2k + 1$ distinct nonzero complex numbers $c_0, \ldots, c_{2k}$ and the corresponding $2k + 1$ values $q_k(c_0), \ldots, \ldots, q_k(c_{2k})$ of the rational function $q_k(z)$.

In what follows for the $2k + 1$ distinct nonzero complex numbers we choose exclusively the $2k + 1$ points equally spaced on a circle of radius r and with center at 0 in the z-plane where r is a positive real number and with r as one of the $2k + 1$ numbers.

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Denoting by c the $2k + 1th$ primitive root of unity, i.e.,

$$c = e^{2\pi i/2k+1} \tag{2}$$

the $2k + 1$ distinct nonzero complex numbers mentioned above are:

$$r, rc, rc^2, \ldots, rc^{2k}. \tag{3}$$

Since $2k + 1$ roots of unity form a cyclic group under multiplication, it is easy to verify that the coefficients $a_{k,}n$ in (1) acquire the following simple form when

Cramer's rule is applied to determin $a_{k,n}$ in terms of (3) and the corresponding $2k + 1$ values of $q_k(z)$ given by

$$(4) \qquad q_k(r), q_k(rc), q_k(rc^2), \ldots, q_k(rc^{2k}).$$

Indeed, for $k = 0, 1, 2, \ldots$ and $n = 0, \pm 1, \pm 2, \ldots$ we readily have:

$$(5)\ a_{k,n} = r^{-(2k+1)n} \cdot \frac{\det(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, q_k(rc^{j-1}), \ldots, c^{k(j-1)})}{\det(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, c^{n(j-1)}, \ldots, c^{k(j-1)})}$$

where the numerator of the fraction in (5) stands for the determinant of a $2k + 1$ by $2k + 1$ matrix whose j-th row is given by the parenthesized expression appearing in the numerator. Analogous notation is used in the denominator of the fraction in (5). We observe also that based on (2) it can be shown that the denominator of the fraction in (5) is never zero.

Next we give a generalization of a result of J. L. Walsh.

Lemma. *Let $f(z)$ be an analytic function in the annulus $R < |z| < R'$ and let a, b, r be positive real numbers such that*

$$(6) \qquad R < r - b < r + a < R' \quad \text{and} \quad (r + a)(r - b) = r^2.$$

Let $q_k(z)$ be the rational function as given by (1) *which coincides* (*interpolates*) *with $f(z)$ in $2k + 1$ points equally spaced on the circle C of radius r and with center at 0 in the z-plane and with r as one of the $2k + 1$ points. Then*

$$(7) \qquad f(z) = \lim_{k \to \infty} q_k(z) \quad \text{uniformly for} \quad r - b \leqq |z| \leqq r + a.$$

Proof. Let us perform in the hypothesis of the Lemma the transformation $T(z)$ of z to z' given by

$$(8) \qquad z' = T(z) = \frac{z}{\sqrt{(r + a)(r - b)}}.$$

From (6) and (8) it follows that

$$(9) \qquad T(r - b) = \sqrt{\frac{r - b}{r + a}} \quad \text{and} \quad T(r + a) = \sqrt{\frac{r + a}{r - b}}.$$

Also, from (8) it follows that $T(z)$ transforms C into the unit circle. Moreover, by (9) we have:

$$T(r - b) = \frac{1}{T(r + a)}.$$

But then (7) follows immediately from III of [3] or Va of [4, p. 201].

Remark. In view of (5), the coefficients $a_{k,n}$ of the (interpolating) rational function $q_k(z)$ mentioned in the Lemma are given by:

$$(10)\quad a_{k,n} = r^{-(2k+1)n} \cdot \frac{\det\left(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, f(rc^{j-1}), \ldots, c^{k(j-1)}\right)}{\det\left(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, c^{n(j-1)}, \ldots, c^{k(j-1)}\right)}$$

which is obtained from (5) by replacing $q_k(rc^{j-1})$ in it by $f(rc^{j-1})$ since for this case (4) is replaced by:

$$f(r), f(rc), f(rc^2), \ldots, f(rc^{2k}).$$

Next, we prove

Theorem. *Let $f(z)$ be an analytic function in the annulus $R < |z| < R'$. Then in the annulus*

$$(1)\qquad f(z) = \ldots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots$$

where for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ we have

$$(12)\quad a_n = \lim_{k\to\infty} r^{-(2k+1)n} \cdot \frac{\det\left(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, f(rc^{j-1}), \ldots, c^{k(j-1)}\right)}{\det\left(c^{-k(j-1)}, \ldots, c^{-2(j-1)}, c^{-(j-1)}, 1, \ldots, c^{n(j-1)}, \ldots, c^{k(j-1)}\right)}$$

and where r is a real number such that $R < r < R'$ and $c = e^{2\pi i/2k+1}$.

Proof. As remarked earlier, the $2k + 1$ by $2k + 1$ determinant appearing in the denominator of (12) is never zero.

Next, we show that there exist always positive real numbers a and b such that

$$(13)\qquad (r + a)(r - b) = r^2 \quad \text{with} \quad a < (R' - r) \quad \text{and} \quad b < (r - R).$$

Indeed, let us choose a positive real number a such that

$$(14)\qquad a < \min\left\{(R' - r), \frac{r(r - H)}{H}\right\} \quad \text{with} \quad R < H < r < R'$$

and let us define

$$(15)\qquad b = \frac{ra}{r + a}.$$

Clearly, (15) immediately implies the equality in (13) and (14) implies the first inequality in (13). On the other hand, by (14) we have $a < \frac{r(r-H)}{H}$ which again by (14) implies $0 < r(r - H) - Ha < r(r - R) - Ra$. Consequently, $ra < r(r - R) - Ra + ra$ and therefore $ra < (r - R)(r + a)$ which by (15) implies the second inequality in (13).

Thus, $f(z)$ is an analytic function in the annulus $R < |z| < R'$ and, in view of (13), the positive real numbers a, b, r satisfy (6). Thus, in view of the Lemma, $f(z)$ satisfies (7). Consequently, for $n = 0, \pm 1, \pm 2, \dots$

$$\frac{f(z)}{z^{n+1}} = \lim_{k\to\infty} \frac{p_k(z)}{z^{n+1}} \qquad \textit{uniformly for} \qquad r - b \leqq |z| \leqq r + a.$$

But then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz = \lim_{k\to\infty} \frac{1}{2\pi i}\int_C \frac{p_k(z)}{z^{n+1}}\,dz$$

where C is the circle mentioned in the Lemma.

However, the left side of the above equality represents [2, p. 77] the coefficient a_n of the Laurent expansion of $f(z)$ given in (11). Similarly, the right side (after the lim sign) of the above equality represents the coefficient $a_{k,n}$ of the rational function $q_k(z)$ given in (10). Thus,

$$a_n = \lim_{k\to\infty} a_{k,n} \qquad \textit{for } n = 0, \pm 1, \pm 2, \dots$$

which, by (10) and (11), implies (12), as desired.

The result of this paper complements an earlier result [1] of the author.

REFERENCES

[1] A. Abian, *The identity theorem for analytic functions*, J. of Math. Anal. and Appl. 45(1974) 682—683.

[2] N. Levinson and R. M. Redheffer: *Complex Variables*, Holden-Day, Inc. (1970), San Francisco.

[3] J. L. Walsh: *Interpolation and an analog of the Laurent development*, Proc. Nat. Acad. Scien. 19 (1933) 203—207.

[4] J. L. Walsh: *Interpolation and approximation*, Amer. Math. Soc. Coll. Pub. XX (1935), New York.

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