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Jitka Ševečková

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THE FIVE-GROUP THEOREM

JITKA ŠEVEČKOVÁ, Brno

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O. Borůvka in [1] proved a theorem called "the special five-group theorem" (25.4). This theorem is a generalization of Zassenhaus lemma for groups (e.g. [2] III 4.3). In Zassenhaus lemma the existence of an isomorphism of factors on certain subgroups is given, in the mentioned five-group theorem there is shown that the above isomorphism is a consequence of the following set-theoretic relation between these factor groups: Every co-set of one factor group meets exactly one co-set of the other factor group. The theorem 25.4 [1] is then stronger than that of Zassenhaus.

In the present note we give a theorem concerning product of congruences in an Ω-group which has 25.4 [1] as its corollary. Our theorem proceeds from 3.5.5 [3], reproduces it partly (see our assertions 1, 2, 3 and 5) and moreover, proves 4 and 6. In contrast to 3.5.5 [3], our proof is not based on Zassenhaus lemma; quite conversely, that lemma follows from our theorem.

As for concepts concerning partitions and congruences "in" see [3, 4]. *The partition in* a set $\mathfrak{G}$ is a family A of nonempty pairwise disjoint subsets of $\mathfrak{G}$. Union $\bigcup A$ of these subsets is called a *domain* of A and every element $A^1 \in A$ is said to be a *block* of A and will be denoted by $A^1 = A(x)$ provided it contains the element $x \in \mathfrak{G}$. The *intersection* of A and a subset $\mathfrak{B}$ ($\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$) is defined as follows $\mathfrak{B} \sqcap A :=$ $:= \{\mathfrak{B} \cap A^1 : A^1 \in A, A^1 \cap \mathfrak{B} \neq \emptyset\}$ [1] 2.3. Two partitions in $\mathfrak{G}$ are said to be *coupled* if every block of one partition meets exactly one block of the other partition [1] 4.1. The system $P(\mathfrak{G})$ of all partitions in $\mathfrak{G}$ is a complete lattice. This system is evidently in a 1-1-correspondence with the family of all symmetric and transitive binary relations in $\mathfrak{G}$. Hence, we need not to distinguish between these both concepts. A stable symmetric and transitive relation in an algebra $(\mathfrak{G}, \Omega)$ is called a *congruence in* $(\mathfrak{G}, \Omega)$. (The congruence *on* $(\mathfrak{G}, \Omega)$ is then a special case of the above concept.) The system $\mathscr{K}(\mathfrak{G}, \Omega)$ of all congruences in $(\mathfrak{G}, \Omega)$ is a complete latiice. In general, $\mathscr{K}$-suprema do not coincide with P-suprema. Let $(\mathfrak{G}, \Omega)$ be an Ω-group. The symbol $\mathfrak{B} \lhd \mathfrak{G}$ or $\mathfrak{B} \subset\!| \ \mathfrak{G}$ means that $\mathfrak{B}$ is an ideal or an Ω-subgroup of $\mathfrak{G}$, respectively. Now, let A be a binary relation in a set $\mathfrak{G}$ and $x \in \mathfrak{G}$; then $A(x)$ denotes the set $\{y \in \mathfrak{G} : yAx\}$ and $\bigcup A = \bigcup\{A(x) : x \in \mathfrak{G}\}$. This notation is in accord with the above introduced symbols $A(x)$ and $\bigcup A$ for a partition A.

Theorem. *Let B and C be congruences in an Ω-group $(\mathfrak{G}, \Omega)$. Then*

(1) $\mathsf{U}BC = \mathsf{U}B \cap [B(0) + \mathsf{U}C] = B(0) + \mathsf{U}B \cap \mathsf{U}C$

(2) $BC(0) = \mathsf{U}B \cap [B(0) + C(0)] = B(0) + \mathsf{U}B \cap C(0)$
in both (1) *and* (2), *the order of summands can be changed*;

(3) $BC(0) \lhd \mathsf{U}BC \subset\!| \, \mathfrak{G}, CB(0) \lhd \mathsf{U}CB \subset\!| \, \mathfrak{G}, \mathsf{U}B \cap C(0) + \mathsf{U}C \cap B(0) \lhd \mathsf{U}B \cap \mathsf{U}C.$

(4) *The partitions* $D := \mathsf{U}BC/BC(0)$, $E := \mathsf{U}CB/CB(0)$ *and*

$$F := \mathsf{U}B \cap \mathsf{U}C/\mathsf{U}B \cap C(0) + \mathsf{U}C \cap B(0)$$

are pairwise coupled and hence

(5) $$D \cong E \cong F.$$

Moreover, there holds

(6) $$\mathsf{U}C \sqcap D = \mathsf{U}B \sqcap E = D \wedge E = F.$$

Proof. (1) and (2) are proved in 3.5.5 [3].

(3) $\mathsf{U}BC$ is an Ω-subgroup of $\mathfrak{G}$ since $\mathsf{U}B \cap \mathsf{U}C$ is an Ω-subgroup and $B(0)$ an ideal of the Ω-group $(\mathsf{U}B, \Omega)$ (e.g. [2] III 4.1). We shall show that $BC(0)$ is an ideal of the Ω-group $(\mathsf{U}BC, \Omega)$. Denote $\mathsf{U}BC/B(0) = \mathrm{P}$, $BC(0)/B(0) = Q$ and for $x \in \mathsf{U}BC$ put $\bar{x} = P(x)$. Evidently, it suffices to show that Q is an ideal of P. The normality of Q in P follows from the stability of the relation BC ([3] 3.2). For if $q \in \bar{q} \in Q$ then $qBC0$; if $p \in \bar{p} \in P$ then $(\pm p)\,BC(\pm p)$. Hence $(p + q - p)\,BC(p + 0 - p)$, $p + q - p \in BC(0)$, $\bar{p} + \bar{q} - \bar{p} \in Q$. We need to prove that for $\omega \in \Omega$ n-ary ($n \geqq 1$), $\bar{q}_i \in Q$ and $\bar{x}_i \in P$ ($i = 1, \ldots, n$) there holds $(\bar{q}_1 + \bar{x}_1) \ldots (\bar{q}_n + \bar{x}_n)\,\omega = \bar{q} + \bar{x}_1 \ldots \bar{x}_n\omega$ for a suitable $\bar{q} \in Q$. For $q_i \in \bar{q}_i$ there exist $a_i \in B(0)$ and $b_i \in \mathsf{U}B \cap C(0)$ with $q_i = a_i + b_i$; hence $\bar{q}_i = \bar{a}_i + \bar{b}_i = \bar{b}_i$. Similarly, there exists $c_i \in \mathsf{U}B \cap \mathsf{U}C$ with $\bar{x}_i = \bar{c}_i$. Finally $(\bar{q}_1 + \bar{x}_1) \ldots (\bar{q}_n + \bar{x}_n)\,\omega = (\bar{b}_1 + \bar{c}_1) \ldots (\bar{b}_n + \bar{c}_n)\,\omega = ((b_1 + c_1) \ldots (b_n + c_n)\,\omega)^- = (b + c_1 \ldots c_n\omega)^-$ for a suitable $b \in \mathsf{U}B \cap C(0)$ since $\mathsf{U}B \cap C(0)$ is an ideal of $\mathsf{U}B \cap \mathsf{U}C$. Thus, we have proved $(\bar{q}_1 + \bar{x}_1) \ldots (\bar{q}_n + \bar{x}_n)\,\omega = \bar{b} + \bar{c}_1 \ldots \bar{c}_n\omega = \bar{b} + \bar{x}_1 \ldots \bar{x}_n\omega$ with $\bar{b} \in Q$.

The assertion for CB can be proved symmetrically. The last assertion in (3) is evident.

(4) First, the following evident relations resulting from (1) and (2) hold

$$\mathsf{U}BC = B(0) + \mathsf{U}B \cap \mathsf{U}C = B(0) + \mathsf{U}B \cap C(0) + \mathsf{U}B \cap \mathsf{U}C = BC(0) + \mathsf{U}B \cap \mathsf{U}C \supseteq \mathsf{U}B \cap \mathsf{U}C$$

$$\mathsf{U}CB = CB(0) + \mathsf{U}B \cap \mathsf{U}C \supseteq \mathsf{U}B \cap \mathsf{U}C.$$

It follows that every block of D meets a block of E, namely in an element of $\mathsf{U}B \cap \mathsf{U}C$. We shall show that it meets only one block of E. Let $CB(0) + x = CB(0) + y$,

$x, y \in \mathsf{U}B \cap \mathsf{U}C$. By (2)

$$y - x \in CB(0) \cap (\mathsf{U}B \cap \mathsf{U}C) = \mathsf{U}C \cap [C(0) + B(0)] \cap (\mathsf{U}B \cap \mathsf{U}C) \subseteq BC(0).$$

Hence if blocks $BC(0) + x$ and $BC(0) + y$ of D meet some block $CB(0) + x$ $(= CB(0) + y)$ of E, then they are equal.

The rest of (4) and (6). The domain of the partition $\mathsf{U}C \sqcap D$ equals to $\mathsf{U}BC \cap \mathsf{U}C = (B(0) + \mathsf{U}B \cap \mathsf{U}C) \cap \mathsf{U}C = \mathsf{U}B \cap \mathsf{U}C = \mathsf{U}F$ since $\mathsf{U}B \cap \mathsf{U}C \subseteq B(0) + \mathsf{U}B \cap \mathsf{U}C \subseteq \mathsf{U}B$. We shall show later that $(\mathsf{U}C \sqcap D)(0) = F(0)$. Put $H = \mathsf{U}B \sqcap C$. Then $(\mathsf{U}C \sqcap D)(0) = BC(0) \cap \mathsf{U}C = \mathsf{U}C \cap [B(0) + \mathsf{U}B \cap C(0)] = (\mathsf{U}C \cap \mathsf{U}B) \cap [B(0) + H(0)]$. Use 3.5.3 [3] for $Q = \mathsf{U}C$. Then the last set equals to $\mathsf{U}B \cap [\mathsf{U}C \cap B(0) + \mathsf{U}C \cap H(0)] = \mathsf{U}B \cap [\mathsf{U}C \cap B(0) + \mathsf{U}B \cap C(0)] = F(0)$. The proof is complete.

From the preceding theorem Borůvka's (special) five-group theorem [1] 25.4 follows.

Corollary. *Let $(\mathfrak{G}, \Omega)$ be an Ω-group, $\mathfrak{B}' \lhd \mathfrak{B} \subset\!| \mathfrak{G}$, $\mathfrak{C}' \lhd \mathfrak{C} \subset\!| \mathfrak{G}$, $\mathfrak{B}' \cap \mathfrak{C} + \mathfrak{C}' \cap \mathfrak{B} \subseteq \mathfrak{V} \lhd \mathfrak{B} \cap \mathfrak{C}$. Then*

(3a) $\mathfrak{B}' + \mathfrak{V} \lhd \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C} \subset\!| \mathfrak{G}$, $\mathfrak{C}' + \mathfrak{V} \lhd \mathfrak{C}' + \mathfrak{C} \cap \mathfrak{B} \subset\!| \mathfrak{G}$.

(4a) *The partitions $K := \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C} / \mathfrak{B}' + \mathfrak{V}$, $L := \mathfrak{C}' + \mathfrak{C} \cap \mathfrak{B} / \mathfrak{C}' + \mathfrak{V}$ and $M := \mathfrak{B} \cap \mathfrak{C} / \mathfrak{V}$ are pairwise coupled and thus*

(5a) $$K \cong L \cong M.$$

Moreover, there holds

(6a) $$\mathfrak{C} \sqcap K = \mathfrak{B} \sqcap L = K \wedge L = M.$$

Putting $\mathfrak{V} = \mathfrak{B}' \cap \mathfrak{C} + \mathfrak{C}' \cap \mathfrak{B}$, we obtain Zassenhaus lemma.

Proof. Put $B = \mathfrak{B} / \mathfrak{B}'$, $C = \mathfrak{C} / \mathfrak{C}'$. By (1) and (2), $\mathsf{U}BC = \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}$, $BC(0) = \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}'$. By (4), the partitions

(4b) $$\begin{cases} D := \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C} / \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}', \qquad E := \mathfrak{C}' + \mathfrak{C} \cap \mathfrak{B} / \mathfrak{C}' + \mathfrak{C} \cap \mathfrak{B}', \\ F := \mathfrak{B} \cap \mathfrak{C} / \mathfrak{B}' \cap \mathfrak{C} + \mathfrak{C}' \cap \mathfrak{A} \\ \text{are pairwise coupled} \end{cases}$$

and there holds

(6b) $$\mathfrak{C} \sqcap D = \mathfrak{B} \sqcap E = D \wedge E = F.$$

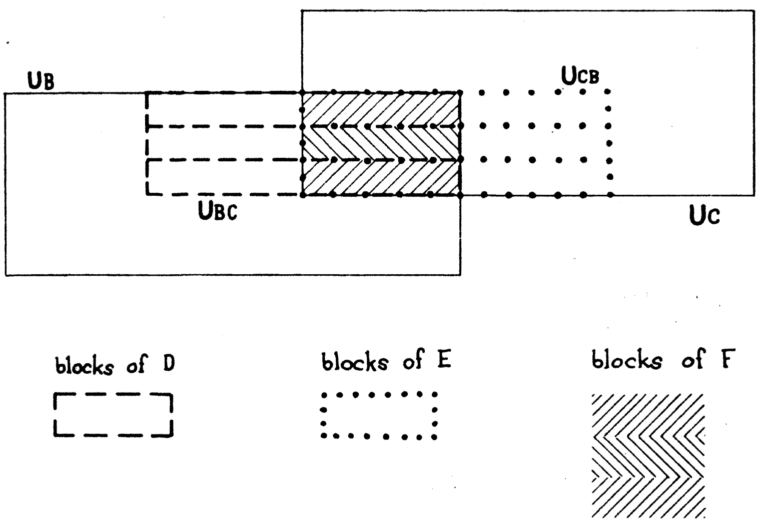
Define $V = \mathfrak{B} \cap \mathfrak{C} / \mathfrak{V}$. Because of $\mathsf{U}M = \mathsf{U}V$ and by (4b) and (6b), the partitions

$$K := D \vee_p V,\ L := E \vee_p V \text{ and } M := F \vee_p V$$

are pairwise coupled, too. We shall express the partitions K, L and M as factor Ω-groups. By [5] 2.1, there holds $D \vee_{\mathscr{X}} V = D \vee_p V$ because of $\mathsf{U}D \supseteq \mathsf{U}V$. By [3]

3.5.7, $(D \vee_P V)(0) = [D(0) + \mathsf{U}D \cap V(0)] \cup [\mathsf{U}V \cap D(0) + V(0)] = [\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}' + (\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}) \cap \mathfrak{B}] \cup [\mathfrak{B} \cap \mathfrak{C} \cap (\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}') + \mathfrak{B}]$. By 3.5.7 [3] again, the first member of the union is an ideal of $D(0) + (\mathsf{U}D \cap \mathsf{U}V) = \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}' + (\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}) \cap \mathfrak{B} \cap \mathfrak{C} = \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C}$ and evidently is equal to $\mathfrak{B}' + \mathfrak{B}$, the other member is contained in the first one. Therefore $D \vee_P V = D \vee_{\mathscr{K}} V = \mathfrak{B}' + \mathfrak{B} \cap \mathfrak{C} / \mathfrak{B}' + \mathfrak{B} = K$. Similarly for L and M. So the assertion (4a) is proved. (6a) follows immediately.

Remark. *Scheme illustrating the set-theoretic relations* (4) *between the partitions D, E and F.*



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J. Ševečková
662 73 Brno, Čechyňská 16
Czechoslovakia