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ON REPRESENTATIONS OF NUMBERS BY SUMS OF SQUARES AND QUADRATIC FIELDS

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This article is concerned with the well-known arithmetical function – the number of representations of naturals by sums of an odd number of integer squares. The famous results are summarized and some new arithmetical properties of this function, its connections with other number – theoretic functions, are proved.

1°. Let be $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ correspondingly the set of natural, integer, rational, real and complex numbers, a prime $p \geqq 3$, $s = 2\sigma + 1$, $\sigma \in \mathbf{N}$,

$$r_s(n) = \sum_{x_1^2 + \ldots + x_s^2 = n} 1 \tag{1.1}$$

is the number of representations of a number $n \in \mathbf{N}$ by the sums of squares of the numbers $x_i \in \mathbf{Z}$, $i = 1, \ldots, s$.

The classic investigations of the function $r_s(n)$ (cf. [14] for historical remarks on this topic) was continued by Lomadze ([14] – [16]). Following to the results of Hardy and Estermann with the help of only elements of the theory of the functions of complex variables, Lomadze proved Mordell's identity (simultaneously for even and odd numbers $s > 8$) in the form

$$\vartheta_3^s(0 \mid \tau) = 1 + \sum_{n=1}^{\infty} \varrho_s(n) \beta^n + \sum_{k=1}^{e} \alpha_s(k) \vartheta_3^{s-8k}(0 \mid \tau) \vartheta_0^{4k}(0 \mid \tau) \vartheta_2^{4k}(0 \mid \tau), \tag{1.2}$$

where the theta-functions

$$\vartheta_0(z \mid \tau) = \sum_{m=-\infty}^{\infty} (-1)^m \beta^{m^2} \exp(2miz),$$

$$\vartheta_2(z \mid \tau) = \sum_{m=-\infty}^{\infty} \beta^{(m+\frac{1}{2})^2} \exp((2m+1)iz),$$

$$\vartheta_3(z \mid \tau) = \sum_{m=-\infty}^{\infty} \beta^{m^2} \exp(2miz),$$

so that

$$\vartheta_3^s(0 \mid \tau) = 1 + \sum_{n=1}^{\infty} r_s(n) \beta^n,$$

and $\beta = \exp(i\pi\tau)$, $\tau = x + iy$, $x, y \in \mathbf{R}$, $y > 0$. The number $e \in \mathbf{N}$ is determined uniquely from the condition (for $s > 8$)

$$8e < s \leqq 8(e+1).$$

The arithmetical properties of the constants $\alpha_s(k)$ are considered below and the main member $\varrho_s(n)$, containing the singular series, in the formulae for the quantity of representations

$$r_s(n) = \varrho_s(n) + \delta_s(n) \tag{1.3}$$

has the form

$$\varrho_s(n) = \frac{2^{2\sigma+1}\sigma!\, n^{\sigma-\frac{1}{2}}}{\pi^\sigma(2^{2\sigma}-1)\,|B_{2\sigma}|}\chi_2(n)T_\sigma(n)\,\overline{\mathscr{L}}(\sigma, \eta t),$$

where $n = f^2 t = 2^\gamma u$, t is squarefree number, $(2, u) = 1$, $\eta = (-1)^\sigma$, $v = nq^{-a}$, if $(v, q) = 1$ and q is a prime odd number, $\left(\frac{d}{u}\right)$ – symbol of Kronecker with $\left(\frac{d}{u}\right) = 0$ for $(d, u) = b > 1$,

$$\overline{\mathscr{L}}(\sigma, \eta t) = \sum_{v=0,\, 2\not\,| v}\left(\frac{\eta t}{v}\right)v^{-\sigma},$$

$$T_\sigma(n) = \prod_{q|f}\left[1 - \left(\frac{\eta t}{q}\right)q^{-\sigma}\right]\prod_{q|n}(1 - q^{1-2\sigma})^{-1}\prod_{q|n,\, 2\not\,| a}\left(1 - q^{(1-2\sigma)\frac{a+1}{2}}\right)\times$$

$$\times\prod_{q|n,\, 2|a}\left(1 + \frac{\left(\frac{\eta v}{q}\right) - q^{1-\sigma}}{1 - \left(\frac{\eta v}{q}\right)q^{-\sigma}}q^{(1-2\sigma)\frac{a}{2}-\sigma}\right), \tag{1.4}$$

and as usual the empty product is supposed to be equal to one,

$$\chi_2(n) = \begin{cases} 1 \mp \Delta \pm \Delta(2 - 2^{1-2\sigma})2^{(1-2\sigma)\frac{\gamma-1}{2}}, & \gamma \geqq 1, 2\not\,|\gamma, \\ 1 \mp \Delta \pm \Delta(2 - 2^{1-2\sigma})2^{(1-2\sigma)\frac{\gamma}{2}}, & \eta u \equiv 3 \pmod 4, 2 | \gamma, \\ 1 \mp \Delta \pm \Delta(2^\sigma - 1 - 2^{1-\sigma})2^{(1-2\sigma)\frac{\gamma+2}{2}}, & \eta u \equiv 5 \pmod 8, 2 | \gamma, \\ 1 \mp \Delta \mp \Delta(2^\sigma - 1 - 2^{1-\sigma})2^{(1-2\sigma)\frac{\gamma+2}{2}}, & \eta u \equiv 1 \pmod 8, 2 | \gamma, \end{cases} \tag{1.5}$$

where $\Delta = (2^\sigma - 2^{1-\sigma})^{-1}$ and the upper signs are taken when $\sigma \equiv 1; 2$ and the lower ones when $\sigma \equiv 3; 0 \pmod 4$.

Further, starting from the famous investigations of Hardy ([5], [6]), Suetuna ([24]) and Bateman ([2]), we conclude that the representation (1.3) is true for every odd $s \geqq$ $\geqq 3$, if we assume $\delta_s(n) = 0$ for $s = 3, 5, 7$.

Let below

$$d = \begin{cases} \eta t, & \eta t \equiv 1 \pmod 4, \\ 4\eta t, & \eta t \equiv 2\,;\,3 \pmod 4, \end{cases}$$

B_k be Bernoulli numbers, satisfying symbolic identity $B_k + k = (B+1)^k$, $k = 2, 3, \ldots$, $B_0 = 1$, $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_k x^{k-i}$ be Bernoulli polynomials (for the identity character

$$\sum_{u=1}^{|d|} \left(\frac{d}{u}\right) B_k\left(\frac{u}{|d|}\right) = B_k, \qquad d = 1,$$

are supposed) and

$$B_m(\chi) = \mathscr{F}^{m-1} \sum_{u=1}^{\mathscr{F}} \chi(u) B_m\left(\frac{u}{\mathscr{F}}\right)$$

be generalized Bernoulli numbers belonging to a primitive residue character χ modulo $\mathscr{F}$. Then in the force

$$\bar{\mathscr{L}}(\sigma, \eta t) = \left[1 - \left(\frac{d}{2}\right) 2^{-\sigma}\right] \mathscr{L}(\sigma, d), \qquad \mathscr{L}(\sigma, d) = \sum_{v=1}^{\infty} \left(\frac{d}{v}\right) v^{-\sigma},$$

and

$$\mathscr{L}(\sigma, d) = (-1)^{\left[\frac{\sigma}{2}\right]+1} \frac{(2\pi)^\sigma}{2\sigma!\,|d|^{\sigma-\frac{1}{2}}} B_\sigma(\chi),$$

where $\chi(x) = \left(\frac{d}{x}\right)$ is a symbol of Kronecker ([13]), we obtain finally

$$\varrho_s(n) = (-1)^{\left[\frac{\sigma}{2}\right]+1} \frac{2^{2\sigma} f^{2\sigma-1}}{(2^{2\sigma}-1)\,|B_{2\sigma}|} \, \frac{2^{\sigma} - \chi(2)}{2^{\alpha(2\sigma-1)}} \chi_2(n) T_\sigma(n) B_\sigma(\chi), \tag{1.6}$$

with

$$\alpha = \begin{cases} 1, & 2 \mid d, \\ 0, & 2 \nmid d. \end{cases}$$

Here as usual $[Z]$ is the integral part of Z. If we propose that n is squarefree so that all representations are proper: in (1.1) $(x_1, \ldots, x_s) = 1$; then in this case $T_\sigma(n) = 1$, as it follows from (1.4), and therefore by (1.6) we obtain

$$\varrho_s(n) = (-1)^{\left[\frac{\sigma}{2}\right]+1} \frac{2^{2\sigma}}{(2^{2\sigma}-1)\,|B_{2\sigma}|} \cdot \frac{2^{\sigma} - \chi(2)}{2^{\alpha(2\sigma-1)}} \chi_2(n) B_\sigma(\chi). \tag{1.7}$$

Since equation (1.3) for $s = 3, 5, 7$ is transformed to $r_s(n) = \varrho_s(n)$ we obtain from (1.7) (or from corresponding results of Bateman, Suetuna or Lomadze with $s = 3$) Gauss formulae for the number of a proper representations of naturals by the sum of three square for $s = 3$ (note that for $n \equiv 7 \pmod 8$ the vanishing of $r_s(n)$ takes place because of $\chi_2(n) = 0$) and Smith–Minkowski formulas for $s = 5$ and 7.

2°. As it is known, Gauss determined the connections between the number $r_3(n)$ of the representations of natural n by the sum of three square and the class number $H(df^2)$ of binary quadratic forms with the determinant df^2 which is some multiple of this natural. The investigations in this way were continued for squarefree naturals by Kiselev ([9]) and Kiselev, Slavutsky ([10], [11]). More general connections of this kind are given below and new facts about the arithmetical structure of the remainder term in (1.3) are received.

Lemma 1. ([10], [11]). *If $H(df^2)$ is the class number of equivalent binary quadratic forms (positive for $d < 0$) of the type $ax^2 + bxy + cy^2$ where $(a, b, c) = 1$, with the determinant df^2, then*

$$H(df^2)\frac{U_{(f),l}}{p^{l-1}} \equiv -\varkappa T_{(f),l}\prod_{q|f}\left[1-\left(\frac{d}{q}\right)\frac{1}{q}\right]\frac{B_m(\chi)}{m} \pmod{p^l} \tag{2.1}$$

for $d = np > 0$, $n \geqq 1$, χ mod n;

$$H(df^2) \equiv -f\prod_{q|f}\left[1-\left(\frac{d}{q}\right)\frac{1}{q}\right]\frac{B_{m+1}(\chi)}{m+1} \pmod{p^l} \tag{2.2}$$

for $d = -np < -4$, $n \geqq 1$, χ mod n;

$$H(df^2)\frac{\overline{U}_{(f),l}}{p^l} \equiv -\varkappa\overline{T}_{(f),l}\prod_{q|f}\left[1-\left(\frac{d}{q}\right)\frac{1}{q}\right]\frac{B_{2m}(\chi)}{2m} \pmod{p^l} \tag{2.3}$$

for $d > 0$, $p \nmid df$, χ mod d;

$$H(df^2)\frac{U_{(f),l}}{p^{l-1}} \equiv -\varkappa T_{(f),l}\prod_{q|f_0}\left[1-\left(\frac{d}{q}\right)\frac{1}{q}\right]\frac{B_{2m}(\chi)}{2m} \pmod{p^l} \tag{2.4}$$

for $d > 0, p \nmid d, (p, f_0) = 1, f = p^c f_0$, $c \in \mathbf{N}$, χ mod d;

$$\left[1-\left(\frac{d}{p}\right)\right]H(df^2) \equiv -f\prod_{q|f}\left[1-\left(\frac{d}{q}\right)\frac{1}{q}\right]\frac{B_{2m+1}(\chi)}{2m+1} \pmod{p^l} \tag{2.5}$$

for $d < -4$, $p \nmid d$, χ mod $|d|$,

where $m = \dfrac{p-1}{2}p^{l-1}$, $\chi(u)$ is coincide with the corresponding symbol of Kronecker, $\varkappa = 1$ for $N(E_1) = -1$ and $\varkappa = 2$ for $N(E_1) = 1$, $N(E_1)$ is the norm of the main unit $E_1 = T_1 + U_1\sqrt{d}$ in the quadratic field $\mathbf{Q}(\sqrt{d})$, T_1, U_1 is the least positive solution of the equation $T^2 - dU^2 = \pm 1$ in integers or halfintegers, with the same suppositions $T_{(f),1}, U_{(f),1}$ is the solution of $T^2 - df^2U^2 = \pm 1$, $E_{(f),l} = T_{(f),l} + U_{(f),l}f\sqrt{d} = E_{(f),1}^{p^{l-1}}$, $\overline{E}_{(f),l} = \overline{T}_{(f),l} + \overline{U}_{(f),l}f\sqrt{d} = E_{(f),1}^{\left[p-\left(\frac{d}{p}\right)\right]p^{l-1}}$.

Remark. The properties of the generalized Bernoulli numbers $B_k(\chi)$ introduced by Leopoldt ([13]) were considered by many of the authors ([3], [23] etc.). In particular, it is known that $\frac{B_k(\chi)}{\varkappa}$ is p-integer number for the character $\chi \bmod \mathscr{F}$, $(p, \mathscr{F}) = 1$ and $\mathscr{F} > 1$. At last, going to the p-adic limit in (2.1)–(2.5) we obtain the local analog of the formulas of Dirichlet for the class number of quadratic fields (in the narrow sense) in terms of p-adic $\mathscr{L}$-functions of Kubota–Leopoldt ([12], [8], [7], [19], [22]).

Lemma 2. *If $e \in \mathbf{N}$ is fixed by the prime $p > 7$ so that $8e < p \leqq 8(e+1)$, $k \leqq e$, $k \in \mathbf{N}$, then the rationals $\alpha_p(k)$ determined by the identity* (1.2) *satisfy the condition*

$$\alpha_p(k) \equiv 0 \pmod p.$$

First of all from (1.6) $\Rightarrow \varrho_s(n) \in \mathbf{Q}$, then also $\delta_s(n) \in \mathbf{Q}$. Further

$$\vartheta_3^{s-8k}(0 \mid \tau)\, \vartheta_0^{4k}(0 \mid \tau)\, \vartheta_2^{4k}(0 \mid \tau) =$$

$$= (1 + 2\beta + 2\beta^4 + \ldots)^{s-8k}(1 - 2\beta + 2\beta^4 - \ldots)^{4k}(1 + \beta^2 + \ldots)^{4k} 2^{4k} \beta^k =$$

$$= \sum_{i=0}^{\infty} h_i(s, k)\, \beta^{k+i}, \tag{2.6}$$

where $h_i(s, k) \in \mathbf{Z}$ for $i, s \in \mathbf{N} \cup \{0\}$ and $s > 8$, $1 \leqq k < \frac{s}{8}$, $h_0(s, k) = 2^{4k}$, so that the identity (1.2) may be written as

$$\sum_{n=1}^{\infty} r_s(n)\, \beta^n = \sum_{n=1}^{\infty} \varrho_s(n)\, \beta^n + \sum_{k=1}^{e} \alpha_s(k) \sum_{i=0}^{\infty} h_i(s, k)\, \beta^{k+i}$$

or

$$\sum_{n=1}^{\infty} r_s(n)\, \beta^n = \sum_{n=1}^{\infty} \varrho_s(n)\, \beta^n + \sum_{n=1}^{e} \beta^n \sum_{k=1}^{n} \alpha_s(k)\, h_{n-k}(s, k) + \sum_{n=e+1}^{n} \beta^n \sum_{k=1}^{e} \alpha_s(k)\, h_{n-k}(s, k). \tag{2.7}$$

Therefore from the three-cornered system

$$\sum_{k=1}^{n} \alpha_s(k)\, h_{n-k}(s, k) = \delta_s(n), \qquad n = 1, \ldots, e, \tag{2.8}$$

about $\alpha_s(k)$ we conclude that $\alpha_s(k) \in \mathbf{Q}$.

Let further $s = 2\sigma + 1 = p$ be a prime. Since $r_p(n) \equiv 0 \pmod p$, then the congruence

$$\delta_p(n) \equiv 0 \pmod p \tag{2.9}$$

is the consequence of $\varrho_p(n) \equiv 0 \pmod p$, as this follows from (1.3).

To prove the lemma it is sufficient to verify the congruence (2.9) for $n \leqq e$ only, where $8e < p \leqq 8(e+1)$.

First of all we turn to (1.4) and show that $T_\sigma(n)$ is p-adic integer for $\sigma = \frac{p-1}{2}$. Indeed, as $n < p$, then $q^{p-2} \not\equiv 1 \pmod p$ for $q \mid n$. Further so $\left(\frac{\eta t}{q}\right) = \left(\frac{\eta v}{q}\right)$ in this case when $2 \mid a$, $q^a \mid n$, but $q^{a+1} \nmid n$, then every multiplier from the denominator in (1.4) is found to be equal to that of the first product.

Now consider $\chi_2(n)$. Here $\sigma = \frac{p-1}{2}$, $1 - 2\sigma = -(p-2)$, so $2^{\pm\sigma} \equiv \left(\frac{2}{p}\right)$, $2^{1-2\sigma} \equiv 2 \pmod p$ and the upper signs in (1.5) are corresponded to the cases $p \equiv 3; 5 \pmod 8$ but the lower ones to $p \equiv 1; 7 \pmod 8$. Then from (1.5) in the case $\gamma \geqq 1$, $2 \nmid \gamma$ we obtain

$$\chi_2(n) = \frac{1 - 2^{1-2\sigma} \mp 2^{-\sigma} \pm 2^{-\sigma}(2 - 2^{1-2\sigma}) . 2^{(1-2\sigma)\frac{\gamma-1}{2}}}{1 - 2^{1-2\sigma}} =$$

$$= \frac{(1 \pm 2^{-\sigma})\left\{1 \mp 2^{-\sigma} \mp 2^{-\sigma} \pm (1 \pm 2^{-\sigma}) . 2^{(1-2\sigma)\frac{\gamma-1}{2}}\right\}}{1 - 2^{1-2\sigma}} \tag{2.10}$$

and by making use of the congruences $1 \pm 2^\sigma \equiv 1 \pm \left(\frac{2}{p}\right) \equiv 1 - \left(\frac{2}{p}\right)^2 \equiv 0 \pmod p$ finally we have

$$\chi_2(n) \equiv 0 \pmod p.$$

By analogy in the case $2 \mid \gamma$ we get

$$\chi_2(n) \equiv 0 \pmod p, \qquad \eta u \equiv 3 \pmod 4,$$

$$\chi_2(n) \equiv \mp\left[1 + \left(\frac{2}{p}\right) - 2\right] . 2^{\frac{\gamma+2}{2}} \pmod p, \qquad \eta u \equiv 5 \pmod 8,$$

$$\chi_2(n) \equiv \pm\left[1 - \left(\frac{2}{p}\right) - 2\right] . 2^{\frac{\gamma+2}{2}} \pmod p, \qquad \eta u \equiv 1 \pmod 8.$$

Hence, always $\chi_2(n) \equiv 0 \pmod p$, except the cases $\eta u \equiv 5 \pmod 8$, $p \equiv 3; 5 \pmod 8$, $2 \mid \gamma$ or $\eta u \equiv 1 \pmod 8$, $p \equiv 1; 7 \pmod 8$, $2 \mid \gamma$. However in these exceptional cases from $2 \mid \gamma \Rightarrow \left(\frac{\eta u}{2}\right) = \left(\frac{\eta t}{2}\right)$, so that $2^\sigma \equiv \left(\frac{2}{p}\right) \pmod p$ implies $2^\sigma - \chi(2) \equiv 0 \pmod p$, where χ is the character with the conductor t. Therefore in view (2.10) and the similar representations for $\chi_2(n)$ in the other cases we observe that the multiplier of the numerator of (1.6), which is divided by p, is equal to $2^\sigma \pm 1$ and

$$\frac{\chi_2(n)(2^\sigma - \chi(2))}{2^{2\sigma} - 1}$$

is p-adic integer.

Since the conductor $f(\chi)$ of the character χ is not exceed $4n$ and $n \leqq e < \frac{p}{8}$, then $f(\chi) < p$ and consequently $B_\sigma(\chi)$ is the p-adic integer. At last, the theorem of

Staudt implies $B_{2\sigma}^{-1} \equiv 0 \pmod p$. So $\varrho_p(n) \equiv 0 \pmod p$ provided $n < \frac{p}{8}$ and then the congruence (2.9) is true for any prime p.

To complete the demonstration of lemma we turn to the system (2.8), where $s = p$ and as we had proved $\delta_p(n) \equiv 0 \pmod p$ for $n = 1, \ldots, e$, the numbers $h_0(p, k) = 2^{4k}$, $h_{n-k}(p, k) \in \mathbf{Z}$. Consecutively surveying the system $(n = 1, \ldots, e)$ we convince that $\alpha_p(k) \equiv 0 \pmod p$ for $k = 1, \ldots, e$ and a prime $p > 7$ (note that the remainder term in (1.3) for $p \leqq 7$ equals zero).

Remark. For the numbers $\delta_p(1)$, and consequently $\alpha_p(1)$, there may be received the evident expression with the help of (1.3) and (1.6). For example, if $p \equiv 5 \pmod 8$, then $r_p(1) = 2p$ and $\sigma \equiv 2 \pmod 4$ imply

$$2p = \frac{2^{p-1}(2^{\frac{p-1}{2}} - 2)}{|B_{p-1}| \,.\, 2^{p-1}} B_{\frac{p-1}{2}} + \delta_p(1)$$

or as $\delta_p(1) = 2^4\alpha_p(1)$ finally

$$\alpha_p(1) = 2^{-3}\left(p - \left(2^{\frac{p-3}{2}} - 1\right) |B_{p-1}^{-1}| \,.\, B_{\frac{p-1}{2}}\right)$$

Further as $2^{\frac{p-3}{2}} \not\equiv 1 \pmod p$ and $p\,|\,B_{p-1}\,| \equiv 1 \pmod p$, then from the last equality it follows the equivalent of famous Ankeny–Artin–Chowla hypothesis ([1], [10], [17], [20], [21]) in the form

$$B_{\frac{p-1}{2}} \not\equiv 0 \pmod p \Leftrightarrow \frac{8\alpha_p(1)}{p} \not\equiv 1 \pmod p. \tag{2.11}$$

Analogous calculations for $p \equiv 1 \pmod 8$ lead to (2.11), too, so that (2.11) is true generally for any prime $p \equiv 1 \pmod 4$.

3°. Starting from above results it can be taken the arithmetical interpretation of the right parts of the congruences (2.1)–(2.5). Namely, the series of the congruences connecting $\varrho_s(n)$ and $H(\mathrm{d}f^2)$ for the discriminant $\mathrm{d}f^2$ (any multiple of n), and hence $r_s(n)$ and $H(\mathrm{d}f^2)$, may be derived from (1.6) and (2.1)–(2.5), if the quantities $\delta_s(n)$ are calculated. Here σ must be chosen as one of the forms m, $m + 1$, $2m$, $2m + 1$, where $m = \frac{p-1}{2} p^{l-1}$, $l \in \mathbf{N}$, p is an odd prime.

Let for example $p \equiv 5 \pmod 8$, $n \equiv 1 \pmod 8$ be a squarefree integer and hence $f = 1$, $n = t$, $T_\sigma(n) = 1$, then choosing $\sigma = m$, so that $\sigma \equiv 2 \pmod 4$, $\eta = 1$, $u = n$, $\chi(2) = 1$, $\alpha = 0$, $\chi_2(n) = (1 + 2^{-\sigma})(1 - 2^{1-\sigma})$, we obtain

$$H(np)\frac{U_l}{p^{l-1}} \equiv \frac{2}{3}\varkappa T_l \frac{\varrho_{2\sigma+1}^{(n)}}{(p-1)p^l}\, p\,|B_{2\sigma}| \pmod{p^l}, \qquad (n, p) = 1,$$

if we take into account that

$$\varrho_{2\sigma+1}(n) = \frac{2(2^{\sigma-1}-1)}{|B_{2\sigma}|} B_\sigma(\chi)$$

and

$$H(np)\frac{U_l}{p^{l-1}} \equiv -\varkappa T_l \frac{B_\sigma(\chi)}{\sigma} (\mathrm{mod}\ p^l),\ \chi \bmod n,$$

as it follows from (1.7) and (2.1). So

$$pB_{(p-1)p^{l-1}} \equiv p - 1\ (\mathrm{mod}\ p^l)$$

and hence in this case

$$p\,|\,B_{(p-1)p^{l-1}}\,| \equiv 1 - p\ (\mathrm{mod}\ p^l)$$

then finally

$$H(np)\frac{U_l}{p^{l-1}} \equiv -\frac{2}{3}\varkappa T_l \frac{1}{p^l}\varrho_{2\sigma+1}(n)\ (\mathrm{mod}\ p^l). \tag{3.1}$$

Further, if we are restricted by the calculations of Lomadze who found $\delta_s(n)$ for $s \leqq 31$, then it may be obtained 43 congruences for $r_s(n)$ with the prime moduli $p \leqq 31$ (and some of these powers for $p = 3, 5$). For squarefree integers n we indicate some examples.

I. We set $p = 3$, $l = 2$, $\sigma = 3$, $n \equiv 1\ (\mathrm{mod}\ 4)$ so that $s = 7$, $\eta = -1$, $u = n$. Since in this case $\eta u \equiv 3\ (\mathrm{mod}\ 4)$ and $\chi(x) = \left(\frac{-4n}{x}\right)$, then (1.5), (1.7) and (2.1) imply $\frac{B_3(\chi)}{3} = \frac{1}{4}\frac{r_7(n)}{7}$, so that finally from

$$H(12n)\frac{U_2}{3} \equiv -2T_2\frac{B_3(\chi)}{3}\ (\mathrm{mod}\ 9),\ (3, n) = 1,$$

we obtain

$$H(12n)\frac{U_2}{3} \equiv 4T_2\frac{r_7(n)}{7}\ (\mathrm{mod}\ 9),\ (3, n) = 1,\ n \equiv 1\ (\mathrm{mod}\ 4). \tag{3.2}$$

II. For $p = 5$, $l = 1$, $\sigma = 3$, $n \equiv 2\ (\mathrm{mod}\ 4)$, that is $s = 7$, $\eta = -1$ and $n = t = 2u$, (1.5), (1.7) and (2.2) with $\chi(x) = \left(\frac{-4n}{x}\right)$ imply

$$H(-20n) \equiv \frac{r_7(n)}{7}\ (\mathrm{mod}\ 5), \qquad (n, 5) = 1, \qquad n \equiv 2\ (\mathrm{mod}\ 4). \tag{3.3}$$

III. Let $p = 5$, $l = 1$, $\sigma = 2$, $n \equiv 1\ (\mathrm{mod}\ 8)$, so that $s = 5$, $\eta = 1$, $n = u$ and $\chi(x) = \left(\frac{n}{x}\right)$. Then from (3.1), in particular, we obtain

$$H(5n)\,U_1 \equiv \varkappa T_1\frac{r_5(n)}{5}\ (\mathrm{mod}\ 5), \qquad (n, 5) = 1, \qquad n \equiv 1\ (\mathrm{mod}\ 8). \tag{3.4}$$

IV. Let $p = 17$, $l = 1$, $\sigma = 9$, $n \equiv 2 \pmod 4$, so that $n = 2u$, $\eta = -1$ and $\chi(x) = \left(\frac{-4n}{x}\right)$. Calculating $\varrho_{19}(n)$ and turning to (2.2) we conclude that

$$H(-68n) \equiv 12\varrho_{19}(n) \pmod{17}, \qquad (n, 17) = 1, n \equiv 2 \pmod 4, \tag{3.5}$$

where in this case

$$\delta_{19}(n) = \frac{835\,216}{43\,867} \sum_{x_1^2 + \ldots + x_{11}^2 = n} (x_1^4 - 3x_1^2x_2^2) -$$
$$- \frac{983\,744}{3\,202\,291} \sum_{x_1^2 + x_2^2 + x_3^2 = n} (x_1^8 - 28x_1^6x_2^2 + 35x_1^4x_2^4),$$

as it shows the calculations ([14]).

V. Let $p = 31$, $l = 1$, $\sigma = 15$, $n \equiv 1 \pmod 4$, so that $s = 31$, $\eta = -1$, $t = u = n$ and $\chi(x) = \left(\frac{-4n}{x}\right)$. Then (1.5), (1.7) and (2.1) imply

$$H(124n)\, U_1 \equiv 4T_1 \frac{\varrho_{31}(n)}{31} \pmod{31}, \qquad (n, 31) = 1, \qquad n \equiv 1 \pmod 4. \tag{3.6}$$

VI. At last, if $p = 11$, $l = 1$, $\sigma = 5$, $n \equiv 7 \pmod 8$ so that $s = 11$, $\eta = -1$ and $\chi(x) = \left(\frac{-n}{x}\right)$, then we obtain analogously

$$H(11n)\, U_1 \equiv \frac{1}{2} T_1 \frac{r_{11}(n)}{11} \pmod{11}, \qquad (n, 11) = 1, \qquad n \equiv 7 \pmod 8, \tag{3.7}$$

where

$$\delta_{11}(n) = \frac{176}{31} \sum_{x_1^2 + x_2^2 + x_3^2 = n} (x_1^4 - 3x_1^2x_2^2) = 0$$

in the examined case.

In the congruences (3.1)$-$(3.7), as it is known $H(d) = 2h(d)$ provided $N(E_1) = +1$ for $d > 0$ and $H(d) = h(d)$ in other cases, so that $H(d)$ coincides with the class number of quadratic field (in the narrow sense). Here $h(d)$ is the class number of quadratic field $\mathbf{Q}(\sqrt{d})$.

4°. So far as

$$\frac{B_k(\chi)}{k} \equiv \frac{B_{k+i}(\chi)}{k+i} \pmod{p^l} \tag{4.1}$$

provided $(p-1)p^{l-1} \mid i$, $k > l$, $p \nmid f(\chi)$, $f(\chi)$ is the conductor of the character χ ([10], [7]), then (1.6) implies the congruences connecting $\varrho_s(n)$ (and consequently $r_s(n)$ in those cases when $\delta_s(n)$ was calculated) for the various values $s = i + k$ and the fixed value of a number n. The natural s is chosen equal to $\frac{p-1}{2} p^{l-1}$, $\frac{p-1}{2} \times$

$\times p^{l-1}+1, (p-1)p^{l-1}$ and $(p-1)p^{l-1}+1$, $l \in \mathbf{N}$. So, for example, if n is squarefree, then (1.7), (1.5) and (4.1) imply

$$\frac{r_7(n)}{7} \equiv \frac{1}{3}\frac{r_{11}(n)}{11} \pmod 3, \qquad 3 \nmid n, \qquad n \equiv 7 \pmod 8,$$

$$\frac{r_5(n)}{5} \equiv \frac{2}{5}\frac{\varrho_{13}(n)}{13} \pmod 5, \qquad 5 \nmid n, \qquad n \equiv 7 \pmod 8,$$

$$\frac{r_7(n)}{7} \equiv -4\frac{r_{19}(n)}{19} \pmod 9, \qquad 3 \nmid n, \qquad n \equiv 7 \pmod 8.$$

In last case $\delta_{19}(n) \equiv 0 \pmod{81}$ (the value of $\delta_{19}(n)$ was given in 3°, case IV).

5°. We fix now $s = 2\sigma + 1$ and consider (in the previous notations) connections between $\varrho_s(t)$ and $\varrho_s(tf^2)$, t is squarefree. As d and consequently $\chi(x)$, α, $B_\sigma(\chi)$ are fully determinated by t and σ, so (1.6) implies

$$\varrho_s(n) = \varrho_s(t)\, f^{2\sigma-1} T_\sigma(n) \frac{\chi_2(n)}{\chi_2(t)}, \qquad n = tf^2, \tag{5.1}$$

If $(2, n) = 1$ in particular, then $f^2 \equiv 1 \pmod 8$ and $t \equiv n \equiv u \pmod 8$, so that $\chi_2(n) = \chi_2(t)$, and hence from (5.1) it follows

$$\varrho_s(n) = \varrho_s(t) f^{2\sigma-1} T_\sigma(n),\ 2 \nmid n. \tag{5.2}$$

And if $n = 2^{2k}t$, $t = u$ is squarefree, $f = 2^k$, then $T_\sigma(n) = 1$ and the equality (5.1) implies

$$\varrho_s(n) = \varrho_s(t)\, 2^{k(2\sigma-1)} \frac{\chi_2(n)}{\chi_2(t)}. \tag{5.3}$$

The correspondences (5.1)–(5.3) contained the results which belong to different authors and had received as elementaly-combinatory methods as analytic ones. So, in particular, if $n = 4t$, t is squarefree, for $s = 5, 7$ we obtain the theorems of Cohen ([4]). Namely, the equality (5.3) implies

$$r_5(4t) = 8\frac{\chi_2(4t)}{\chi_2(t)} r_5(t),$$

so that from (1.5) finally

$$r_5(4t) = \begin{cases} 5r_5(t), & t \equiv 2; 3 \pmod 4, \\ 9r_5(t), & t \equiv 1 \pmod 8, \\ \dfrac{47}{7} r_5(t), & t \equiv 5 \pmod 8, \end{cases}$$

and

$$r_7(4t) = 32\frac{\chi_2(4t)}{\chi_2(t)} r_7(t),$$

so that

$$r_7(4t) = \begin{cases} 47r_7(t), & t \equiv 1; 2 \pmod 4, \\ 33r_7(t), & t \equiv 3 \pmod 8, \\ \dfrac{1\,157}{37} r_7(t), & t \equiv 7 \pmod 8. \end{cases}$$

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