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ON CLOSURE OPERATORS ON MONOIDS

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INTRODUCTION

The essential part of gramatical categories theory is based on the idea of Galois connection using the induced closure operator.

A *groupoid* is a set G with a binary operation. If x, y are elements of G, then we denote by xy the element which is obtained by applying the operation to the ordered pair (x, y); xy is the product of x, y. An element $e \in G$ is called an *identity* if $ex = xe = x$ for each $x \in G$. Clearly each groupoid has at most one identity. A groupoid with an identity and with an associative operation is called a *monoid.* If x_i is an element of a groupoid G for $i = 1, 2, \dots, n$, where $n \geqq 0$ is an integer, then it is possible to form products of these elements in the given order in several ways, e.g. $(\dots((x_1x_2)\, x_3 \dots x_{n-1})\, x_n$ or $x_1(x_2 \dots (x_{n-2}(x_{n-1}x_n)) \dots)$. If the operation of G is associative, then all these products are equal; we shall denote them by $x_1x_2 \dots x_n$. If $x_i = x$ for $i = 1, 2, \dots, n$, then we write x^n instead of $x_1x_2 \dots x_n$.

Let V be an arbitrary set. We denote by V^* the set of all finite sequences of elements of V including the empty sequence Λ; these sequences are called *strings.* For any $x \in V$, we identify x with the string $(x) \in V^*$. We define the operation of *concatenation* in V^*: If $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_n)$ where $m, n \geqq 0$ are integers and $x_i, y_i \in V$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, then we put $xy = (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$. It is easy to see that Λ is an identity and that this operation is associative. Thus, V^* is a monoid, if provided by the operation of concentration; this monoid is called the *free monoid on* V. We have $(x_1, x_2, \dots, x_m) = (x_1)(x_2) \dots (x_m) = x_1x_2 \dots x_m$ foı each integeı $m \geqq 0$ and for arbitrary elements $x_i \in V$ $(i = 1, 2, \dots, m)$, which implies that each element $x \in V^*$ is of the form $x = x_1x_2 \dots x_m$ where $m \geqq 0$ is an integer and $x_i \in V$ for $i = 1, 2, \dots, m$. We put $|x| = m$ and $|x|$ is called the *length of* x. Let V be a set, $L \subseteq V^*$ a subset of the free monoid V^*. Then the ordered pair (V, L) is a called a *language*. Let (V, L) be a language, $x \in V^*$, $(u, v) \in V^* \times V^*$. If $uxv \in L$, then we put $(x, (u, v)) \in \varrho \subseteq V^* \times (V^* \times V^*)$. We say that (u, v) is a *context* accepting x. The correspondence ϱ from V^* to $V^* \times V^*$ induces a Galois connection between 2^{V^*} and $2^{V^* \times V^*}$. The last defines a closure operator on 2^{V^*}.

In [2], necessary and sufficient conditions have been found for obtaining a Galois connection between 2^{V^*} and $2^{V^* \times V^*}$ by means of some language (V, L). This paper solves a similar problem for closure operators.

At first, we study some basic properties of the closure operators mentioned above. It has appeared that this study can be generalized and transferred from a free monoid to a general one. In solving the basic problem we start from general closure operators on monoids. We are looking for necessary and sufficient condition s for a closure operator to be derived from a Galois connection given by means of contexts. From the standpoint of linguistic interpretation of these results the following question formulated by prof. Novotný, is answered: Which are necessary and sufficient conditions for closure operator c on 2^{V^*} having the property $c(M)\, c(N) \subseteq c(MN)$ for all $M, N \subseteq V^*$, to be derived from a language (V, L) by constructing the Galois connection by means of its contexts.

1. PRINCIPAL CLOSURE OPERATORS

1.1. Definition. Let G be a set, $(2^G, \subseteq)$ the set of all its subsets partially ordered by inclusion, φ a mapping of 2^G into 2^G. Let the following three conditions be satisfied for arbitrary $X, Y \subseteq G$:

(A) $\varphi(X) \supseteq X$.
(B) $\varphi(\varphi(X)) = \varphi(X)$.
(C) $X \subseteq Y$ implies $\varphi(X) \subseteq \varphi(Y)$.

Then φ is called a *closure operator on* 2^G. The set $\varphi(X)$ is called the *φ-closure of the set X*.

1.2. Definition. Let G be a set, φ be a closure operator on 2^G. A set $X \subseteq G$ is called *φ-closed* if $\varphi(X) = X$.

We denote by Φ_G the set of all closure operators on 2^G.

1.3. Remark. If G is a set then we say a "*closure operator on* G" instead of a "*closure operator on* 2^G", too.

In this paper we shall study the closures, which can belong to various closure operators on a given set. Therefore the distinction, introduced in 1.2, is necessary.

1.4. Theorem. (See [1], § 23). *Let G be a set, φ a closure operator on G. Then the following assertions hold:*

(A) *G is φ-closed.*
(B) *φ is defined, in a unique way, by the system of all φ-closed subsets of G.*
(C) *The φ-closure of each subset X of G is the least φ-closed subset of G including X.*

1.5. Lemma. *Let G be a set. A subset Φ of 2^G is the system of all φ-closed subsets for a closure operator φ iff Φ is closed with respect to intersections.*

Proof. See [1], p. 75.

1.6. Definition. Let G be a monoid, $P_1, P_2, \ldots, P_n$ subsets of G where n is a natural number. Then we put $P_1 P_2 \ldots P_n = \{x_1 x_2 \ldots x_n;\ x_i \in P_i,\ i = 1, 2, \ldots, n\}$.

1.7. Definition. Let S and T be a pair of partially ordered sets, σ a mapping of S into T and τ a mapping of T into S. We say that the ordered pair of mappings (σ, τ) establishes a Galois connection between the partially ordered sets S and T, if the following conditions (1)–(4) are satisfied:

(A) $x_1 \leqq x_2$ implies $\sigma(x_1) \geqq \sigma(x_2)$ for arbitrary $x_1, x_2 \in S$.
(B) $y_1 \leqq y_2$ implies $\tau(y_1) \geqq \tau(y_2)$ for arbitrary $y_1, y_2 \in T$.
(C) $x \leqq \tau\sigma(x)$ for every element x of S.
(D) $y \leqq \sigma\tau(y)$ for every element y of T.

1.8. Theorem. *If the ordered pair of mappings (σ, τ) establishes a Galois connection between the partially ordered sets S and T, then $\tau\sigma$ is a closure operator on S, and $\sigma\tau$ is a closure operator on T.*

Proof. See [1], Theorem 16.

1.9. Remark. Let G be a monoid, $L \subseteq G$ its subset. For $X \subseteq G$ we put $\sigma_L(X) = \{(u, v);\ (u, v) \in G \times G,\ uxv \in L$ for each $x \in X\}$. For $Y \subseteq G \times G$ we put $\tau_L(Y) = \{x;\ x \in G,\ uxv \in L$ for each $(u, v) \in Y\}$. Then the ordered pair of mappings (σ_L, τ_L) is a Galois connection between 2^G and $2^{G \times G}$.

Indeed, if $X_1, X_2 \in 2^G$ are arbitrary sets such that $X_1 \subseteq X_2$, and $(u, v) \in \sigma_L(X_2)$, then $uxv \in L$ for each $x \in X_2$. However, $X_1 \subseteq X_2$ implies $uxv \in L$ for each $x \in X_1$. Thus, $(u, v) \in \sigma_L(X_1)$; we obtain $\sigma_L(X_1) \supseteq \sigma_L(X_2)$. Further, let $X \in 2^G$ be an arbitrary set, $x \in X$ its element. Then $uxv \in L$ for each $(u, v) \in \sigma_L(X)$, which implies $x \in \tau_L(\sigma_L(X))$. Therefore we have $\tau_L(\sigma_L(X)) \supseteq X$. Thus, we have verified the validity of (A) and (C) from 1.7. Similarly, we can prove that (B) and (D) holds true, too. Thus, (σ_L, τ_L) establishes a Galois connection between partially ordered sets $(2^G, \subseteq)$ and $(2^{G \times G}, \subseteq)$.

1.10. Corollary. *Let G be a monoid, $L \subseteq G$ its subset, (σ_L, τ_L) a Galois connection between 2^G and $2^{G \times G}$. We put $\tau_L(\sigma_L(X)) = \varphi_L(X)$ for arbitrary $X \subseteq G$. Then φ_L is a closure operator on G.*

1.11. Definition. Let G be a monoid, φ a closure operator on G. φ is called *principal*, if there exists $L \subseteq G$ with the property $\varphi = \varphi_L$.

We denote by Φ_{G_p} the set of all principal closure operators on G.

1.12. Theorem. *Let G be a monoid, $L \subseteq G$ its subset, φ_L a principal closure operator on G. Then L is φ_L-closed.*

Proof. By 1.1. (A) we obtain $L \subseteq \varphi_L(L)$.

Let us have $x \in \varphi_L(L)$. Then $uxv \in L$ for each $(u, v) \in \sigma_L(L)$. As $(e, e) \in \sigma_L(L)$, we have $x = exe \in L$ which implies $\varphi_L(L) \subseteq L$.

1.13. Theorem. *Let G be a monoid, $L \subseteq G$ its subset. Then the following conditions are equivalent:*

(i) $\varphi_L(X) = G$ *for each* $X \subseteq G$.

(ii) $L = G$.

Proof. Let us have $L = G$. Then $\sigma_L(X) = G \times G$ for each $X \subseteq G$ and further $\tau_L(Y) = G$ for each $Y \subseteq G \times G$. Thus, $\varphi_L(X) = G$ for each $X \subseteq G$.

Let us have $\varphi_L(X) = G$ for each $X \subseteq G$. If $L \neq G$ then, according to 1.12, we have $\varphi_L(L) = L \neq G$, which is a contradiction. Thus $L = G$.

1.14. Theorem. *Let G be a monoid, $L \subseteq G$ its subset. Let $M, N \subseteq G$ be arbitrary sets. Then $\varphi_L(M)\, \varphi_L(N) \subseteq \varphi_L(MN)$.*

Proof. Let $x \in \varphi_L(M)$, $y \in \varphi_L(N)$, $(u, v) \in \sigma_L(MN)$. If $m \in M$ and $n \in N$ are arbitrary elements, then $mn \in MN$. It yields $umnv \in L$. Thus $um(nv) \in L$ for each $m \in M$. Hence $(u, nv) \in \sigma_L(M)$; we have $uxnv \in L$ seeing that $x \in \tau_L(\sigma_L(M))$. It implies $(ux)\, nv \in L$ for each $n \in N$. We have proved that $(ux, v) \in \sigma_L(N)$. Since $y \in \tau_L(\sigma_L(N))$, we obtain $uxyv \in L$. It follows $xy \in \tau_L(\sigma_L(MN)) = \varphi_L(MN)$.

1.15. Example. Let (V, L) be a language where $V = \{a\}$ and $L = \{a^2, a^3\}$. We put $M = \{a^3\}$, $N = \{\Lambda, a\}$.

Evidently, $M, N \subseteq V^*$. We have $\sigma_L(M) = \sigma_L(\{a^3\}) = \{(\Lambda, \Lambda)\}$, $\varphi_L(M) = \tau_L(\{(\Lambda, \Lambda)\}) = \{a^2, a^3\}$. Further, $\sigma_L(N) = \sigma_L(\{\Lambda, a\}) = \{(\Lambda, a^2), (a, a), (a^2, \Lambda)\}$, $\varphi_L(N) = \tau_L(\{(\Lambda, a^2), (a, a), (a^2, \Lambda)\}) = \{\Lambda, a\}$. Thus, $\varphi_L(M)\, \varphi_L(N) = \{a^2, a^3\} \times \{\Lambda, a\} = \{a^2, a^3, a^4\}$. Clearly, $MN = \{a^3, a^4\}$. It follows that $\sigma_L(MN) = \sigma_L(\{a^3, a^4\}) = \emptyset$, $\varphi_L(MN) = \tau_L(\emptyset) = V^*$, which implies $\varphi_L(M)\, \varphi_L(N) = \{a^2, a^3, a^4\} \subset V^* = \varphi_L(MN)$.

2. ADMISSIBLE CLOSURE OPERATORS

2.1. Definition. Let G be a monoid, φ a closure operator on G. We say that φ is *admissible* if $\varphi(M)\, \varphi(N) \subseteq \varphi(MN)$ for arbitrary $M, N \subseteq G$.

We denote by Φ_{G_a} the set of all admissible closure operators on G.

2.2. Remark. By 1.14, we see that every principal closure operator is admissible on a monoid.

2.3. Theorem. *Let G be a monoid. Let elements a, x in G exist such that $a \neq e$ and $ax \neq a$.*

Then $\Phi_{G_a} \subset \Phi_G$.

Proof. We put $\mathfrak{A}_\varphi = \{X; X \subseteq G, e \notin X\}$. If $\emptyset \neq \mathfrak{M} \subseteq \mathfrak{A}_\varphi \cup G$ then $\bigcap_{A \in \mathfrak{M}} \in \mathfrak{A}_\varphi \cup G$. Thus, by 1.4.(C), $\mathfrak{A}_\varphi \cup G$ is a system of all φ-closed subsets from G, where φ is a suitable closure operator on G. According to 1.4.(B), the closure operator φ is defined by this system.

By 1.4.(C) we have, for every $M \subseteq G$, that $\varphi(M) = M$ when $e \notin M$, and $\varphi(M) = G$ when $e \in M$.

Let $M = \{a\}$, $N = \{e\}$. Then $\varphi(M) = \{a\}$, $\varphi(N) = G$, $MN = \{a\}$, $\varphi(MN) = \{a\}$. Thus, $\varphi(M)\,\varphi(N) = \{a\}\, G \nsubseteq \{a\} = \varphi(MN)$.

2.4. Theorem. *There exists an admissible closure operator not principal on a monoid.*

Proof. Let V be a set, $a \in V$. We put $\mathfrak{S}_\varphi = \{\emptyset, \{\Lambda\}, \{a\}, \{\Lambda, a\}, V^*\}$.
It is easy to see that $\mathfrak{S}_\varphi$ is a system of all φ-closed sets, where φ is a suitable closure operator. This system defines φ.

Let $M \subseteq V^*$ be a set.

(a) Let us have $M = \emptyset$. Then $\varphi(M)\,\varphi(N) = M\varphi(N) = \emptyset = \varphi(MN)$ for arbitrary $N \subseteq V^*$.

(b) Let us have $\emptyset \neq M \subseteq V^*$.

Let us suppose that $M = \{\Lambda\}$ and $N \subseteq V^*$. Then $\varphi(M)\,\varphi(N) = M\varphi(N) = \varphi(N) = \varphi(\{\Lambda\}\,N) = \varphi(MN)$ for an arbitrary $N \subseteq V^*$.

Let us suppose that $M \neq \{\Lambda\}$ and $N \subseteq V^*$. If $N = \emptyset$ or $= \{\Lambda\}$, then we have $\varphi(M)\,\varphi(N) = \varphi(M)\,N = \varphi(MN)$. If $\emptyset \neq N \neq \{\Lambda\}$, then the set $MN \subseteq V^*$ contains a string having the length greater than 1. Thus, by 1.4.(C),

$$\varphi(MN) = V^* \supseteq \varphi(M)\,\varphi(N).$$

We have proved $\varphi(M)\,\varphi(N) \subseteq \varphi(MN)$ for any $M, N \subseteq V^*$. Therefore φ is an admissible closure operator on V^*.

Let us suppose that φ is principal; we put $\varphi = \varphi_L$ for a suitable $L \subseteq V^*$. By 1.11, $L \in \mathfrak{S}_\varphi$.

(1) Let $L = \emptyset$ or $= \{\Lambda\}$ or $= \{a\}$.
We obtain $\sigma_L(\{\Lambda, a\}) = \emptyset$ and $\varphi_L(\{\Lambda, a\}) = \tau_L(\emptyset) = V^* \neq \{\Lambda, a\} = \varphi(\{\Lambda, a\})$ which is a contradiction.

(2) Let us have $L = \{\Lambda, a\}$.
Then $\sigma_L(\{a\}) = \{(\Lambda, \Lambda)\}$, $\varphi_L(\{a\}) = \tau_L(\{(\Lambda, \Lambda)\}) = \{\Lambda, a\} \neq \{a\} = \varphi(\{a\})$, which is a contradiction.

(3) Let us have $L = V^*$.
By 1.11, $\varphi_L(X) = V^*$ holds for each $X \subseteq V^*$. It follows that $\varphi_L(X) = V^* \neq X = \varphi(X)$ for $X \in \mathfrak{S}_\varphi - \{V^*\}$, which is a contradiction.

We have proved that φ is not principal.

2.5. Corollary. *Let $V \neq \emptyset$ be a set.*

Then $\Phi_{V_p^*} \subset \Phi_{V_a^*} \subset \Phi_{V^*}$.

Proof. 1. Let us have $a \in V^*$, $a \neq \Lambda$. Then $ax \neq a$ for each $x \in V^*$. Thus, according to 2.3, we have $\Phi_{V_a^*} \subset \Phi_{V^*}$.

2. V is not empty. Thus, by proof of 2.4, $\{\emptyset, \{\Lambda\}, \{\Lambda, a\}, \{a\}, V^*\}$ is the system of all φ-closed subsets from V^*, where φ is an admissible closure operator not principal on V^*. Therefore, by 2.2, the second part of our assertion holds true, too.

3. CHARACTERIZATION OF PRINCIPAL CLOSURE OPERATORS

3.1. Lemma. *Let G be a monoid, $L \subseteq G$ its subset. Let there exist φ_L-closed sets $X, Y \subseteq G$ such that $Y \nsubseteq X$. Then there exist φ_L-closed sets $U, V \subseteq G$, such that $UXV \subseteq L$ and $UYV \nsubseteq L$.*

Proof. There exist $(u_0, v_0) \in \sigma_L(X)$ and $y_0 \in Y$, such that $u_0 y_0 v_0 \notin L$. Namely, if $uyv \in L$ for each $(u, v) \in \sigma_L(X)$ and each $y \in Y$, then $Y \subseteq \tau_L(\sigma_L(X)) = \varphi_L(X) = X$, which is a contradiction.

We put $U = \varphi_L(\{u_0\})$, $V = \varphi_L(\{v_0\})$. Then we have $u_0 y_0 v_0 \in UYV$ and $u_0 y_0 v_0 \notin L$. Thus, $UYV \nsubseteq L$.

On the contrary, $u_0 x v_0 \in L$ holds for each $x \in X$. We obtain $(e, xv_0) \in \sigma_L(\{u_0\})$ for each $x \in X$. Then we have $uxv_0 \in L$ for each $x \in X$ and each $u \in \tau_L(\sigma_L(\{u_0\}) = \varphi_L(\{u_0\}) = U$. It implies $(ux, e) \in \sigma_L(\{v_0\})$ for each $u \in U$ and each $x \in X$. Thus, $uxv \in L$ for each $u \in U$, $x \in X$, $v \in \tau_L(\sigma_L(\{v_0\})) = \varphi_L(\{v_0\}) = V$, which implies $UXV \subseteq L$.

3.2. Definition. Let G be a monoid, $L \subseteq G$ its subset, φ a closure operator on G. We say that L is a *disjunctive set for* φ if, for arbitrary φ-closed sets $X, Y \subseteq G$ with the property $Y \nsubseteq X$, there exist φ-closed sets $U, V \subseteq G$, such that $UXV \subseteq L$ and $UYV \nsubseteq L$.

3.3. Theorem. *There exists a disjunctive closed set for any principal closure operator on a monoid.*

Proof. It follows from 1. and 3.1.

3.4. Theorem. *Let G be a monoid, φ an admissible closure operator on G. If there exists a φ-closed set disjunctive for φ, then φ is principal.*

Proof. Let $X \subseteq G$ be an arbitrary set.

(A) Let us suppose that $y \in \varphi_L(X) - \varphi(X)$.

Clearly, $\varphi(X)$ and $\varphi(\{y\})$ are φ-closed sets with the properties $y \in \varphi(\{y\})$ and $y \notin \varphi(X)$. Thus, $\varphi(\{y\}) \nsubseteq \varphi(X)$. Since L is a disjunctive closed set for φ, there exist φ-closed $U, V \subseteq G$ such that $U\varphi(X)\, V \subseteq L$ and $U\varphi(\{y\})\, V \nsubseteq L$. Evidently, $U \neq \emptyset \neq V$. Further, there exist $u_0 \in U$, $y_0 \in \varphi(\{y\})$ and $v_0 \in V$ such that $u_0 y_0 v_0 \notin L$. But $u_0 x v_0 \in L$

for each $x \in X$, thus, $(u_0, v_0) \in \sigma_L(X)$. Moreover, $y \in \varphi_L(X) = \tau_L(\sigma_L(X))$ which implies $u_0 y v_0 \in L$. It follows $u_0 y_0 v_0 \in \varphi(\{u_0\})\, \varphi(\{y\})\, \varphi(\{v_0\}) \subseteq \varphi(\{u_0 y v_0\}) \subseteq L$ seeing that φ is an admissible closure operator and L is a φ-closed set. Thus we have a contradiction. Hence, we have $\varphi_L(X) \subseteq \varphi(X)$.

(B) Let us suppose that $y \in \varphi(X) - \varphi_L(X)$.

Then there exists an ordered pair $(u_0, v_0) \in \sigma_L(X)$, such that $u_0 y v_0 \notin L$. Indeed, from the fact that $uyv \in L$ for each $(u, v) \in \sigma_L(X)$ it follows that $y \in \tau_L(\sigma_L(X)) = \varphi_L(X)$, which is a contradiction. It implies $u_0 y v_0 \in \varphi(\{u_0\}\, \varphi(X)\, \varphi(\{v_0\}) \subseteq \varphi(\{u_0\}\, X \{v_0\})$, because φ is an admissible closure operator. The fact that $(u_0, v_0) \in \sigma_L(X)$ implies $\{u_0\}\, X\{v_0\} \subseteq L$. It follows $\varphi(\{u_0\}\, X\{v_0\}) \subseteq \varphi(L) = L$ seeing that L is φ-closed. Thus, we obtain $u_0 y v_0 \in L$, which is a contradiction. Therefore we have $\varphi(X) \subseteq \varphi_L(X)$.

We have proved $\varphi(X) = \varphi_L(X)$ for each $X \subseteq G$.

3.5. Main Theorem. *Let G be a monoid, φ a closure operator on G. Then the following assertions are equivalent:*

(A) *φ is principal.*

(B) *φ is admissible and there exists a disjunctive φ-closed subset in G.*

Proof. It follows from 2.2, 3.3 and 3.4.

3.6. Example. Let V^* be a free monoid over $V = \{a\}$. We put $\mathscr{A}_\psi = \{\emptyset, \{\Lambda\}, \{a\}, V^*\}$. It is easy to see that $\mathscr{A}_\psi$ is a system closed with respect to intersections, which defines a closure operator Ψ on V^*.

1. We put $L = \{a\}$.

Let $X, Y \in \mathscr{A}_\psi$ be sets with the property $Y \nsubseteq X$.

(a) Let us have $X = \emptyset$. Then $Y = \{\Lambda\}$ or $= \{a\}$ or $= V^*$. We put $U = \{a\} = W$. Then we obtain $UXW = \emptyset \subseteq L$ and $UYW = \{a^2\}$ in the first case, $= \{a^3\}$ in the second case, and $= \{a^2\}\, V^*$ in the third. None of these sets is a subset of L.

(b) Let us have $X = \{\Lambda\}$. Then $Y = \{a\}$ or $= V^*$. If $U = \{\Lambda\}$, $W = \{a\}$ then $UXW = \{\Lambda\}\{\Lambda\}\{a\} = \{a\} = L$. If $Y = \{a\}$ then $UYV = \{\Lambda\}\{a\}\{a\} = \{a^2\} \nsubseteq L = \{a\}$. At last, if $Y = V^*$ then $UYV = \{\Lambda\}\, V^*\{a\} = V^* - \{\Lambda\} \neq \{a\} = L$.

(c) Let us have $X = \{a\}$. Then $Y = V^*$ or $= \{\Lambda\}$. If $U = \{\Lambda\}$ and $W = \{\Lambda\}$, then $UXW = \{\Lambda\}\{a\}\{\Lambda\} = \{a\} = L$. Further, $UYW = \{\Lambda\}\, V^*\{\Lambda\} = V^*$ or $= \{\Lambda\}\{\Lambda\}\{\Lambda\} = \{\Lambda\}$. It follows that $UYW \nsubseteq \{a\} = L$.

We have proved that to each Ψ-closed sets $X, Y \subseteq V^*$ with the property $Y \nsubseteq X$ there exist Ψ-closed sets $U, W \subseteq V^*$ such that $UXW \subseteq$ and $UYW \nsubseteq L$. Thus $L = \{a\}$ is a disjunctive set for Ψ.

Let $R \subseteq V^*$ be a Ψ-closed set, i.e. $R \in \mathscr{A}_\psi$.

(i) Let us have $R = \emptyset$. Then $\sigma_L(\emptyset) = V^* \times V^*$, $\tau_L(V^* \times V^*) = \emptyset$, $\varphi_L(\emptyset) = \tau_L(\sigma_L(\emptyset)) = \emptyset$.

(ii) Let us have $R = \{\Lambda\}$. Then $\sigma_L(\{\Lambda\}) = \{(\Lambda, a), (a, \Lambda)\}$, $\varphi_L(\{\Lambda\}) = \tau_L(\{(\Lambda, a), (a, \Lambda)\}) = \{\Lambda\}$.

(iii) Let us have $R = \{a\}$. Then $\sigma_L(\{a\}) = \{(\Lambda, \Lambda)\}$, $\varphi_L(\{a\}) = \tau_L(\{(\Lambda, \Lambda)\}) = \{a\}$.
(iv) Let us have $R = V^*$. Then $\sigma_L(V^*) = \emptyset$, $\varphi_L(V^*) = \tau_L(\emptyset) = V^*$.

We have proved that $\varphi_L(R) \in \mathscr{A}_\psi$.

Let $Z \subseteq V^*$ be a set with the property $Z \notin \mathscr{A}_\psi$. By 1.4.(D) we have $\Psi(Z) = V^*$. Clearly it follows that $\sigma_L(Z) = \emptyset$ and $\varphi_L(Z) = \tau_L(\emptyset) = V^*$.

From this analysis it follows that $\Psi = \varphi_L$. Simultaneously, we have proved that Ψ is obtained by constructing the Galois connection by means of contexts of the language (V, L), where $L = \{a\}$ is a disjunctive set for Ψ.

2. We put $L = \{\Lambda\}$.

Let us denote $\mathfrak{D} = \{UXW;\ X = \{a\},\ U, W \in \mathscr{A}_\psi\}$. It is easy to see that $\mathfrak{D} = \{\emptyset, \{a\}, \{a^2\}, \{a^3\}, \{V^* - \{a, \Lambda\}, \{V^* - \{\Lambda\}\}\}$, thus $UXW \nsubseteq L$ for any not empty Ψ-closed sets $U, W \subseteq V^*$. It follows that $UXW \subseteq L$ implies either $U = \emptyset$ or $W = \emptyset$. Thus $UYW = \emptyset \subseteq L$ for each $Y \subseteq V^*$. Therefore L is not a disjunctive set for Ψ.

We have $\sigma_L(\{a\}) = \emptyset$ and $\varphi_L(\{a\}) = \tau_L(\emptyset) = V^* \neq \{a\} = \Psi(\{a\})$. Thus, we obtain $\Psi \neq \varphi_L$.

We have proved that $L = \{\Lambda\}$ is not a disjunctive set for Ψ, and this closure operator on V^* cannot be obtained by constructing the Galois connection by means of contexts of the corresponding language (V, L).

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