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AUTONOMOUS AUTOMATA AND CLOSURES WITH THE SAME ENDOMORPHISM MONOIDS

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There is given a characterization of autonomous automata with algebraic closures on set of their states such that endomorphism monoids of automata coincide with endomorphism monoids of corresponding closure spaces. This problem is also studied for algebraic preclosures and the question of the unicity is treated, too.

1. INTRODUCTION

The paper [9] is concerned with linear realizations of autonomous automata without output function, i.e. there is studied an existence of monomorphisms from autonomous automata into such ones, where sets of states are vector spaces over finite fields and nextstate functions are endomorphisms of these spaces. This paper deals with a certain modification of the mentioned problem, namely when the next-state function is an endomorphism of algebraic closure space or algebraic pre-closure space and moreover when endomorphism monoids of the considered structures coincide. This is a problem of a realization in the categorical sense (cf. [5]). Algebraic pre-closure spaces can be considered as a special type of non-deterministic autonomous automata. From results of this paper it follows that the structure of deterministic autonomous automata can be described (with respect to the preservation of endomorphism monoids) by algebraic pre-closure operations and not by closure operations, in general. Autonomous automata endomorphism monoids of which coincide with those of closure spaces are of a special type.

2. BASIC DEFINITIONS AND NOTATIONS

By an autonomous automaton, we mean an ordered pair (A,f), where A is a non-void finite set (set of states) and f is a mapping of the set A into itself (next-state function); cf. [9].

2.1. Let (A_1, f_1), (A_2, f_2) be autonomous automata. A mapping $g: A_1 \to A_2$ is said to be a homomorphism of the automaton (A_1, f_1) into the automaton (A_2, f_2) if for each $a \in A_1$ there holds $gf_1(a) = f_2g(a)$. The set of all homomorphisms of (A_1, f_1) into (A_2, f_2) is denoted by $\mathbf{H}((A_1, f_1), (A_2, f_2))$. A homomorphism of an autonomous automata (A, f) into itself is called an endomorphism and the monoid of all endomorphisms (with respect to the composition of mappings) is denoted by $\mathbf{E}(A, f)$. A construction of the set $\mathbf{H}((A_1, f_1), (A_2, f_2))$, especially $\mathbf{E}(A, f)$ is a special case of the construction of all homomorphisms of a unary algebra into another one given in papers [7] and [8]. From there (and also from [6]) we take some notions necessary for our purpose. An autonomous automaton (A, f) is said to be connected if to every pair of states $a, b \in A$ there exists a pair m, n of non-negative integers such that $f^n(a) = f^m(b)$. A maximal (with respect to the set inclusion) connected subautomaton of an automaton (A, f) is called a component of (A, f). Components of (A, f) will be denoted by (A_ι, f_ι). A set of states $\{a \in A_\iota : f^k(a) = a$ for some natural $k\}$ is called a cycle of (A_ι, f_ι) and is denoted by $Z(A_\iota, f_\iota)$. The cardinal number $|Z(A_\iota, f_\iota)|$ is called the rang of the component (A_ι, f_ι) and it is denoted by $R(A_\iota, f_\iota)$. (The cardinal number of a set X is denoted by $|X|$). Clearly, an automaton is connected iff it has exactly one component. By def. 7 in [8], we say that a connected automaton (A_2, f_2) is admissible to a connected automaton (A_1, f_1) if $R(A_2, f_2)$ divides $R(A_1, f_1)$.

Notions of theory of closure spaces are taken from papers [1], [2] and [3]. Let S be a set. A mapping $C: \exp S \to \exp S$ is said to be a pre-closure operation (or briefly a pre-closure) on the set S if $X \subseteq C(X)$ and $X \subseteq Y \subseteq S$ implies $C(X) \subseteq C(Y)$ for each $X, Y \subseteq S$. If, moreover, $CC(X) = C^2(X) \subseteq C(X)$ for each $X \subseteq S$, then C is called a closure operation (a closure). An ordered pair (S, C) is called a pre-closure space (a closure space) if C is a pre-closure (a closure) on S. A pre-closure C is called algebraic and the corresponding space as well if $C(X) = \cup\{C(Y) : Y \subseteq X, |Y| < \aleph_0\}$. A homomorphism of a pre-closure space (S_1, C_1) into another one (S_2, C_2) is a mapping $g: S_1 \to S_2$ such that $g(C_1(X)) = C_2(g(X))$ for every set $X \subseteq S_1$. (See [2]). The set of all closure homomorphisms from (S_1, C_1) into (S_2, C_2) is denoted by $\mathbf{H}((S_1, C_1), (S_2, C_2))$ and the endomorphism monoid of the space (S, C) by $\mathbf{E}(S, C)$. It is to be noted that additive pre-closure spaces with $C(\phi) = \phi$, (i.e. from the topological point of view) are studied in detail in [4]. We put $[a)_f = \{f^n(a) : n = 0, 1, 2, \ldots\}$ for $a \in A$. By [9] we say that an automaton (A_2, f_2) realizes (A_1, f_1) if there exists a monomorphism g: $(A_1, f_1) \to (A_2, f_2)$. The class of all autonomous automata is detoned by $\mathfrak{A}$.

3. CYCLIC AUTOMATA

3.1. Definition. An autonomous automaton (A, f) is called a periodic automaton if f^k is the identity map of A for a positive integer k. The smallest positive integer k, for which f^k is the identity map, is called the period $\pi(f)$ of (A, f). A connected periodic automaton will be called a cyclic automaton.

Remark 1. It is to be noted that in [9] periodic automata in our sense are called permutations. There are as autonomous automata considered mappings (of finite sets) only. Evidently, every periodic automaton (A, f) can be written in the form $\sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$, where (A_ι, f_ι) are cyclic automata (components of (A, f)) and $\pi(f)$ is l.c.m. of $|A_\iota| = R(A_\iota, f_\iota)$, $1 \leqq \iota \leqq n$, cf. [9]. In what follows $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$ means that (A_ι, f_ι), $1 \leqq \iota \leqq n$ are exactly all components of the autonomous automaton (A, f).

3.2. Proposition. *Let (A, f) be a cyclic autonomous automaton of the rang $r \geqq 3$. Then there exist at least $2r - 4$ different algebraic preclosures C_k on A such that $\mathbf{E}(A, f) = \mathbf{E}(A, C_k)$, $k = 1, 2, \ldots, 2r - 4$.*

Proof. Let (A, f) be a cycle with $r = |A| \geqq 3$. For $k = 1, 2, \ldots, r - 2$, $a \in A$, we put $C_k(\phi) = C'_k(\phi) = \phi$, $C_k(a) = \{f^i(a) : i = 0, 1, \ldots, k\}$, $C'_k(a) = \bigcup_{i=1}^{k} f^{-i}(a) \cup \{a\}$ and $C_k(X) = \bigcup_{a \in X} C_k(a)$, $C'_k(X) = \bigcup_{a \in X} C'_k(a)$ for $X \subseteq A$, $X \neq \phi$. Evidently, $\mathfrak{C}_A = \{C : k = 1, \ldots, r - 2\} \cup \{C'_k : k = 1, \ldots, r - 2\}$ is a system of different algebraic pre-closures on A such that $|\mathfrak{C}_A| = 2r - 4$. Let k be an integer, $1 \leqq k \leqq 2r - 4$, $g \in \mathbf{E}(A, f)$, $a \in A$. There is $\mathbf{E}(A, f) = \{f^n : n = 0, 1, \ldots, r - 1\}$ and with respect to $f^n g = gf^n$ we have $g(C_k(a)) = C_k(g(a))$, $g(C'_k(a)) = C'_k(g(a))$, thus $g(C_k(X)) = C_k(g(X))$, $g(C'_k(X)) = C'_k(g(X))$ for each $X \subseteq A$. Hence $\mathbf{E}(A, f) \subseteq \mathbf{E}(A, C)$ for every pre-closure $C \in \mathfrak{C}_A$. Let $1 \leqq k \leqq r - 2$, $g \in \mathbf{E}(A, C_k)$, $a \in A$. Suppose first that $k = 1$. Then $\{g(a), gf(a)\} = gC_1(a) = C_1(g(a)) = \{g(a), fg(a)\}$. Since $g(a) \neq fg(a)$, there is $g(a) \neq gf(a)$ and we have $gf(a) = fg(a)$. Now, let $2 \leqq k \leqq r - 2$. Assume that $gf(a) = f^l g(a)$, where $2 \leqq l \leqq k$. Since g is a pre-closure endomorphism there exists an integer $p \in \{2, 3, \ldots, k\}$ with $gf^p(a) = f^{l-1}g(a)$. There holds $f^p(a) \in C_k(f(a))$, i.e. $gf^p(a) \in C_k(gf(a))$, however $gf^p(a) = f^{l-1}g(a) \notin C_k(f^l g(a)) = C_k(gf(a))$. This is a contradiction, hence $l \leqq 1$. Since the equality $gf(a) = g(a)$ leads also to a contradiction, because of $|C_k(g(a))| = |C_k(a)|$ and $gC_k(a) = C_k(g(a))$, we have $gf(a) = fg(a)$. This equality holds for each $a \in A$, thus $g \in \mathbf{E}(A, f)$. In a similar way we get the same result for $C = C'_k$, $1 \leqq k \leqq r - 2$, therefore $\mathbf{E}(A, f) = \mathbf{E}(A, C)$ for each $C \in \mathfrak{C}_A$, q.e.d.

Remark 2. A question of a description of all pre-closures on a cyclic autonomous automaton (A, f) (or a periodic automaton in general), endomorphism monoid of which coincides with $\mathbf{E}(A, f)$, seems to be open.

A pre-closure space (A, C) is called discrete (trivial) if $C(X) = X$ for each $X \subseteq A$ (if $C(X) = A$ for each $X \subseteq A$, $X \neq \phi$ and $C(\phi) = \phi$ or $C(\phi) = A$).

3.3. Lemma. *Let (A_1, f_1), (A_2, f_2) be cyclic autonomous automata, where $|A_2|$ is a prime number and the divisor of $|A_1|$. Let (A_1, C_1), (A_2, C_2) be algebraic closure spaces such that $\mathbf{H}((A_1, f_1), (A_2, f_2)) \subseteq \mathbf{H}((A_1, C_1), (A_2, C_2))$. Then the space (A_2, C_2) is either discrete or trivial.*

Proof. (A_2, f_2) is admissible to (A_1, f_1) according to the assumption, thus $\mathbf{H}((A_1, f_1), (A_2, f_2)) \neq \phi$. If $g : (A_1, f_1) \to (A_2, f_2)$ is a homomorphism then $\varphi_1 = gf_1^n$, $\varphi_2 = f_2^n g$ are homomorphisms of (A_1, f_1) into (A_2, f_2) for each non-negative integer n. Let C_1, C_2 be closures on A_1, A_2 respectively, such that $\mathbf{H}((A_1, f_1), (A_2, f_2)) \subseteq \mathbf{H}((A_1, C_1), (A_2, C_2))$. It is easy to show that $f_2^n : (A_2, C_2) \to (A_2, C_2)$ is an endomorphism for each non-negative integer n. Indeed, let on the contrary Y be a subset of A_2 with $f_2^n(C_2(Y)) \neq C_2(f_2^n(Y))$. Put $X = g^{-1}(Y)$, where $g \in \mathbf{H}((A_1, f_1), (A_2, f_2))$. Then $\varphi_2 C_1(X) = f_2^n g C_1(X) = f_2^n C_2(g(X)) = f_2^n C_2(Y) \neq C_2 f_2^n(Y) = C_2 f_2^n g(X) = C_2 \varphi_2(X)$, which is in a cotradiction with $\varphi_2 \in \mathbf{H}((A_1 C_1), (A_2, C_2))$. Thus $f_2^n \in \mathbf{E}(A_2, C_2)$. Let $b_0 \in A_2$ be an arbitrary element. If $C_2(b_0) = \{b_0\}$ then, since $f_2^n \in \mathbf{E}(A_2, C_2)$ for each integer n and C_2 is algebraic, C_2 is discrete. Let $C_2(b_0) = Y \neq \{b_0\}$. Put $Y = \{b_0, b_1, \ldots, b_{k-1}\}$, where $k \leqq |A_2|$, $b_i \neq b_j$ for $i \neq j$. Assume the notation is choosen in such a way that $f^{l_i}(b_i) = b_{i+1}$ for $i = 0, 1, \ldots, k-2$ and $f^{l_{j-1}}(b_{k-1}) = b_0$, where l_i are the least non-negative integers with this property $f^l(b_i) \bar{\in} Y$ for $l < l_i$. Let $b_i \in Y$ be arbitrary. Since g is an automorphism of (A_2, C_2), there holds $|C_2(b_i)| = |Y|$. Further, Y is a closed set in (A_2, C_2) hence $C_2(b_i) = Y$. Let $l_i < l_{i+1}$. Then $f^{l_i}(b_{i+1}) \bar{\in} Y$. On the other hand $b_{i+1} \in C_2(b_i)$, thus $f^{l_i}(b_{i+1}) \in f^{l_i} C_2(b_i) = C_2(f^{l_i}(b_i)) = C_2(b_{i+1}) = Y$, which is a contradiction. Thus $l_i = l_{i+1}$ for each i. Let l be the least natural number with the property $f^l(b_0) = b_1$. Then $Y = \{f^0(b_0), f^l(b_0), f^{2l}(b_0), \ldots, f^{(k-1)l}(b_0)\}$, $b_0 = f^l f^{(k-1)l}(b_0) = f^{kl}(b_0)$, thus $k \cdot l = |A_2| = R(A_2, f_2)$. By the supposition $|A_2|$ is a prime number, $1 < k \leqq |A_2|$, hence $l = 1$ and $k = |A_2|$, i.e. $Y = A_2$. Consequently (A_2, C_2) is a trivial closure space.

Remark 2. If (A, f) is a cyclic automaton, $a_0 \in A$ an arbitrary element and for $a, b \in A$ we put $a \,.\, b = f^{n+m}(a_0)$, where n, m are the least non-negative integer such that $f^n(a_0) = a$, $f^m(a_0) = b$, we get that $(A, .)$ is a finite cyclic group of the order $|A|$ $(= R(A, f))$ with the unit a_0. It was shown in the above proof that if C is an algebraic closure on A such that $\mathbf{E}(A, f) \subseteq \mathbf{E}(A, C)$, then $(C(b_0), .)$ is a subgroup of $(A, .)$. If $(A, .)$ is of a prime order, then the subgroup $(C(b_0), .)$ is either trivial or non-proper (i.e. $C(b_0) = A$).

3.4. Theorem. *Let (A_1, f_1), (A_2, f_2) be cyclic autonomous automata. There exist algebraic closures C_1, C_2 on A_1, A_2 respectively with the property $\mathbf{H}((A_1, f_1), (A_2, f_2)) = \mathbf{H}((A_1, C_1), (A_2, C_2)) \neq \phi$ iff either $|A_2| = 1$ or $|A_1| = |A_2| = 2$.*

Proof. If $|A_2| = 1$, then $|A_2^{A_1}| = 1$, thus $\mathbf{H}((A_1, f_1), (A_2, f_2)) = \mathbf{H}((A_1, C_1), (A_2, C_2)) \neq \phi$. If $|A_1| = |A_2| = 2$, then the above equality is also valid for C_1, C_2 trivial.

Let C_i be such a closure on A_i $(i = 1, 2)$ that considered sets of homomorphisms are non empty and coincide. Assume $|A_2| \geqq 3$. If $|A_2|$ is a prime number, then with respect to lemma 3.3 (A_2, C_2) is trivial. C_2 cannot be discrete because no constant mapping of A_1 into A_2 belong to $\mathbf{H}((A_1, f_1), (A_2, f_2))$. Suppose $|A_2| \geqq 3$ and that it is not a prime number. Choose $b_1, b_2 \in A_2$ such that $b_1 \neq f_2(b_2)$ and $b_2 \in C(b_1)$. Let $g \in \mathbf{H}((A_1, f_1), (A_2, f_2)) = \mathbf{H}((A_1, C_1), (A_2, C_2))$. Define a bijection $h : A_2 \to A_2$ in the following way: $h(b_1) = b_2$, $h(b_2) = b_1$, $h(b) = b$ for $b \in A_2$, $b_1 \neq b \neq b_2$. According to remark 3 we have that $(C_2(b), .)$ is a subgroup of $(A_2, .)$, it holds $C_2(b_1) = C_2(b_2)$ hence $h \in \mathbf{E}(A_2, C_2)$ and thus $hg \in \mathbf{H}((A_1, C_1), (A_2, C_2)$. On the other hand, let $b_0 \in A_2$ be such an element that $f_2(b_0) = b_1$. Choose $a \in g^{-1}(b_0)$. Then $b_0 \neq b_2$ and there holds $hgf_1(a) = hf_2g(a) = hf_2(b_0) = h(b_1) = b_2 \neq b_1 = f_2(b_0) = f_2h(b_0) = f_2hg(a)$, hence $hg \bar{\in} \mathbf{H}((A_1, f_1), (A_2, f_2))$, which is a contradiction. Therefore $|A_2| \leqq 2$. Suppose $A_2 = \{b_1, b_2\}, b_1 \neq b_2$. Admit that $|A_1| > 2$. There is $C_2(b_1) = C_2(b_2) = A_2$. Let $g \in \mathbf{H}((A_1, f_1), (A_2, f_2))$, $a_1 \in g^{-1}(b_1)$. Since g is a homomorphism of (A_1, C_1) onto (A_2, C_2), we have $C_1(a_1) \cap g^{-1}(b_2) \neq \phi$. Let $a_2 \in C_1(a_1)$ be such an element that $g(a_2) = b_2$. It holds $C_1(a_2) \subseteq C_1(a_1)$. If $C_1(a) \neq C_1(a_2)$, then there exists a point $a_3 \in C(a_2)$ with the property $g(a_3) = b_1$. There is $C_1(a_3) \subseteq C_1(a_2)$ again. Since $|C_1(x)| \geqq 2$ for each $x \in A_1$, there exists at least one pair of elements $a_1, a_2 \in A_1$ such that $C_1(a_1) = C_1(a_2)$, $x \in C_1(a_1)$ implies $C_1(x) = C_1(a_1)$ and $g(a_1) = b_1$, $g(a_2) = b_2$. Now, define a mapping $h : A_1 \to A_2$ by $h(a_1) = b_2$, $h(a_2) = b_1$ and $h(x) = g(x)$ for $x \in A_1 - \{a_1, a_2\}$. If $X \subseteq A_1 - C_1(a_1)$, then $C_1(X) \subseteq A_1 - C_1(a_1)$, thus $hC_1(X) = gC_1(X) = C_2(g(X)) = C_2(h(X))$. If $X \subseteq A$ and $X \cap C_1(a_1) \neq \phi$, then $C_1(X) = C_1(X - C_1(a_1)) \cup C_1(a_1)$ and thus $hC_1(X) = hC_1(X - C_1(a)) \cup hC_1(a_1) = C_2(h(X - C_1(a_1))) \cup A_2 = C_2(h(X - C_1(a_1))) \cup C_2(h(C_1(a_1) \cap X)) = C_2(h(X))$. Especially, if $X \subseteq C(a_1)$, then $C_1(X) = C(a_1)$ and thus $hC_1(X) = h(C_1(a_1)) = A_2 = C_2(h(X))$. Therefore h is a homomorphism of (A_1, C_1) onto $(A_2, C_2.)$ However, if $f_1(a_1) = a_2$, then $hf_1(a_2) = gf_1(a_2) = f_2g(a_2) = f_2(b_2) = b_1 \neq b_2 = f_2(b_1) = f_2h(a_2)$ and $f_1(a_1) \neq a_2$ implies $hf_1(a_1) = gf_1(a_1) = f_2g(a_1) = f_2(b_1) = b_2 \neq b_1 = f_2(b_2) = f_2h(a_1)$. Hence we get $h \bar{\in} \mathbf{H}((A_1, f_1), (A_2, f_2))$, which is a contradiction. Therefore it holds $|A_1| = 2$, q.e.d.

3.5. Corollary. *Let (A, f) be a cyclic autonomous automaton, C an algebraic closure operation on the set A. Then* $\mathbf{E}(A, f) = \mathbf{E}(A, C)$ *iff* $|A| \leqq 2$.

4. TREES

One of the types of connected autonomous automata which is studied in more detail in [9] is a tree.

4.1. Definition. An autonomous automaton (A, f) is called a tree if there exists a non-negative integer k such that f^k is a constant map (cf. [9], p. 68).

Evidently, a tree is a connected autonomous automaton.

4.2. Definition. Let (A, f) be a tree. The smallest non-negative k, for which f^k is a constant map, is called the height of (A, f) and is denoted by $H(f)$, the uniquely determined constant is called the base of (A, f) and is denoted by z_f, i.e. $\{z_f\} = Z(A, f)$. Let $a \in A$. The smallest non-negative integer k such that $f^k(a) = z_f$ is called the level of a in (A, f) and will be denoted by $\lambda(a)$. An element $a \in A$ is said to be extremal in (A, f) if $\lambda(a) = H(f)$.

4.3. Theorem. *Let (A, f) be a tree with the base z_f. If $H(f) = 1$, then there are precisely two algebraic pre-closures C_1, C_2 on the set A with $\mathbf{E}(A, f) = \mathbf{E}(A, C_i)$, $i = 1, 2$, where $C_1(X) = X \cup f(X)$, $C_2(X) = C_1(X) \cup \{z_f\}$ for each $X \subseteq A$. If $H(f) \geqq 2$, then C_1 is the only algebraic pre-closure on the set A such that $\mathbf{E}(A, f) = \mathbf{E}(A, C_1)$.*

Proof. Let $g \in \mathbf{E}(A, f)$, $X \subseteq A$. Then $gC_1(X) = g(X \cup f(X)) = g(X) \cup \{gf(x) : x \in X\} = g(X) \cup \{fg(x) : x \in X\} = g(X) \cup fg(X) = C_1(g(X))$, hence $\mathbf{E}(A, f) \subseteq \mathbf{E}(A, C_1)$. Let $h \in \mathbf{E}(A, C_1)$. It holds $\{h(z_f)\} = hC_1(z_f) = C_1(h(z_f)) = \{h(z_f), fh(z_f)\}$, thus $h(z_f) = fh(z_f) = z_f$. Then for $X \subseteq A$ arbitrary we have $hC_2(X) = h(C_1(X) \cup \{z_f\}) = C_1(h(X)) \cup \{z_f\} = C_2(h(X))$, i.e. $h \in \mathbf{E}(A, C_2)$. Hence $\mathbf{E}(A, f) \subseteq \mathbf{E}(A, C_1) \subseteq \mathbf{E}(A, C_2)$.

Let $g \in \mathbf{E}(A, C_1)$, $a \in A$. Then $\{g(a), gf(a)\} = g\{a, f(a)\} = gC_1(a) = C_1(g(a)) = \{g(a), fg(a)\}$. Thus either $gf(a) = g(a) = fg(a)$ or $gf(a) \neq g(a)$ and $gf(a) \neq fg(a)$, hence $g \in \mathbf{E}(A, f)$. Suppose that (A, f) is a tree of the height $H(f) = 1$, $g \in \mathbf{E}(A, C_2)$. It holds $\{g(z_f)\} = gC_2(\phi) = C_2(g(\phi)) = C_2(\phi) = \{z_f\}$. If $a \in A$, we have $\{g(a), gf(a), z_f\} = gC_2(a) = C_2(g(a)) = \{g(a), z_f\}$ for $f(x) = z_f$ if $x \in A$. It follows from here that either $g(a) = z_f = gf(a)$ or $g(a) \neq z_f$ and since $f(a) = z_f$, we have $gf(a) = g(z_f) = z_f$. Thus $gf(a) = z_f = fg(a)$ for each $a \in A$, i.e. $g \in \mathbf{E}(A, f)$. Consequently $\mathbf{E}(A, f) = \mathbf{E}(A, C_1) = \mathbf{E}(A, C_2)$ for a tree (A, f) of the height 1 and $\mathbf{E}(A, f) = \mathbf{E}(A, C_1)$ for each tree (A, f).

Now, consider an algebraic pre-closure C on the set A satisfying the condition $\mathbf{E}(A, f) = \mathbf{E}(A, C)$. Since the constant transformation of A with the value z_f belongs to $\mathbf{E}(A, f)$, we have $C(z_f) = \{z_f\}$. Let a_0 be an extremal element of the tree (A, f), a an arbitrary element of A. According to definition 9. and 2.12. in [8] there exists an endomorphism g of the tree (A, f) such that $g(a_0) = a$ and $g(A) = [a)_f$. Then $C(a) = C(g(a_0)) = gC(a_0) \subseteq [a)_f$. If $H(f) = 1$, because of $f(a) = z_f$ and $C(\phi) \subset C(a)$

for each $a \in A$, we have $C(\phi) \in \{\phi, \{z_f\}\}$. The pre-closure C is algebraic thus either $C = C_1$ or $C = C_2$.

Let $H(f) \geqq 2$. Assume there exist $a \in A$ of the level $\lambda(a) \geqq 2$ and an integer $k \geqq 2$ such that $f^k(a) \in C(a)$. Since $f^k \in \mathbf{E}(A, f) = \mathbf{E}(A, C)$ for each non-negative integer n, using the above considered endomorphism g, we get that $f^k(x) \in C(x)$ holds for each $x \in A$. Choose $b \in f^{-1}(z_f) - \{z_f\}$ and put $B_0 = \{x \in A : \lambda(x) = H(f)\}$, $B_1 = \{x \in A : : \lambda(x) = H(f) - 1\}$, i.e. B_0 is the set of all extremal elements in (A, f). Define a transformation $\varphi : A \to A$ by $\varphi(x) = b$ for $x \in B_0 \cup B_1$ and $\varphi(x) = z_f$ for $x \in \in A - (B_0 \cup B_1)$. Since $f(x) \in B_1$ for each $x \in B_0$, we have $\varphi \bar{\in} \mathbf{E}(A, f)$. On the other hand, consider a constant transformation of A with the value z_f. We get $C(\phi) \in \in \{\emptyset, \{z_f\}\}$. Let $X \subseteq A$, $X \neq \phi$. If $X \cap (B_0 \cup B_1) = \phi$, then $\varphi(X) = \{z_f\}$ and $C(X) \cap \subset (B_0 \cup B_1), = \varnothing$ thus $\varphi(C(X)) = \{z_f\} \doteq C(z_f) = C(\varphi(X))$ in this case. Let $X \cap (B_0 \cup B_1) \neq \varnothing$. Since $f^k(X) \subseteq C(X)$, we have $C(X) \cap (A - (B_0 \cup B_1)) \neq \varnothing$ and since any constant transformation of A with the value different from z_f does not belong to $\mathbf{E}(A, f) = \mathbf{E}(A, C)$ we get $\varphi(C(X)) = \{z_f, b\} = C\{z_f, b\} = C(b)$. If $X \subseteq (B_0 \cup B_1)$, then $\varphi(X) = \{b\}$, if $X \nsubseteq (B_0 \cup B_1)$, then $\varphi(X) = \{z_f, b\}$ hence $\varphi C(X) = C(\varphi(X))$. If $C(\phi) = \phi$, then $\varphi C(\phi) = \phi = C(\varphi(\phi))$, if $C(\phi) = \{z_f\}$, then $\varphi C(\phi) = \{z_f\} = C(\varphi(\phi))$. Therefore $\varphi \in \mathbf{E}(A, C)$, which is a contradiction. Hence $k < 2$ and since $C(a) = \{a\}$ for $a \neq z_f$ leads to a contradiction (as is stated above), we have $C(a) = \{a, f(a)\}$ for each $a \in A$ thus $C(\phi) = \phi$. Since C is algebraic we get $C = C_1$, q.e.d.

Let us prove the corresponding theorem for closures.

4.4. Theorem. *Let (A, f) be a tree with the base z_f, C be an algebraic closure operation on the set A. Then $\mathbf{E}(A, f) = \mathbf{E}(A, C)$ iff $H(f) = 1$. In this case there are precisely two algebraic closures C_1, C_2 on A with the property $\mathbf{E}(A, f) = \mathbf{E}(A, C_i)$, $i = 1, 2$, where $C_1(X) = X \cup f(X)$, $C_2(X) = C_1(X) \cup \{z_f\}$ for each $X \subseteq A$.*

Proof. Let $H(f) > 1$, C be an algebraic closure on A such that $\mathbf{E}(A, f) = \mathbf{E}(A, C)$. Let $a \in A$, $a \neq z_f$. As in the proof of theorem 4.3. denote by g such an endomorphism of (A, f) that $g(A) = [a)_f$. The existence of g follows from [8], 9., 2.12. Then $g \in \in \mathbf{E}(A, C)$, hence $C(a) \subseteq [a)_f$. Assume there exists $a \in A$ with $z_f \bar{\in} C(a)$. Let $k \geqq 1$ be an integer such that $f^k(a) \in C(a)$, $f^l(a) \bar{\in} C(a)$ for each $l \geqq k$. Since $f^k \in \mathbf{E}(A, f) = = \mathbf{E}(A, C)$, we have $C(f^k(a)) \subseteq [f^k(a))_f$ hence $C(a) \cap C(f^k(a)) = \{f^k(a)\}$. Since a closure of each singleton is the least closed set containing its element and intersection of two closed sets is a closed set, there is $C(f^k(a)) = \{f^k(a)\}$. This is a contradiction because $f^k(a) \neq z_f$ and the constant transformation of A with the value $f^k(a)$ does not belong to $\mathbf{E}(A, f)$. Hence $z_f \in C(a)$ for each $a \in A$. Choose $b \in f^{-1}(z_f)$, $b \neq z_f$ and define a mapping $h : A \to \{b, z_f\}$ by $h(x) = b$ for each $x \neq z_f$ and $h(z_f) = = z_f$. Let $X \subseteq A$. If $\phi \neq X \neq \{z_f\}$, we have $h(X) = \{b, z_f\}$ and $C(X) \neq \{z_f\}$. Then $C(h(X)) = \{b, z_f\} = hC(X)$. Further, $C(h(z_f)) = C(z_f) = \{z_f\} = hC(z_f)$ and since $C(\phi) \in \{\phi, \{z_f\}\}$, there holds $C(h(\phi)) = hC(\phi)$. Thus we have $h \in \mathbf{E}(A, C)$. However,

$H(f)$ is assumed greater than 1, hence there exists $x_0 \in A$ with $\lambda(x_0) \geqq 2$, consequently $h \in \mathbf{E}(A, f)$ for $hf(x_0) = b \neq z_f = fh(x_0)$. This contradiction implies $H(f) = 1$.

If $H(f) = 1$, then by theorem 4.3. there exist precisely two algebraic pre-closures C_1, C_2 on A such that $\mathbf{E}(A,f) = \mathbf{E}(A, C_i)$, $i = 1, 2$. However, $C_1^2(X) = C_1(X) \cup \cup C_1(f(X)) = X \cup \{z_f\} = C_1(X)$ for $X \subseteq A$, $X \neq \phi$ and $C_1^2(\phi) = \phi = C_1(\phi)$ and similarly $C_2^2(X) = C_2(X)$ for each $X \subseteq A$. Hence C_1, C_2 are closure operations. The proof is complete.

5. GENERAL CASE

In this paragraph there will be considered arbitrary autonomous automata $(A, f) = = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$, where $(A_\iota, f_\iota) : 1 \leqq \iota \leqq n$ is the system of all components of (A, f). If (A_ι, f_ι) is a tree, then the base of (A_ι, f_ι) will be denoted by z_ι instead of z_{f_ι}.

5.1. Definition. Let (A_1, C_1), (A_2, C_2) be pre-closure spaces. We shall say that (A_1, C_1) is embedded into (A_2, C_2) if there is a one-to-one mapping $\varphi : A_1 \to A_2$ such that for each set $X \subseteq A$ there holds $\varphi C_1(X) = C_2(\varphi(X)) \cap \varphi(A_1)$, (cf. [2], p. 183).

5.2. Proposition. *To every autonomous automaton (A, f), there can be assigned an algebraic pre-closure C on the set A such that*

$1°$ $\mathbf{E}(A, f) = \mathbf{E}(A, C)$

$2°$ *If (A_2, f_2) realizes (A_1, f_1), then (A_1, C_1) is embedded into (A_2, C_2).*

Proof. Let $(A, f) \in \mathfrak{A}$. Put $C(X) = X \cup f(X)$ for each $X \subseteq A$. Evidently, $\mathbf{E}(A, f) \subseteq \subseteq \mathbf{E}(A, C)$. From 3.2. and 4.3. it follows that $\mathbf{E}(A_\iota, f_\iota)$ for each component (A_ι, f_ι) of (A, f), where $C_\iota(X) = C(X) \cap A_\iota = C(X)$ for every set $X \subseteq A_\iota$. If (A_ι, f_ι), $(A_\varkappa, f_\varkappa)$ are different components of (A, f), $\mathbf{H}((A_\iota, C_\iota), (A_\varkappa, C_\varkappa)) \neq \phi$, where C_ι; $C_\varkappa$ are relativizations of C onto A_ι, $A_\varkappa$ respectively, then it is easy to show (similar to the proofs of 3.2. and 4.3.) that $\mathbf{H}((A_\iota, f_\iota), (A_\varkappa, f_\varkappa)) \neq \phi$ and $\mathbf{H}((A_\iota, C_\iota), (A_\varkappa, C_\varkappa)) \subseteq \subseteq \mathbf{H}((A_\iota, f_\iota), (A_\varkappa, f_\varkappa))$. Then we get the equality $\mathbf{E}(A,f) = \mathbf{E}(A, C)$, thus $1°$ holds. Let $(A_1, f_1) \in \mathfrak{A}$, $(A_2, f_2) \in \mathfrak{A}$, C_1, C_2 be the above defined pre-closures for (A_1, f_1), (A_2, f_2) respectively. Let $g : (A_1, f_1) \to (A_2, f_2)$ be a monomorphism, $X \subseteq A_1$ Since $g(A_1)$ is an f_2-stable set in (A_2, f_2), we have $C_2(g(X)) \cap g(A_1) = C_2(g(X)) = g(X) \cup \cup f_2 g(X) = gC_1(X)$ hence g is an embedding of (A_1, C_1) into (A_2, C_2).

Remark. Condition $2°$ in 5.2. cannot be replaced by this stronger condition $2°'$ (A_2, f_2) realizes (A_1, f_1) iff (A_1, C_1) is embedded into (A_2, f_2), because embeddings of algebraic pre-closure spaces in the sense of [3] (defined above) are not morphisms corresponding to monomorphism of autonomous automata in the sense of a realization of concrete categories. However, if we define an embedding g of (A_1, C_1) into (A_2, C_2) by the requirement that g is injective and $gC_1(X) = C_2 g(X)$ for each $X \subseteq A$, we can write $2°'$ instead of $2°$ in 5.2.

5.3. Theorem. *Let* $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$ *be an autonomous automaton. There exists an algebraic closure* C *on the set* A *with the property* $\mathbf{E}(A, f) = \mathbf{E}(A, C) \neq \phi$ *iff for each* $\iota \in \{1, 2, \ldots, n\}$ *there is* $f_\iota^2 \in \{f_\iota, id_{A_\iota}\}$, *i.e.* (A_ι, f_ι) *is a tree of the height* 1 *or a periodic component with* $R(A_\iota, f_\iota) \leqq 2$.

Proof. Let $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$ be an autonomous automaton satisfying the condition given in the theorem. Put $C(X) = X \cup f(X)$ for each $X \subseteq A$. Evidently (A, C) is an algebraic closure space and $\mathbf{E}(A, f) \subseteq \mathbf{E}(A, C)$. Let $g \in \mathbf{E}(A, C)$, $a \in A_\iota$, $g(a) \in A_\varkappa$. If $R(A_\iota, f_\iota) = 2$, then $(A_\varkappa, f_\varkappa)$ is admissible to (A_ι, f_ι). Let $R(A_\iota, f_\iota) = 1$. Then $C(z_\iota) = \{z_\iota\}$, $\{g(z_\iota), f_\varkappa g(z_\iota)\} = Cg(z_\iota) = gC(z_\iota) = \{g(z_\iota)\}$ hence $R(A_\varkappa, f_\varkappa) = 1$. Then we get, in the similar way as in the proof of theorem 4.3., that $gf_\iota(a) = f_\varkappa g(a)$ i.e. $gf(a) = fg(a)$. Let $R(A_\iota, f_\iota) = 2$. If $R(A_\varkappa, f_\varkappa) = 1$, since $g \in \mathbf{E}(A, C)$, $Z(A_\iota, f_\iota) = C(a)$ for each $a \in A_\iota$ and $C(z_\varkappa) = \{z_\varkappa\}$, we have $gZ(A_\iota, f_\iota) = \{z_\varkappa\}$ thus $gf_\iota(a) = z_\varkappa = f_\varkappa g(a)$. If $R(A_\varkappa, f_\varkappa) = 2$, then $gZ(A_\iota, f_\iota) = Z(A_\varkappa, f_\varkappa)$, simultaneously $x, y \in Z(A_\iota, f_\iota)$, $x \neq y$ implies $g(x) \neq g(y)$, thus we have $gf_\iota(a) = f_\varkappa g(a)$, too. Hence $gf(a) = fg(a)$ for every $a \in A$, i.e. $g \in \mathbf{E}(A, f)$.

Now, let C be an algebraic closure on A such that $\mathbf{E}(A, C) = \mathbf{E}(A, f)$, where $(A, f) = \sum_{\leqq \iota \leqq n} (A_\iota, f_\iota)$. Let' $\iota \in \{1, 2, \ldots, n\}$ be arbitrary. Consider such endomorphisms $g \in \mathbf{E}(A, f)$, for which $g(x) = x$ if $x \in A - A_\iota$. If $R(A_\iota, f_\iota) > 1$, then according to 3.5. it holds $R(A_\iota, f_\iota) = 2$. If (A_ι, f_ι) is a tree, then with respect to theorem 4.4 we have $H(f_\iota) = 1$. Let $R(A_\iota, f_\iota) = 2$. Assume $A_\iota \neq Z(A_\iota, f_\iota)$. Let $a \in A_\iota - Z(A_\iota, f_\iota)$ be such that $f_\iota(a) \in Z(A_\iota, f_\iota)$. Consider a mapping $h : A \to A$ defined by $h(x) = a$ for $x \in A_\iota - Z(A_\iota, f_\iota)$, $h(f_\iota(a)) = f_\iota^2(a)$, $h(f_\iota^2(a)) = f_\iota(a)$ and $h(x) = x$ for each $x \in A - A_\iota$. Since $Z(A_\iota, f_\iota) \subset C(x)$ for each $x \in A_\iota$, we have $h \in \mathbf{E}(A, C)$. However, $h(f(a)) = f_\iota^2(a) \neq f_\iota(a) = f_\iota(h(a))$, thus $h \bar{\in} \mathbf{E}(A, f)$, which is a contradiction. Hence $A_\iota = Z(A_\iota, f_\iota)$ and we have $\iota \in \{1, 2, \ldots, n\}$ implies $f_\iota^2 = f_\iota$ or $f_\iota^2 = \mathrm{id}_{A_\iota}$, q.e.d.

Now, describe all closures on A having the property $\mathbf{E}(A, f) = \mathbf{E}(A, C)$, where $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$. Put $N_1 = \{\iota : 1 \leqq \iota \leqq n, R(A_\iota, f_\iota) = 1\}$, $N_2 = \{\iota : 1 \leqq \iota \leqq n, R(A_\iota, f_\iota) = 2\}$. With respect to theorem 5.3. it holds $N_1 \cup N_2 = \{1, 2, \ldots, n\}$. Put $C_1(X) = X \cup f(X)$ for each $X \subseteq A$.

5.4. Theorem. *Let* $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$ *be an autonomous automaton,* $\mathfrak{C}_A$ *be the system of all closure operations on* A *such that* $\mathbf{E}(A, f) = \mathbf{E}(A, C)$. *Let* N_1, N_2, C_1 *be symbols defined as above. Then it holds:*

(i) *If* $|N_1| = 1$, $N_2 \neq \phi$ *and* $(A, f) = (A_1, f_1) + \sum_{\iota \in N_2} (A_\iota, f_\iota)$, *then* $\mathfrak{C}_A = \{C_1, C_2, C_3\}$, *where* $C_2(X) = C_1(X) \cup \{z_1\}$ *for* $\phi \neq X \subseteq A$, $C_2(\phi) = \phi$ *and* $C_3(X) = C_1(X) \cup \{z_1\}$ *for each* $X \subseteq A$.

(ii) *If either* $N_1 = \phi$, $|N_2| \geqq 2$ *or* $|N_1| \geqq 2$, *then* $\mathfrak{C}_A = \{C_1\}$.

Remark. For the case $N_1 = \phi$, $|N_2| = 1$ see 3.5., for the case $|N_1| = 1$, $N_2 = \phi$ see theorem 4.3.

Proof of theorem 5.4. Let C be an algebraic closure operation on A such that $\mathbf{E}(A, f) = \mathbf{E}(A, C)$. Let $|N_1| = 1$, $N_2 = \phi$ and $(A,f) = (A_1, f_1) + \sum_{\iota \in N_2} (A_\iota, f_\iota)$ i.e. $N_1 = \{1\}$ and (A_1, f_1) is either a tree of a singleton. With respect to theorem 4.4. there is $z_1 \in C(a)$ for each $a \in A_1$ and $H(f_1) = 1$ whenever (A_1, f_1) is a tree. If $a, b \in A_1$, $a \neq b \neq z_1$, then $b \bar{\in} C(a)$. Assume $C(a) \cap A_\varkappa \neq \phi$ for some $a \in A_1$, $\varkappa \in N_2$. Then $C_1(z_1) \cap A_\varkappa \neq \phi$ and since the constant self map of A with the value z_1 belongs to $\mathbf{E}(A, C)$, we get a contradiction. Hence $X \subseteq A_1$, $X \neq \phi$ implies $C(X) = X \cup f(X)$, $C(\phi) \in \{\phi, \{z_1\}\}$. Let $\iota \in N_2$ be arbitrary, $a \in A_\iota$. Then $Z(A_\iota, f_\iota) \subseteq C(a)$. Since (A_ι, f_ι) is admissible to each $(A_\varkappa, f_\varkappa)$, $\varkappa \in N_2$ and thus for each $\varkappa \in N_2$ there exists an endomorphism of (A, C) which maps $A_\varkappa$ onto A_ι, we have that $X \subseteq A_\iota, \varkappa \in N_2, \varkappa \neq \iota$ implies $C(X) \cap A_\varkappa = \phi$. Let there exists $a \in A_\iota$ with $C(a) \cap A_1 \neq \phi$. Since the mapping h which is an identity mapping onto $A - A_1$ and $h(x) = f_1(x)$ for $x \in A_1$ belongs to $\mathbf{E}(A,f) = \mathbf{E}(A, C)$, we have $C(a) \cap A_1 = \{z_1\}$. Then $C(X) \cap A_1 = \{z_1\}$ for each $X \subseteq A$, $X \neq \phi$. Consequently, we get these possibilities: Either $C(X) = X \cup f(X)$ for each $X \subseteq A$ or $C(X) = X \cup f(X) \cup \{z_1\}$ for $\phi \neq X \subseteq A$ and $C(\phi) = \phi$, or $C(X) = X \cup f(X) \cup \{z_1\}$ for each $X \subseteq A$. If $N_1 = \phi$, $|N_2| \geqq 2$, then each two components of (A,f) are mutually admissible, thus $X \subseteq A_\iota$ implies $C(X) \cap A_\varkappa = \phi$ for each $\varkappa \neq \iota$. Then the closure operation C having the property $\mathbf{E}(A,f) = \mathbf{E}(A, C)$ is equal to C_1. Let $|N_1| \geqq 2$. Let (A_1, f_1), (A_2, f_2) be such components of (A, f) that $R(A_1, f_1) = R(A_2, f_2) = 1$. Since (A_1, f_1), (A_2, f_2) are mutually admissible, then as above $X \subseteq A_1$ implies $C(X) \cap A_2 = \phi$ and $Y \subseteq A_2$ implies $C(Y) \cap A_1 = \phi$ and also $C(X) \cap A_1 = C(X) \cap A_2 = \phi$ for each $X \subseteq \bigcup_{\iota \in N_2} A_\iota$. Since $C(\phi) \subseteq C(X)$ for each $X \subseteq A$ it holds $C(\phi) = \phi$. Hence $C = C_1$, q.e.d.

A closure C on A is said to be topological if $C(X \cup Y) = C(X) \cup C(Y)$ for $X \subseteq A$, $Y \subseteq A$. Of course, a topological closure is algebraic and vice versa whenever the underlying set is finite, which is our case. A topological closure C such that $C(\phi) = \phi$ is also called a topology; we shall denote it by τ. If (A, τ) is a topological space, then $\mathbf{E}(A, \tau)$ is the monoid of all closed deformations of (A, τ), i.e. of all closed continuous self maps of the space (A, τ). We write $\mathbf{S}(A, \tau)$ instead of $\mathbf{E}(A, \tau)$. If (A, τ) is a finite T_1-space (i.e. $\tau\{x\} = \{x\}$ for each $x \in A$), then τ is the discrete topology hence the problem treated here becomes trivial. For τ being T_0-topology (i.e. feebly semi-separated in the terminology of [4]) we get, using above results, the following theorem.

5.5. Theorem. *Let $(A,f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$ be an autonomous automaton. There exists a T_0-topology τ on the set A such that $\mathbf{S}(A, \tau) = \mathbf{E}(A, f)$ iff each component (A_ι, f_ι) of (A, f) is either a tree of the height* 1 *or a singleton. Such a topology τ is unique and it is given by $\tau X = X \cup f(X)$ for each $X \subseteq A$.*

Proof. Let (A, f) be an autonomous automaton each component of which is either a tree or a singleton and let τ be a T_0-topology on the set A. According to the theorem 5.3. and (ii) of 5.4. it holds $\mathbf{S}(A, \tau) = \mathbf{E}(A, f)$.

Let $(A, f) \in \mathfrak{A}$, $(A, f) = \sum_{1 \leqq \iota \leqq n} (A_\iota, f_\iota)$. Let τ be a T_0-topology on A such that $\mathbf{E}(A, f) = \mathbf{S}(A, \tau)$. With respect to theorem 5.3. we have $R(A_\iota, f_\iota) \leqq 2$ for each $\iota \in \{1, 2, \ldots, n\}$ and (A_ι, f_ι) is either a tree of the height 1 or a periodic component (i.e. component without non-cyclic elements). Let there exist $\varkappa \in \{1, 2, \ldots, n\}$ such that $R(A_\varkappa, f_\varkappa) = 2$, let $A_\varkappa = \{a_1, a_2\}$. Denote by $\Omega(a_i)$ the system of all neighbourhoods of the point a_i, $(i = 1, 2)$ in the space (A, τ). Since a set $X \subseteq A$ is a neighourhood of a point $a \in A$ in the topological space (A, τ) iff $a \in A - \tau(A - X)$, we have $\cap\, \Omega(a_1) = \cap\, \Omega(a_2) = \{a_1, a_2\} = A_\varkappa$, thus points a_1, a_2 are not T_0-separated in (A, τ). This is a contradiction hence $R(A_\iota, f_\iota) = 1$ for each ι. The last assertion follows from theorem 5.4.

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