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# ON THE ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONLINEAR SYSTEMS

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Abstract. The method of variation of constants. BIHARI inequality [2] and the SCHAUDER–TYCHONOV fixed point theorem [3] are used to study the asymptotic relations between the solutions of the systems (1) $\frac{dx}{dt} = A(t)\,x + f(t, x)$ and (2) $\frac{dy}{dt} = A(t)\,y$. The application of the results deduced here to an n-th order differential equation yields a generalization of a result for the second order differential equation by Mehri and Zarghamee [4].

## 1. INTRODUCTION

The paper is devoted to the study of the system

$$x' = A(t)\,x + f(t, x), \tag{1}$$

where $A(t)$ is an $n \times n$ continuous matrix defined on $J = [0, \infty)$ and $f(t, x)$ is an $n$-dimensional vector function defined on the domain $D: t \geqq t_0,\ |x| < \infty$, where $|\,.\,|$ denotes any appropriate vector norm.

Moreover, it is assumed that $f(t, x)$ is "small" in some sense so that we can consider the system (1) as a perturbation of the linear system

$$\frac{dy}{dt} = A(t)\,y. \tag{2}$$

Let $Y(t)$ be a fundamental matrix of solutions of (2). In the present paper sufficient conditions are established for the following:

(1) every solution $x(t)$ of (1) whose initial condition satisfies a given inequality can be expressed in the form $x(t) = Y(t)\,c(t)$ where $c(t)$ is a suitable differentiable vector-function such that $\int_{t_0}^{\infty} |\,c'(t)\,|\,dt < \infty$;

(2) for every constant vector $\xi$ there exists a solution $x(t)$ of (1) such that $\lim_{t\to\infty} x(t) = \xi$.

## 2. MAIN-RESULTS

Theorem 1. Let the function $f(t, x)$ satisfy the condition

(3) $$|Y^{-1}(t)f(t, Y(t)z)| \leqq g(t)\omega(|z|)$$

for every $n$-vector $z$.

Here $g(t)$ and $\omega(r)$ are functions with the following properties:

(4) $g(t)$ is continuous and nonnegative for $t \geqq t_0$.

(5) $\omega(r)$ is continuous, positive and nondecreasing for $r > 0$.

(6) $$\int_{t_0}^{\infty} g(t)\,dt < \Omega(\infty),$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\omega(s)}, \qquad r_0 > 0.$$

Then every solution $x(t)$ of (1) such that

(7) $$|Y^{-1}(t_0)x(t_0)| < \Omega^{-1}[\Omega(\infty) - \int_{t_0}^{\infty} g(t)\,dt]$$

($\Omega^{-1}$ means the inverse function of $\Omega(r)$) can be expressed in the form $x(t) = Y(t)c(t)$ where $c(t)$ is a suitable differentiable vector function such that

(8) $$c(t_0) = Y^{-1}(t_0)x(t_0), \qquad \int_{t_0}^{\infty} |c'(t)|\,dt < \infty.$$

Proof. Using the formula of the variation of constants, any solution $x(t)$ of (1) can be written in the form $x(t) = Y(t)c(t)$, where $c(t)$ satisfies the following differential equation

(9) $$c' = Y^{-1}(t)f(t, Y(t)c), \qquad c(t_0) = Y^{-1}(t_0)x(t_0).$$

Integrating (9) in norm and applying (3) we get

$$\int_{t_0}^{t} |c'(s)|\,ds = \int_{t_0}^{t} |Y^{-1}(s)f(s, Y(s)c(s))|\,ds \leqq$$

$$\leqq \int_{t_0}^{t} g(s)\omega(|c(s)|)\,ds.$$

From the monotonity of $\omega(r)$ and the fact that

$$(10)\qquad |c(t)| \leqq |c(t_0)| + \int_{t_0}^{t} |c'(s)|\,ds$$

we get

$$(11)\qquad \int_{t_0}^{t} |c'(s)|\,ds \leqq \int_{t_0}^{t} g(s)\,\omega[|c(t_0)| + \int_{t_0}^{s} |c'(\tau)|\,d\tau]\,ds, \qquad t \geqq t_0 .$$

Now, let us define a continuous function $Q(t)$ by

$$(12)\qquad Q(t) = |c(t_0)| + \int_{t_0}^{t} |c'(r)|\,dr.$$

Then (11) may be rewritten in the form

$$(13)\qquad Q(t) \leqq |c(t_0)| + \int_{t_0}^{t} g(s)\,\omega(Q(s))\,ds, \qquad t \geqq t_0 .$$

Hence by the Lemma of BIHARI [2, p. 83]

$$(14)\qquad Q(t) \leqq \Omega^{-1}[\Omega(|c(t_0)|) + \int_{t_0}^{t} g(s)\,ds], \qquad t_0 \leqq t \leqq b_1 \leqq \infty,$$

where the constant $b_1$ is determined by the requirement

$$(15)\qquad \Omega(|c(t_0)|) + \int_{t_0}^{b_1} g(s)\,ds \leqq \Omega(\infty).$$

From the fact that $c(t_0) = Y^{-1}(t_0)\,x(t_0)$ and from the conditions (6) and (7) it is seen that (14) is valid for all $b_1 \geqq 0$. Since the argument of $\Omega^{-1}$ in (14) is an increasing function and $\Omega(|c(t_0)|) + \int_{t_0}^{\infty} g(s)\,ds < \Omega(\infty)$ by (7), $Q(t)$ is bounded. Hence $\int_{t_0}^{\infty} |c'(s)|\,ds < \infty$ and (8) is proved.

**Remark 1.** If $\displaystyle\int_1^{\infty} \frac{dt}{\omega(t)} = \infty$, which means that $\Omega(\infty) = \infty$, the condition (6) may be replaced by $\int_{t_0}^{\infty} g(t)\,dt < \infty$ and the restriction (7) on $x(t_0)$ may be omitted.

**Remark 2.** From (8) it follows that $\lim_{t\to\infty} c(t)$ exists and is finite.

**Theorem 2.** Let the function $f(t, x)$ satisfy for every $n$-vector $z$ the condition

(16) $$| Y^{-1}(t) f(t, Y(t) z) | \leqq F(t, |z|),$$

where $F(t, r)$ has the following properties:

(17) $F(t, r)$ is continuous and non-decreasing in $r$ for each $t$ on $t \geqq t_0$, $r \geqq 0$.

(18) $$\int_{t_0}^{\infty} F(t, a)\, dt < \infty \qquad \text{for each constant } a \geqq 0.$$

Then for every constant n-vector $\xi$ there exists a $t^*$, $t^* \geqq t_0$ and a solution $x(t)$ of (1) defined for $t \geqq t^*$, which can be expressed in the form $x(t) = Y(t)\, c(t)$, where $c(t)$ is a differentiable n-vector function such that

(19) $$\lim_{t\to\infty} c(t) = \xi.$$

Proof. Using the formula of the variation of constants, any solution $x(t)$ of (1) can be written in the form $x(t) = Y(t)\, c(t)$, where $c(t)$ satisfies (9).

Consider the integral equation

(20) $$c(t) = \xi - \int_t^{\infty} Y^{-1}(s) f(s, Y(s)\, c(s))\, ds, \qquad t \geqq t^*.$$

By direct differentiation one can show that each solution $c(t)$ of (20) if it exists, is a solution of (9) for $t \geqq t^*$.

Using Schauder–Tichonov fixed point theorem [3, p. 9], we shall prove the existence of a solution of (20) for $t \geqq t^*$.

Let $\varkappa > 0$ be any constant, $\varkappa > |\xi|$. Let $t^*$ be chosen in such a way that $\int_{t^*}^{\infty} F(s, \varkappa) \times$ $\times\, ds < \varkappa - |\xi|$; this is possible with respect to (18).

Let $E$ denote the set of all $n$-vector valued functions $h(t)$ continuous on $[t^*, \infty)$ and $|h(t)| \leqq \varkappa$.

Using (16), (17) and (18), we get

$$\left| \int_{t^*}^{\infty} Y^{-1}(s) f(s, Y(s)\, h(s))\, ds \right| \leqq \int_{t^*}^{\infty} F(s, |h(s)|)\, ds \leqq \int_{t^*}^{\infty} F(s, \varkappa)\, ds \leqq \varkappa - |\xi|.$$

This insures that the operator

$$Th = \xi - \int_t^\infty Y^{-1}(s) f(s, Y(s) h(s)) \, ds$$

is defined on $E$ and maps $E$ into $E$.

To prove that $T$ is continuous on $E$, let $h_n \in E$ $(n = 1, 2, \ldots)$ be a sequence of functions in $E$, which converges uniformly on every finite interval $[t^*, t_1]$ to a function $h$, $h \in E$. By (16) we have

$$| Th - Th_n | = | \int_t^\infty Y^{-1}(s) f(s, Y(s) h(s)) \, ds - \int_t^\infty Y^{-1}(s) f(s, Y(s) h_n(s) \, ds | \leqq$$

$$\leqq \int_{t^*}^\infty | Y^{-1}(s)(f(s, Y(s) h(s)) - f(s, Y(s) h_n(s))) | \, ds \leqq$$

$$\leqq \int_{t^*}^{t_1} | Y^{-1}(s)(f(s, Y(s) h(s)) - f(s, Y(s) h_n(s))) | \, ds +$$

$$+ \int_{t_1}^\infty | Y^{-1}(s) f(s, Y(s) h(s)) | \, ds + \int_{t_1}^\infty | Y^{-1}(s) f(s, Y(s) h_n(s) | \, ds \leqq$$

$$\leqq \int_{t^*}^{t_1} | Y^{-1}(s)(f(s, Y(s) h(s)) - f(s, Y(s) h_n(s)) | + 2 \int_{t_1}^\infty F(s, \varkappa) \, ds.$$

Given any $\varepsilon > 0$, by (18) we can choose $t_1$ such that

$$\int_{t_1}^\infty F(s, \varkappa) \, ds < \frac{\varepsilon}{4}.$$

From the continuity of $f(t, x)$ and the uniform convergence of $h_n(s)$ to $h(s)$ on $[t^*, t_1]$ we get that for $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that for each $n \geqq n_0(\varepsilon)$

$$| f(s, Y(s) h(s)) - f(s, Y(s) h_n(s)) | < \frac{\varepsilon}{2 \int_{t^*}^{t_1} | Y^{-1}(s) | \, ds}.$$

Hence for $n \geqq n_0(\varepsilon)$ we have

$$| Th_n - Th | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for } t \geqq t^*$$

so that $T$ is continuous on $E$.

From the fact that $TE \subset E$ it follows that the functions in $TE$ are uniformly bounded.

Now, we shall prove that the functions in the image set $TE$ are equicontinuous at every point of $[t^*, \infty)$. Let $t_1, t_2$ be any pair of numbers, $t^* \leqq t_1 < t_2 < \infty$, we get

$$| Th(t_1) - Th(t_2) | = | \int_{t_2}^{t_1} Y^{-1}(s) f(s, Y(s) h(s)) \, ds | \leqq$$

$$\leqq \int_{t_1}^{t_2} | Y^{-1}(s) f(s, Y(s) h(s)) | \, ds.$$

Applying (16), (17) and (18), we get

$$| Th(t_1) - Th(t_2) | \leqq \int_{t_1}^{t_2} F(s, \varkappa) \, ds$$

and the right hand side of the inequality does not depend on $h$. Hence the functions in $TE$ are equicontinuous.

Now all the assumptions of Schauder–Tichonov fixed point theorem are satisfied, hence the mapping $T$ has at least one fixed point in $E$, say $h_0(t)$ so that

$$h_0(t) = T h_0(t) = \xi - \int_t^{\infty} Y^{-1}(s) f(s, Y(s) h_0(s)) \, ds, \qquad t \geqq t^*.$$

This means that $h_0(t)$ is a solution of (20).

Consequently $x(t) = Y(t) h_0(t)$ is a solution of (1) existing for $t \geqq t^*$.

Further we have to prove that $\lim_{t \to \infty} h_0(t) = \xi$ but this is a direct consequence of (16), (17) and (18) since we have

$$| h_0(t) - \xi | = | \int_t^{\infty} Y^{-1}(s) f(s, Y(s) h_0(s)) \, ds | \leqq \int_t^{\infty} F(s, \varkappa) \, ds \to 0$$

as $t \to \infty$. This completes the proof.

**Remark.** If it is assumed instead of (18) that

$$\int_{t_0}^{\infty} F(t, a)\,\mathrm{d}t \leqq M < \infty \qquad \text{for every } a \geqq 0,$$

one can choose $\varkappa$ such that $\varkappa > |\xi| + M$ and then the statement of the theorem is valid for the whole interval $[t_0, \infty)$ that means we can take $t^* = t_0$.

A direct consequence of Theorem 2 is the following corollary which we shall use in the proof of Theorem 3.

**Corollary 1.** Let the hypotheses of Theorem 1 be satisfied. Then for every constant $n$-vector $\xi$ there exists a $t^*$, $t^* \geqq t_0$ and a solution $x(t)$ of (1) defined for $t \geqq t^*$, which can be expressed in the form $x(t) = Y(t)\,c(t)$, where $c(t)$ is differentiable $n$-vector function such that $\lim_{t\to\infty} c(t) = \xi$.

Proof. To prove Corollary 2, one needs to observe that the conditions (4)–(7) imply the assumptions of Theorem 2, which can be easily proved.

**Theorem 3.** Let the hypotheses of Theorem 1 be satisfied and let, in addition,

$$\int_{1}^{\infty} \frac{\mathrm{d}t}{\omega(t)} = \infty \qquad (\text{that means } \Omega(\infty) = \infty). \tag{21}$$

Then for every solution $x(t)$ of (1) on $[t_0, \infty)$ there exists a solution $y(t)$ of (2) such that

$$|\,Y^{-1}(t)\,(x(t) - y(t))\,| \to 0 \quad \text{for} \quad t \to \infty \tag{22}$$

and vice versa.

Proof. Let $x(t)$ be any solution of (1). Respecting (21), the restriction (7) on $x(t_0)$ may be omitted. By Theorem 1 $x(t)$ can be expressed in the form $x(t) = Y(t)\,c(t)$ and there exists a constant $n$-vector $\xi$ such that $\lim_{t\to\infty} c(t) = \xi$. Consider the solution $y(t) = Y(t)\,\xi$ of (2). We get

$$|\,Y^{-1}(t)(x(t) - y(t))\,| = |\,Y^{-1}(t)(Y(t)\,c(t) - Y(t)\,\xi)\,| = |\,c(t) - \xi\,| \to 0$$

for $t \to \infty$. Then (22) holds.

Now, let $y(t)$ be a solution of (2). Then there exists a constant $n$-vector $\xi_0$ such that $y(t) = Y(t)\,\xi_0$. By Corollary 1, given $\xi_0$, there exists a $t^*$, $t^* \geqq t_0$ and a solution of (1) of the form $x(t) = Y(t)\,c(t)$ and $\lim_{t\to\infty} c(t) = \xi_0$. This implies also that $|\,Y^{-1}(t)\,(x(t) - y(t))\,| = |\,c(t) - \xi_0\,| \to 0$ for $t \to \infty$. This completes the proof.

## 3. COROLLARIES AND APPLICATION TO AN $n$-th ORDER SCALAR EQUATION

Now we shall apply Theorem 1 and Corollary 1 on an $n$-th order scalar differential equation. For $n = 2$ the result yields a generalization of a result of MEHRI and ZARGHAMEE [4].

Consider the $n$-th order scalar differential equation

$$u^{(n)} = h_1(t)\, u^{(n-1)} + \ldots + h_n(t)\, u + h(t, u, u', \ldots, u^{(n-1)}), \tag{23}$$

where $h_i(t) \in C[0, \infty)$ $(i = 1, \ldots, n)$ and $h(t, u, u', \ldots, u^{(n-1)}) \in C([0, \infty) \times R^n)$.

Let $v_1(t), \ldots, v_n(t)$ be a set of $n$-linearly independent solutions of the linear equation

$$\begin{aligned} v^{(n)} &= h_1(t)\, v^{(n-1)} + \ldots + h_n(t)\, v, \\ v_i^{(j-1)}(0) &= \delta_{i_j} \qquad (i, j = 1, \ldots, n). \end{aligned} \tag{24}$$

Let $W(t)$ be the Wronskian of the functions $v_1(t), \ldots, v_n(t)$ and let $W_k(t)$ be the determinant obtained from $W(t)$ by replacing the $k$-th column by $(0, \ldots, 0, 1)$. We define the functions $\varphi(t)$ and $\eta_i(t)$ $(i = 1, \ldots, n)$ as follows

$$\eta_i(t) = \max\left(| v_1^{(i)}(t) |, \ldots, | v_n^{(i)}(t) |\right) \qquad (i = 0, 1, \ldots, n-1),$$

and

$$\varphi(t) = \max\left(| W_1(t) |, \ldots, | W_n(t) |\right).$$

Let us suppose that $h(t, u, u', \ldots, u^{(n-1)})$ satisfies the following condition.

H: If the functions $u^{(i)}(t)$ are such that there exists a nonnegative continuous function $\gamma(t)$ such that $| u^{(i)}(t) | \leqq \eta_i(t)\, \gamma(t)$ for $t \geqq 0$, $(i = 0, \ldots, n-1)$, then $h(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ satisfies the following estimate

$$| h(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) | \leqq \psi(t)\, \omega(\gamma(t)). \tag{25}$$

Here $\omega(s)$ is a continuous function which is positive and nondecreasing for $s > 0$, and $\psi(t)$ is continuous and nonnegative for $t \geqq 0$.

Now we shall prove the following theorem:

**Theorem 4.** Suppose that $h(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ satisfies H and that

$$\int_0^\infty \psi(t)\, \varphi(t) \exp\left[-\int_0^t h_1(s)\, ds\right] dt < \frac{1}{n}\, \Omega(\infty). \tag{26}$$

Then every solution $u(t)$ of (23) satisfying at $t = 0$ the inequality

$$\sum_{i=0}^{n-1} | u^{(i)}(0) | < \Omega^{-1}\left\{\Omega(\infty) - n \int_0^\infty \psi(t)\, \varphi(t) \exp\left[-\int_0^t h_1(s)\, ds\right] dt\right\} \tag{27}$$

can be expressed in the form $u(t) = \sum_{i=1}^{n} c_i(t)\, v_i(t)$, where $c_i(t)$ $(i = 1, \ldots, n)$ are continuous scalar functions such that

$$\int_0^\infty | c_i'(t) |\, dt < \infty. \tag{28}$$

Further if $a_1, \ldots, a_n$ are $n$-arbitrary constants, there is a solution $u(t)$ of (23), which can be written in the form $u(t) = \sum_{i=1}^{n} c_i(t)\, v_i(t)$ with

$$\lim_{t\to\infty} c_i(t) = a_i. \tag{29}$$

Proof. Let equations (23) and (24) be put into system forms (1) and (2) respectively, with

$$x(t) = \big(u(t), u'(t), \ldots, u^{(n-1)}(t)\big)^T,$$
$$y(t) = \big(v(t), v'(t), \ldots, v^{(n-1)}(t)\big)^T,$$
$$f(t, x(t)) = (0, 0, \ldots, 0, h)t, u(t), u'(t), \ldots, u^{(n-1)}(t))^T$$

and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ h_n(t) & h_{n-1}(t) & h_{n-2}(t) & \ldots & h_1(t) \end{pmatrix}.$$

Hence the fundamental matrix of (2) will be given by

$$Y(t) = \begin{pmatrix} v_1(t) & \ldots & v_n(t) \\ v_1'(t) & \ldots & v_n'(t) \\ \ldots & \ldots & \ldots \\ v_1^{(n-1)}(t) & \ldots & v_n^{(n-1(}(t) \end{pmatrix}$$

with $Y(o) = I$.

Now we shall prove that $Y(t)$ and $f(t, x(t))$ satisfy the conditions of Theorem 1.

In the proof we shall use a specific matrix (vector) norm $\|\,.\,\|$ defined by the sum of the absolute values of the elements.

Using the formula of the variation of constants, any solution $x(t)$ of (1) can be written in the form

$$x(t) = Y(t)\, c(t), \tag{30}$$

where $c(t)$ is an $n$-vector function satisfying the equation (9).

Let $c_i(t)$, $(i = 1, \ldots, n)$ denote the components of $c(t)$.

Writing (30) in terms of its components, we have

$$u^{(i)}(t) = \sum_{j=1}^{n} c_i(t)\, v_j^{(i)}(t), \qquad i = 0, \ldots, n-1.$$

Hence

$$(31) \qquad |u^{(i)}(t)| \leqq \sum_{j=1}^{n} |v_j^{(i)}(t)|\; |c_j(t)| \leqq \eta_i(t) \sum_{j=1}^{n} |c_j(t)| = \eta_i(t)\, \| c(t) \|,$$

in view of the definition of $\eta_i(t)$.

Using (25), (30) and the definition of $\varphi(t)$, we get

$$\| Y^{-1}(t) f(t, Y(t) c(t)) \| = \left\| \frac{1}{W(t)} \begin{pmatrix} \ldots W_1(t) \\ \ldots W_2(t) \\ \ldots\ldots\ldots \\ \ldots W_n(t) \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ h(t, u, u', \ldots, u^{(n-1)}) \end{pmatrix} \right\| =$$

$$= \left\| \frac{h(t, u, u', \ldots, u^{(n-1)})}{W(t)} \sum_{k=1}^{n} W_k(t) \right\| \leqq \frac{n\psi(t)\, \varphi(t)}{|W(t)|} \omega(\| c(t) \|).$$

Since

$$W(t) = \det Y(t) = \det Y(0) \exp\left[\int_0^t Tr(A(s))\, ds\right] = \exp\left[\int_0^t h_1(s)\, ds\right],$$

we obtain

$$\| Y^{-1}(t) f(t, Y(t) c(t)) \| \leqq n\psi(t)\, \varphi(t) \exp\left[-\int_0^t h_1(s)\, ds\right] \omega(\| c(t) \|)$$

hence (3) and (4) are fulfilled with $t_0 = 0$ and

$$g(t) = n\psi(t)\, \varphi(t) \exp\left[-\int_0^t h_1(s)\, ds\right].$$

The conditions (26) and (27) imply (6) and (7), respectively. Now, all conditions of Theorem 1 are satisfied. Hence (28) follows from Theorem 1.

The conclusion (29) follows from Corollary 1 by taking $\xi = (a_1, \ldots, a_n)^T$. This completes the proof.

Let us note that Theorem 4 assumes less restrictive conditions than those given in [4] since the condition $\int_{s_0}^{\infty} \frac{ds}{\omega(s)} = \infty$ is omitted and instead of the existence of the limit $\lim_{t \to \infty} c_i(t)$, the stronger result $\int_0^{\infty} |c_i'(t)|\, dt < \infty$ is proved.

Theorem 1 implies the following useful corollary.

**Corollary 2.** Let the hypotheses of Theorem 1 be satisfied and let all solutions of (2) be bounded for $t \geqq t_0$. Then every solution of (1) such that (7) is satisfied, exists and is bounded for $t \geqq t_0$. The bound will be given explicitly.

If, in addition, $f(t, 0) = 0$ for $t \geqq t_0$ and

$$\int_{\varepsilon}^{1} \frac{dt}{\omega(t)} \to \infty \quad \text{as} \quad \varepsilon \to 0, \tag{32}$$

the solution $x \equiv 0$ is stable.

Proof. If all solutions of (2) are bounded, then

$$| Y(t) | \leqq d, \qquad t \geqq t_0 \tag{33}$$

for some constant $d > 0$.

Let $x(t)$ be a solution of (1) satisfying (7), then the boundedness of $x(t)$ for $t \geqq t_0$ follows directly from Theorem 1.

Really, by (6), (10), (14) and (33), the solution $x(t)$ of (1), if it exists, is bounded on $[t_0, \infty)$ for

$$| x(t) | \leqq | Y(t) | \; | c(t) | \leqq d\Omega^{-1}\left[\Omega(| c(t_0) |) + \int_{t_0}^{\infty} g(s)\, ds\right]. \tag{34}$$

The existence of $x(t)$ for $t \geqq t_0$ is assured by its boundedness and the assumption that $f(t, x)$ is continuous on $[t_0, \infty)\, xR_n$.

In order to prove the stability of the solution $x \equiv 0$, we proceed as follows:

Since (32) implies $\Omega^{-1}(r) \to 0$ for $r \to -\infty$, we can choose for a given $\varepsilon > 0$ an $M < 0$ such that $\Omega^{-1}(r) \leqq \frac{\varepsilon}{d}$ for $r \leqq M$.

Now, if $| c(t_0) | = | Y^{-1}(t_0)\, x(t_0) |$ is sufficiently small, it is

$$\Omega \,| Y^{-1}(t_0)\, x(t_0) |) + \int_{t_0}^{t} g(s)\, ds \leqq M$$

for $t \geqq t_0$ so that (33) and (34) imply $| x(t) | \leqq d \,.\, \frac{\varepsilon}{d} = \varepsilon$.

This completes the proof.

**Remark.** Theorem 3 applied to the special case $f(t, x) = G(t)\, x$, where $G(t)$ is an $n \, x \, n$ matrix-function, continuous on $[0, \infty)$, yields the result of BEBERNES and VINH [2].

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