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MONOTONICITY THEOREMS CONCERNING DIFFERENTIAL EQUATIONS $y'' + f(t, y, y') = 0$

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1. Consider a differential equation

$$
(1) \qquad \begin{cases} y'' + f(t, y, y') = 0, \\ \text{where } f(t, y, y') \text{ is continuous on } D = \{(t, y, v): \\ t \in [a, \infty),\ -\infty \langle y, v < \infty\}, f(t, y, v)\, y > 0 \text{ for } y \neq 0. \end{cases}
$$

It is evident that the Cauchy initial problem for (1) has a continuous solution but we do not suppose its uniqueness. In all the work we shall omit the trivial solution $y \equiv 0$ from our considerations.

A non-trivial solution y of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_1^\infty$ such that $a \leqq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ on (t_k, t_{k+1}), $k = 1, 2, \ldots$, $\lim\limits_{k \to \infty} t_k = \infty$ holds.

In the present paper we shall deal only with oscillatory solutions of (1), especially, we shall prove some monotonicity theorems concerning the amplitudes and the distribution of zeros of these solutions and their derivatives.

The special cases of the differential equation (1) were studied in the above mentioned direction in [3], [4], [5], [6], [7].

Let y be an oscillatory solution of (1) and $\{t_k\}_1^\infty$ the sequence of all its zeros. Then there exists exactly one sequence of numbers $\{\tau_k\}_1^\infty$, called the sequence of extremants of y, such that $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$ holds. This a consequence of the following lemma (see [1], [2]):

Lemma 1. *Let y be an arbitrary non-trivial solution of* (1) *and $t_1 < t_2$ its consecutive zeros $\big(y(t) \neq 0$ for $t \in (t_1, t_2)\big)$. Then t_1 and t_2 are the simple zeros of y, there exists exactly one number τ such that $t_1 < \tau < t_2$, $y'(\tau) = 0$ holds. Further,*

$$
f\big(t, y(t), y'(t)\big)\, y'(t) > 0, \qquad t \in (t_1, \tau),
$$

$$
f\big(t, y(t), y'(t)\big)\, y'(t) < 0, \qquad t \in (\tau, t_2),
$$

$$
|\, y'(t_1) \,|\, (\tau - t_1) > |\, y(\tau) \,|.
$$

Put: $\Delta_i = t_{i+1} - t_i$, $\delta_i = \tau_i - t_i$, $\gamma_i = t_{i+1} - \tau_i$, $i = 1, 2, \ldots$ Thus $\Delta_i = \gamma_i + \delta_i$. Our aim consists of finding of conditions under which the sequences

$$\{|y(\tau_i)|\}_1^\infty, \qquad \{|y'(t_i)|\}_1^\infty, \qquad \{\Delta_i\}_1^\infty$$

are monotone ($\{|y'(t_i)|\}_1^\infty$ is the sequence of the absolute values of all local extremes of y' because of $(y'' =) - f(t, y, y') = 0 \Leftrightarrow y = 0 \Leftrightarrow t = t_k$).

2. We start with the investigation of a monotonity of $\{|y'(t_i)|\}_1^\infty$.

Theorem 1. *Let y be an oscillatory solution of* (1). *Let the function $|f(t, y, v)|$ is non-increasing with respect to t and non-increasing with respect to v in D and further let $f(t, y, v) = f(t, y, -v)$ in D. Let $z \in [0, |y(\tau_k)|]$ be an arbitrary number. Denote by ${}^1t$, ${}^2t$ such numbers that ${}^1t \in [t_k, \tau_k]$, ${}^2t \in [\tau_k, t_{k+1}]$, $|y({}^1t)| = |y({}^2t)| = z$. Then*

$$|y'({}^1t)| \geqq |y'({}^2t)|, \qquad \tau_k - {}^1t \leqq {}^2t - \tau_k, \qquad k = 1, 2, 3, \ldots$$

holds so that, in particular, the sequence $\{|y'(t_k)|\}_1^\infty$ is non-increasing and

$$\delta_k \leqq \gamma_k, \qquad k = 1, 2, \ldots$$

holds.

Proof. By multiplying the equation (1) by $-2y'$ and by the integration in the limits from t to τ_k we obtain

$$y'^2(t) = 2 \int_t^{\tau_k} f(t, y(t), y'(t))\, y'(t)\, \mathrm{d}t, \qquad t \in [t_k, t_{k+1}]. \tag{2}$$

Let $y(t) > 0$ on (t_k, t_{k+1}). If $y < 0$, the proof is similar. Thus especially $f(t, y, y') > 0$ on this interval, $y'(t) > 0$ for $t \in [t_k, \tau_k)$, $y'(t) < 0$ for $t \in (\tau_k, t_{k+1}]$ (see Lemma 1).

Denote by ${}^1t(y)$ the inverse function to $y(t)$, $t \in [t_k, \tau_k]$ and ${}^2t(y)$ the inverse function to $y(t)$, $t \in [\tau_k, t_{k+1}]$. These functions exist as $y'(t) \neq 0$ for $t \in [t_k, t_{k+1}]$, $t \neq \tau_k$. Performing the substitution of the integral in (2) $y = y(t)$, the equation (2) is transformed into

$$y'^2({}^it) = 2 \int_y^{y(\tau_k)} f({}^it, y, y'({}^it))\, \mathrm{d}y, \qquad y \in [0, y(\tau_k)],\ i = 1, 2.$$

From this and when using the assumptions of the theorem we obtain: $y \in [0, y(\tau_k)]$,

$$\begin{aligned} \frac{d}{\mathrm{d}y}\{y'^2({}^1t(y)) - y'^2({}^2t(y))\} &= -2\{f({}^1t, y, y'({}^1t)) - f({}^2t, y, y'({}^2t))\} = \\ &= -2\{[f({}^1t, y, y'({}^1t)) - f({}^2t, y, y'({}^1t))] + [f({}^2t, y, y'({}^1t)) - f({}^2t, y, |y'({}^2t)|)]\} \leqq \\ &\leqq -2[f({}^2t, y, |y'({}^1t)|) - f({}^2t, y, |y'({}^2t)|)]. \end{aligned} \tag{3}$$

With regard to the function $|f|$ being non-increasing with respect to y' the following relation is valid

$$(4)\qquad |y'(^1t)| - |y'(^2t)| \leqq 0 \Rightarrow \frac{d}{dy}[y'^2(^1t) - y'^2(^2t)] \leqq 0,$$

$$y \in [0, y(\tau_k)].$$

Suppose that there exists a number $\bar{y} \in [0, y(\tau_k)]$ such that

$$|y'(^1t(\bar{y}))| < |y'(^2t(\bar{y}))|.$$

Then it follows from (4) that

$$|y'(^1t(y))| - |y'(^2t(y))| \leqq |y'(^1t(\bar{y}))| - |y'(^2t(\bar{y}))| < 0, \qquad y \in [\bar{y}, y(\tau_k)].$$

But it is a contradiction because for $y = y(\tau_k)$ we have

$$|y'(^1t)| - |y'(^2t)| = |y'(\tau_k)| - |y'(\tau_k)| = 0.$$

Thus finally

$$(5)\qquad |y'(^1t)| \geqq |y'(^2t)|, \qquad y \in [0, y(\tau_k)].$$

Consider two functions $f_1(y) = \tau_k - {}^1t(y) \geqq 0, f_2(y) = {}^2t(y) - \tau_k \geqq 0, y \in [0, y(\tau_k)]$. From the proved part (5) of the theorem it follows that

$$\frac{d}{dy}[f_1(y) - f_2(y)] = -\frac{1}{y'(^1t)} - \frac{1}{y'(^2t)} \geqq 0, \qquad y \in [0, y(\tau_k)).$$

Thus the function $f_1 - f_2$ is non-decreasing and with regard to $f_1(y) = f_2(y) = 0$ for $y = y(\tau_k)$ we can conclude that $f_1 \leqq f_2$, i.e.

$$\tau_k - {}^1t(y) \leqq {}^2t(y) - \tau_k, \qquad y \in [0, y(\tau_k)]$$

and the first part of the theorem is proved. The special case follows from it for $y = y(\tau_k)$.

The following Theorem can be proved in the same way as Theorem 1.

Theorem 2. *Let y be an oscillatory solution of (1). Let the function $|f(t, y, v)|$ is non-decreasing with respect to t and non-decreasing with respect to v in D and further let $f(t, y, v) = f(t, y, -v)$ in D. Let $z \in [0, |y(\tau_k)|]$ be an arbitrary number. Denote by ${}^1t$, ${}^2t$ such numbers that ${}^1t \in [t_k, \tau_k]$, ${}^2t \in [\tau_k, t_{k+1}]$, $|y(^1t)| = |y(^2t)| = z$. Then*

$$|y'(^1t)| \leqq |y'(^2t)|, \qquad \tau_k - {}^1t \geqq {}^2t - \tau_k, \qquad k = 1, 2, \ldots$$

holds so that, in particular, the sequence $\{|y'(t_k)|\}_1^\infty$ is non-decreasing and

$$\delta_k \geqq \gamma_k, \qquad k = 1, 2, \ldots$$

holds.

Two next theorems deal with monotony of the sequence $\{|y(\tau_k)|\}_1^\infty$. The second of them can be proved similarly to the first.

Theorem 3. *Let y be an oscillatory solution of* (1). *Let the function $|f(t, y, v)|$ is non-increasing with respect to t and non-decreasing with respect to v in D and further let $f(t, y, v) = -f(t, -y, v)$, $(t, y, v) \in D$.*

(*i*) *Let $z \in [0, |y(\tau_k)|]$ be an arbitrary number. Denote by t^1, t^2 such numbers that $t^1 \in [\tau_k, t_{k+1}]$, $t^2 \in [t_{k+1}, \tau_{k+1}]$, $|y(t^1)| = |y(t^2)| = z$. Then*

$$|y'(t^1)| \leqq |y'(t^2)|.$$

(*ii*) *The sequence $\{|y(\tau_k)|\}_1^\infty$ is non-decreasing.*

Proof. By multiplying the equation (1) by $-2y'$ and by integration we obtain

$$y'^2(t) - y'^2(t_{k+1}) = 2 \int_t^{t_{k+1}} f(t, y, y')\, y'\, \mathrm{d}t.$$

Let ${}^1t(y)({}^2t(y))$ be the inverse function to $y(t)$, $t \in [\tau_k, t_{k+1}](t \in [t_{k+1}, \tau_{k+1}])$. The substitution $y = y(t)$ in the integral gives us the relation

$$y'^2({}^it) - y'^2(t_{k+1}) = -2 \int_0^y f({}^it, y, y'({}^it))\, \mathrm{d}y,$$

$$i = 1, 2; \qquad |y| \in [0, \min(|y(\tau_k)|, |y(\tau_{k+1})|)] \overset{\text{def.}}{=} J.$$

From this (we must use the assumption of f being odd with respect to y)

$$\frac{d}{\mathrm{d}|y|}(y'^2({}^1t) - y'^2({}^2t)) = -2[f({}^1t, |y|, y'({}^1t)) - f({}^2t; |y|, y'({}^2t))].$$

So we have the same situation as in the proof of Theorem 1 and we can derive by the same way that the similar relation to (3) holds:

$$\frac{d}{\mathrm{d}|y|}(y'^2({}^1t) - y'^2({}^2t)) \leqq -2[f({}^2t, |y|, y'({}^1t)) - f({}^2t, |y|, y'({}^2t))], \ |y| \in J. \tag{6}$$

For $y = 0$ we have $y'({}^1t) = y'({}^2t) = y'(t_{k+1})$. Assume that there exists a number $\bar{y} \in J$, $\bar{y} \neq 0$ that

$$|y'({}^1t)| > |y'({}^2t)| \tag{7}$$

holds for $|y| = \bar{y}$. Then there exists an interval $J_1 = (z, \bar{y}]$ such that the relation (7) is valid on J_1 and $|y'(^1t)| = |y'(^2t)|, |y| = z$. From this there exists a number $\xi \in J_1$ such that

$$\frac{d}{d|y|}(y'^2(^1t) - y'^2(^2t))|_{|y|=\xi} > 0.$$

But it is in contradiction with (6) because for $|y| = \xi$ we have (by using (7) and the assumption that $|f|$ is non-decreasing with respect to y')

$$\frac{d}{d|y|}(y'^2(^1t) - y'^2(^2t)) \leqq 0.$$

Thus

$$|y'(^1t)| \leqq |y'(^2t)|, \qquad |y| \in J. \tag{8}$$

Now suppose that $|y(\tau_k)| > |y(\tau_{k+1})|$. Then for $|y| = |y(\tau_{k+1})|$ we have $|y'(^1t)| > > 0$ (see Lemma 1), $|y'(^2t)| = 0$ and thus

$$|y'(^1t)| > |y'(^2t)| \qquad \text{for} \qquad |y| = |y(\tau_{k+1})|.$$

But according to (8) $|y'(^1t)| \leqq |y'(^2t)|$, which is a contradiction. So we can conclude that

$$|y(\tau_k)| \leqq |y(\tau_{k+1})|, \qquad k = 1, 2, \ldots, \qquad J = [0, |y(\tau_k)|]$$

and the statement of the theorem is proved.

Theorem 4. *Let y be an oscillatory solution of* (1). *Let the function $|f(t, y, v)|$ is non-decreasing with respect to t and non-increasing with respect to v in D and further let $f(t, y, v) = -f(t, -y, v)$ in D.*

(*i*) *Let $z \in [0, |y(\tau_{k+1})|]$ be an arbitrary number. Denote by ^1t, ^2t such numbers that $^1t \in [\tau_k, t_{k+1}]$, $^2t \in [t_{k+1}, \tau_{k+1}]$, $|y(^1t)| = |y(^2t)| = z$ holds. Then*

$$|y'(^1t)| \geqq |y'(^2t)|.$$

(*ii*) *The sequence $\{|y(\tau_k)|\}_1^\infty$ is non-increasing.*

3. This paragraph deals with the special type of the differential equation (1):

$$\text{(9)} \quad \begin{cases} y'' + f(t, y)\, g(y') = 0 \\ \text{where } f(t, y) \text{ is continuous on } D_1 = \{(t, y) : t \in [a, \infty), \ -\infty < y < \infty\}, \\ f(t, y)\, y > 0 \text{ for } y \neq 0, g \text{ is continuous on } (-\infty, \infty), g(v) > 0, v \in (-\infty, \infty). \end{cases}$$

For (9) we can derive the more substantial results for the solutions.

Theorem 5. *Let y be an oscillatory solution of* (9). *Let the function $|f(t, y)|$ is non-increasing with respect to t in D_1. Let $z \in [0, |y(\tau_k)|]$ be an arbitrary number. Denote by ^1t, ^2t, ^3t such numbers that $^1t \in [t_k, \tau_k]$, $^2t \in [\tau_k, t_{k+1}]$, $^3t \in [t_{k+1}, \tau_{k+1}]$, $|y(^1t)| = |y(^2t)| = |y(^3t)| = z$.*

(*i*) *If* $g(v) = g(-v)$ *for* $v \in (-\infty, \infty)$, *then*

(10) $$|y'(^1t)| \geqq |y'(^2t)|, \qquad \tau_k - {}^1t \leqq {}^2t - \tau_k, k = 1, 2, \ldots$$

In particular, the sequence $\{|y'(t_k)|\}_1^\infty$ *is non-increasing and*

$$\delta_k \leqq \gamma_k, \qquad k = 1, 2, \ldots$$

(*ii*) *If* $f(t, y) = -f(t, -y)$, $(t, y) \in D_1$, *then the sequence* $\{|y(\tau_k)|\}_1^\infty$ *is non-decreasing and*

(11) $$|y'(^2t)| \leqq |y'(^3t)|.$$

Proof. By multiplying the equation (9) by $-y'(g(y'))^{-1}$ and by integration in the limits from t to τ_k we obtain

(12) $$\int_0^{|y'(t)|} \frac{t}{g(t)} \, dt = -\int_t^{\tau} \frac{y''y'}{g(y')} \, dt = \int_t^{\tau} f(t, y) \, y' \, dt = \int_{y(t)}^{y(\tau_k)} f(t, y) \, dy,$$

$t \in [t_k, t_{k+1}]$

If ${}^1t(y)({}^2t(y))$ is the inverse function to $y(t)$, $t \in [t_k, \tau_k]$ ($[\tau_k, t_{k+1}]$), then it follows from (12) that for the fixed $|y| \in [0, |y(\tau_k)|]$

$$\int_0^{|y'(^1t)|} \frac{t}{g(t)} \, dt - \int_0^{|y'(^2t)|} \frac{t}{g(t)} \, dt = \int_y^{y(\tau_k)} [f(^1t, z) - f(^2t, z)| \, dz \geqq 0$$

holds because of $|f|$ being the non-increasing function with respect to t. Thus

(13) $$|y(^1t)| \geqq |y'(^2t)|.$$

The validity of the relation $\tau_k - {}^1t \leqq {}^2t - \tau_k$ can be proved (by use of (13)) by the same considerations as the same relation in Theorem 1.

Let ${}^3t$ be the inverse function to $y(t)$, $t \in [t_{k+1}, \tau_{k+1}]$. By multiplying the equation (9) by $y'(g(y'))^{-1}$ and by integration in the limits from $t \in [\tau_k, \tau_{k+1}]$ to t_{k+1} we can get:

$$\int_{y'(t)}^{y'(t_{k+1})} \frac{t}{g(t)} \, dt = \int_t^{t_{k+1}} \frac{y''y'}{g(y')} \, dt = -\int_t^{t_{k+1}} f(t, y) \, y' \, dt = \int_0^{|y(t)|} f(t, y) \, dy.$$

From this for the fixed $y \in [0, \min(|y(\tau_k)|, |y(\tau_{k+1})|)] \stackrel{\text{def.}}{=} J$ there holds

$$\int_{y'(^2t)}^{y'(t_{k+1})} \frac{t}{g(t)} \, dt - \int_{y'(^3t)}^{y'(t_{k+1})} \frac{t}{g(t)} \, dt = \int_0^{|y|} [f(^2t, z) - f(^3t, z)] \, dz \geqq 0$$

because $|f|$ is non-increasing with respect to t. Thus $|y'({}^2t)| \leqq |y'({}^3t)|$ and the first part of the statement of the theorem is proved. The rest can be derived in the same way as the same result in Theorem 3.

The following theorem can be proved similarly.

Theorem 6. *Let y be an oscillatory solution of* (9). *Let the function $|f(t, y)|$ be non-decreasing with respect to t in D_1. Let $z \in [0, |y(\tau_{k+1})|]$ be an arbitrary number and let ${}^1t$, ${}^2t$, ${}^3t$ be of the same meaning as in Theorem* 5.

(*i*) *If $g(v) = g(-v)$ for $v \in (-\infty, \infty)$, then*

$$|y'({}^1t) \leqq |y'({}^2t)|, \qquad \tau_k - {}^1t \geqq {}^2t - \tau_k. \tag{14}$$

In particular, the sequence $\{|y'(t_k)|\}_1^\infty$ is non-decreasing and $\delta_k \geqq \gamma_k$, $k = 1, 2, \ldots$

(*ii*) *If $f(t, y) = -f(t, -y)$, $(t, y) \in D_1$, then the sequence $\{|y(\tau_k)|\}_1^\infty$ is non-increasing and*

$$|y'({}^2t)| \geqq |y'({}^3t)|. \tag{15}$$

Remark 1. Theorems 5 and 6 are the generalizations of some results in [3], [4], [5], [6], [7]. Das [4] studied the equation

$$y'' + f(t, y) = 0,$$

Bihari [5] proved the monotonity of the sequences $\{|y(\tau_k)|\}_1^\infty$, $\{|y'(t_k)|\}_1^\infty$ for the equation

$$y'' + \varphi(t) f(y)\, h(y') = 0, \tag{16}$$

but under many other assumptions ($\varphi > 0$ is continuous, increasing and bounded for $t \geqq a$, $f(y)$ is odd, non-decreasing and $f(y) \in \operatorname{Lip}(1)$ for $|y| \leqq b$, $h(u) > 0$, $|h|$ is non-increasing, $h(u) \in \operatorname{Lip}(1)$ for all u, h is even). The papers [3], [6] deals with the special types of the differential equation (16), Katranov [7] proved some results of Theorem 6.

Remark 2. It is evident from the proofs that the theorems 5 and 6 are valid, too if we replace the words and symbols non-decreasing, non-increasing, $\leqq$, $\geqq$ by increasing, decreasing, $<$, $>$, respectively with the exceptions of (11), (15) for $z = 0$, (10) for $z = |y(\tau_k)|$ and (14) for $z = |y(\tau_{k+1})|$, of course.

Remark 3. If the function $f(t, y) \equiv f(y)$ is constant with respect to t and $f(y) = -f(-y)$, $g(v) = g(-v)$, then every oscillatory solution of (9) is a half-periodic function. From the proofs of Theorems 5 and 6 we can see that

$$|y'({}^1t)| = |y'({}^2t)| = |y'({}^3t)|, \qquad \tau_k - {}^1t = {}^2t - \tau_k,$$

and the required relation $t_{k+1} - {}^2t = {}^3t - t_{k+1}$ can be proved in the same way as the relation $\tau_k - {}^1t = {}^2t - \tau_k$. So the above mentioned conclusion is valid. This situation was studied in [5] for the differential equation (16).

Theorem 7. *Let y be an oscillatory solution of* (9). *Let*

(i) $\frac{\partial}{\partial t} f(t, y)$, $\frac{\partial}{\partial y} f(t, y)$ *exist in* D_1,

(ii) $f(t, y) = -f(t, -y)$ *in* D_1,

(iii) $\frac{\partial}{\partial t} |f(t, y)| \leqq 0$ *in* D_1,

(iv) $\frac{\partial}{\partial y} f(t, y) \geqq 0$ in D_1, let $\frac{\partial}{\partial y} f(t, y)$ *be non-increasing with respect to* y *in* $D_2 = \{(t, y) : (t, y) \in D_1, y \geqq 0\}$, $\frac{\partial}{\partial y} f(t, y)$ *be non-increasing with respect to* t *in* D_2.

Then

$$\gamma_k \leqq \delta_{k+1}, \qquad k = 1, 2, \ldots$$

If, in addition, $g(v) = g(-v)$, $v \in (-\infty, \infty)$, then

$$\Delta_k \leqq \Delta_{k+1}, \qquad k = 1, 2, \ldots$$

Proof. Denote by ${}^1t(y')({}^2t(y'))$ the inverse function to $y'(t)$, $t \in [\tau_k, t_{k+1}]$ ($t \in [t_{k+1}, \tau_{k+1}]$). These functions exist because $y''(t) = 0 \Leftrightarrow y(t) = 0 \Leftrightarrow t = t_{k+1}$ on $[\tau_k, \tau_{k+1}]$. Suppose that $y'(t) > 0$ on (τ_k, τ_{k+1}). The statement for the case $y'(t) < 0$ can be proved similarly. Then $y(t) < 0$, $f < 0$, $y''(t) > 0$ on $[\tau_k, t_{k+1})$, $y(t) > 0$, $f > 0$, $y''(t) < 0$ on $(t_{k+1}, \tau_{k+1}]$ (use (9) and Lemma 1).
According to (9) and by use of f being odd with respect to y the following estimation holds for the fixed $y' \in [0, y'(t_{k+1})]$

$$\frac{d}{dy'}\left\{\frac{|y''({}^1t)|}{g(y')} - \frac{|y''({}^2t)|}{g(y')}\right\} = \frac{d}{dy'}\{|f({}^1t, y({}^1t))| - |f({}^2t, y({}^2t))|\} =$$

$$= \frac{\partial}{\partial t}|f({}^1t, y({}^1t))|\frac{1}{y''({}^1t)} + \frac{\partial}{\partial y}|f({}^1t, y({}^1t))|\frac{y'}{y''({}^1t)} -$$

$$- \frac{\partial}{\partial t}|f({}^2t, y({}^2t))|\frac{1}{y''({}^2t)} - \frac{\partial}{\partial y}|f({}^2t, y({}^2t))|\frac{y'}{y''({}^2t)} \leqq$$

$$\leqq -\frac{\partial}{\partial y} f({}^1t, |y({}^1t)|)\frac{y'}{|y''({}^1t)|} + \frac{\partial}{\partial y} f({}^2t, y({}^2t))\frac{y'}{|y''({}^2t)|} =$$

$$= \frac{y'}{|y''({}^1t)\, y''({}^2t)|}\left\{\left[\frac{\partial}{\partial y} f({}^2t, y({}^2t)) - \frac{\partial}{\partial y} f({}^1t, y({}^2t))\right]|y''({}^1t)| + \right.$$

$$+\left[\frac{\partial}{\partial y}f({}^1t, y({}^2t)) - \frac{\partial}{\partial y}f({}^1t, |y({}^1t)|)\right]|y''({}^1t)| + \frac{\partial}{\partial y}f({}^1t, |y({}^1t)|)\times$$

$$\times(|y''({}^1t)| - |y''({}^2t)|)\Big\} \leqq H(y')\cdot\left(\frac{|y''({}^1t)|}{g(y')} - \frac{|y''({}^2t)|}{g(y')}\right),$$

$$H(y') = \frac{y'q(y')}{|y''({}^1t)\,y''({}^2t)|}\,\frac{\partial}{\partial y}f({}^1t, |y({}^1t)|) \geqq 0,$$

because according to (11) $|y({}^1t)| \leqq |y({}^2t)|$. Suppose that there exists a number $z \in [0, y'(t_{k+1})]$ such that

$$G(y') = \frac{|y''({}^1t)|}{g(y')} - \frac{|y''({}^2t)|}{g(y')} < 0$$

holds for $y' = z$. Then according to the above mentioned estimation $G(y') \leqq G(z) < 0$, $y' \in [z, y'(t_{k+1})]$ is valid, but this is a contradiction because $G(y'(t_{k+1})) = 0$. Thus $G(y') \geqq 0$ on $[0, y'(t_{k+1})]$ and finally

(17) $$|y''({}^1t)| \geqq |y''({}^2t)|, \qquad y' \in [0, y'(t_{k+1})].$$

Consider two functions $h_1(y') = t_{k+1} - {}^1t$, $h_2(y') = {}^2t - t_{+k1}$, $y' \in [0, y'(t_{k+1})]$. According to (17)

$$\frac{\mathrm{d}}{\mathrm{d}y'}[h_1 - h_2] = -\frac{1}{y''({}^1t)} - \frac{1}{y''({}^2t)} \geqq 0, \qquad y' \in [0, y'(t_{k+1})).$$

Thus $h_1 - h_2$ is non-decreasing and with regard to $h_1(y') = h_2(y') = 0$ for $y' = y'(t_{k+1})$ we can conclude that $h_1 \leqq h_2$. So the first part of the statement of the theorem is proved and the rest follows from the Theorem 5.

The following theorem can be proved similarly to Theorem 7.

Theorem 8. *Let y be an oscillatory solution of* (9). *Let*

(i) $\frac{\partial}{\partial t}f(t, y)$, $\frac{\partial}{\partial y}f(t, y)$ *exist in* D_1,

(ii) $f(t, y) = -f(t, -y)$ *in* D_1,

(iii) $\frac{\partial}{\partial t}|f(t, y)| \geqq 0$ *in* D_1,

(iv) $\frac{\partial}{\partial y}f(t, y) \geqq 0$ *in* D_1, *let* $\frac{\partial}{\partial y}f(t, y)$ *be non-increasing with respect to* y *in* $D_2 = \{(t, y) : (t, y) \in D_1, y \geqq 0\}$, $\frac{\partial}{\partial y}f(t, y)$ *be non-decreasing with respect to* t *in* D_2.

Then

$$\gamma_k \geqq \delta_{k+1}, \qquad k = 1, 2, \ldots$$

If, in addition, $g(v) = g(-v)$, $v \in (-\infty, \infty)$, *then*

$$\Delta_k \geqq \Delta_{k+1}, \qquad k = 1, 2, \ldots$$

Remark 4. The statement of Theorem 8 was proved by Bihari [3] for the differential equation (16) but under the more restrictive assumptions on the functions $h(y')$ ($h > 0$ is an even, non-increasing function for $y' > 0$, non-decreasing for $y' < 0$, $h \in \text{Lip}(1)$) and $\varphi(t)$ (φ is increasing) and under the different assumptions on the function $f(y)$ ($f \in \text{Lip}(1)$, $f(y)$ is increasing, $f(y)\, y^{-1} = 0\,(1)\,(y \to 0)$, further $f(y)\, y^{-1}$ is non-increasing for $y > 0$ and non-decreasing for $y < 0$).

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