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DIRECT PRODUCTS OF WEAK HOMOMORPHISMS

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1. Introduction. Weak homomorphisms and weak isomorphisms are studied by many authors, see for example [2], [3], [4], [5] and [11]. It was proved that weak homomorphisms have many of "nice" properties analogous to ordinary homomorphisms (e.g. they give relations between subalgebras and kernels of weak homomorphisms, they preserve map superposition and restriction on subalgebras etc.). But for direct products of weak homomorphisms this analogy is rather complicated, i.e. the direct product of weak homomorphisms need not be a weak homomorphism. One can easily state the conditions securing this analogy, as it is shown in this paper. However, we shall rather be concerned with the converse problem, for which algebras are weak homomorphisms decomposable into direct products of weak homomorphisms. The analogical problem for ordinary homomorphisms was solved in [6], [7], [8] and [9]. In the present paper there are given some sufficient conditions for solving this problem.

2. Basic concepts. Let $\mathfrak{A} = (A, \mathbf{F})$ be an algebra with the support A and a set $\mathbf{F}$ of fundamental operations. We use the notation introduced in [10]. By $\mathbf{A}$ is denoted the set of all algebraic operations of the algebra $\mathfrak{A}$, i.e. $\mathbf{A}$ contains all operations from F, all trivial operations and all operations derived from fundamental and trivial operations as successive superpositions of these (see [10]). Let $\mathfrak{B} = (B, \mathrm{G})$ be also an algebra and $\mathbf{B}$ the set of all algebraic operations of $\mathfrak{B}$. Let h be a mapping of A into B. Making use of the mapping h we introduce a relation R_h between $\mathbf{A}$ and $\mathbf{B}$ setting for $f \in \mathbf{A}$ and $f^* \in \mathbf{B}$

$$fR_hf^* \quad \text{if and only if} \quad f^* . h = h . f,$$

i.e. $h(f(x_1, \dots, x_n)) = f^*(h(x_1), \dots, h(x_n))$ for each $x_1, \dots, x_n \in A$.

Definition 1. A mapping h of A into B is called a *weak homomorphism* of $\mathfrak{A} = (A, \mathbf{F})$ into $\mathfrak{B} = (B, \mathbf{G})$ if to every fundamental operation $f \in \mathbf{F}$ there exists an algebraic operation $f^* \in \mathbf{B}$ such that fR_hf^* and, vice versa, to each $g^* \in \mathbf{G}$ there exists $g \in \mathrm{A}$ such that gR_hg^*. If h is a one-to-one mapping of A onto B and h is a weak homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then h is called a *weak isomorphism*.

Definition of weak homomorphism is usualy formulated for algebraic operation only. However, it was proved in [5] that it can be also formulated for fundamental f, g as in the definition 1.

Let $\mathfrak{A}_\tau$ be a set of algebras for $\tau \in T$. If all $\mathfrak{A}_\tau$ are of the same type, then they have the same set of fundamental operations. Let us denote it by the same symbol $\mathbf{F}$, i.e. $\mathfrak{A}_\tau = (A_\tau, \mathbf{F})$ for $\tau \in T$, and not make any difference between operations from $\mathfrak{A}_\tau$ for different $\tau \in T$. In other words, we shall not index these operations by indices τ of the algebras $\mathfrak{A}_\tau$.

By the symbol $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ is denoted the direct product of algebras $\mathfrak{A}_\tau$, i.e. $\mathfrak{A} = (A, \mathbf{F})$, where $A = \prod_{\tau \in T} A_\tau$ is a Cartesian product of sets A_τ and operations are performed component by component.

Definition 2. Let $\mathfrak{A}_\tau = (A_\tau, \mathbf{F}), \mathfrak{B}_\tau = (B_\tau, \mathbf{G})$ be algebras for $\tau \in T$ and h_τ be a mapping of $\mathfrak{A}_\tau$ into $\mathfrak{B}_\tau$ for each $\tau \in T$. The *direct product of mappings* h_τ, denoted by $h = \prod_{\tau \in T} h_\tau$, is a mapping h of $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ into $\mathfrak{B} = \prod_{\tau \in T} \mathfrak{B}_\tau$ such that

$$pr_\tau(h(a)) = h_\tau(pr_\tau(a)) \qquad \text{for each} \qquad \tau \in T, a \in A, \tag{V}$$

where pr_τ denotes the τ-th projection of $\mathfrak{A}$ (or $\mathfrak{B}$) onto $\mathfrak{A}_\tau$ (or $\mathfrak{B}_\tau$, respectively), and A is the support of $\mathfrak{A}$.

It is clear that the direct product of homomorphic mappings is a homomorphic mapping (see [6], Theorem 1). This cannot be true for weak homomorphisms in general. If $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$ are Boolean algebras and if h_1 is an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{B}_1$ and h_2 is an antiisomorphism of $\mathfrak{A}_2$ onto $\mathfrak{B}_2$, then h_1, h_2 are weak homomorphisms of Boolean algebras (see [11]), but $h = h_1 \times h_2$ is not because $h(0) \neq 0$, $h(0) \neq 1$.

3. Direct products. Let $\mathfrak{A}_\tau = (A_\tau, \mathbf{F}), \mathfrak{B}_\tau = (B_\tau, \mathbf{G})$ be algebras for $\tau \in T$ and h_τ be a weak homomorphism of $\mathfrak{A}_\tau$ into $\mathfrak{B}_\tau$ for each $\tau \in T$. By the definition of direct products of algebras, $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ has the same fundamental (and also algebraic) operations as each $\mathfrak{A}_\tau$, i.e. $\mathfrak{A} = (A, \mathbf{F})$, $\mathfrak{B} = (B, \mathbf{G})$. Weak homomorphisms h_τ for $\tau \in T$ are called *similar* if $R_{h_{\tau'}} = R_{h_{\tau''}}$ for each $\tau'\ \tau'' \in T$.

Theorem 1. *Let $\mathfrak{A}_\tau = (A_\tau, \mathbf{F}), \mathfrak{B}_\tau = (B_\tau, \mathbf{G})$ be algebras and h_τ be a weak homomorphism of $\mathfrak{A}_\tau$ into $\mathfrak{B}_\tau$ for each $\tau \in T$. If h_τ are similar for $\tau \in T$, then the mapping $h = \prod_{\tau \in T} h_\tau$ is a weak homomorphism of $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ into $\mathfrak{B} = \prod_{\tau \in T} \mathfrak{B}_\tau$.*

Proof. Let h_τ be similar, then $fR_{h_\tau}f^*$ implies fR_hf^* for $h = \prod_{\tau \in T} h_\tau$, because the operations on $\mathfrak{A}, \mathfrak{B}$ are performed componentwise. Then for each $f \in \mathbf{F}$ there exists $f^* \in \mathbf{B}$ such that fR_hf^*, because for each $f \in \mathbf{F}$ there exists $f^* \in \mathbf{B}$ such that $fR_{h_\tau}f^*$ for each $\tau \in T$, and vice versa, for each $g^* \in \mathbf{G}$ there exists $g \in \mathrm{A}$ such that gR_hg^*, thus h is a weak homomorphism.

It is clear that for Boolean algebras and weak isomorphisms also the converse statement is true. Thus we obtain:

Corollary 2. *Let $\mathfrak{A}_\tau$, $\mathfrak{B}_\tau$ be Boolean algebras and h_τ be a weak isomorphism of $\mathfrak{A}_\tau$ onto $\mathfrak{B}_\tau$ for each $\tau \in T$. Then $h = \prod_{\tau \in T} h_\tau$ is a weak isomorphism of Boolean algebra $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ onto Boolean algebra $\mathfrak{B} = \prod_{\tau \in T} \mathfrak{B}_\tau$ if and only if either $h_\tau(0) = 0$ or $h_\tau(0) = 1$ for all $\tau \in T$.*

Proof. By [1], each weak isomorphism h_τ of Boolean algebras fulfils $h_\tau(0) = 0$ or $h_\tau(0) = 1$. A direct product of Boolean algebras is a Boolean algebra (see [1]) and each weak isomorphism is a one-to-one homomorphism (see [5]); then the assertion follows directly from the theorem 1 and foregoing contraexample.

Lemma A. *Each chain considered as a lattice has exactly two binary algebraic operations which are not trivial. These operations are fundamental.*

Proof. On a chain considered as a lattice there exist two binary fundamental operation only, namely $\vee$ and $\wedge$, and two binary trivial operations (see [10]), namely e_1, e_2, where $e_1(a, b) = a$, $e_2(a, b) = b$ for each a, b. Further (see [10]), each algebraic operation can be obtained by a successive superposition of fundamental and trivial operations. Denote as f, g the fundamental binary operations, i.e. $f = \vee$, $g = \wedge$ or $f = \wedge$, $g = \vee$. We can easily prove $e_1(f, g) = f$, $e_2(f, g) = g$, $f(e_i, e_j) = f$ for $i \neq j$ and $f(e_i, e_i) = e_i$, $f(e_i, g) = e_i$ for $f \neq g$ and $f(e_i, f) = f$, $f(g, e_i) = e_i$, $f(f, e_i) = f$, $f(f, f) = f$, $g(f, f) = f$. Because each two elements are comparable, we obtain $f(f, g) = f(g, f) = f$. Accordingly, superpositions of binary fundamental and trivial operations are fundamental and trivial operations only which completes the proof.

Let L_1, L_2 be lattices. A mapping h of L_1 into L_2 is said to be a *dual homomorphism* if $h(a \vee b) = h(a) \wedge h(b)$ and $h(a \wedge b) = h(a) \vee h(b)$ for each $a, b \in L_1$.

Lemma B. *Each weak homomorphism of a chain into a chain is either a homomorphism or a dual one.*

Proof. This follows directly from the definition 1 and lemma A.

Corollary 3. *Let $\mathfrak{A}_\tau$ be chains for $\tau \in T$. Then there exist exactly two algebraic binary operations which are not trivial on the distributive lattice $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$.*

Proof. The algebras $\mathfrak{A}_\tau$, $\mathfrak{A}$ have the same set of algebraic operations, i.e. $A^{(2)} = \{\vee, \wedge\}$ by the lemma A. By [1], $\mathfrak{A}$ is the distributive lattice. (The symbol $A^{(n)}$ is introduced in [10].)

Theorem 4. *Let $\mathfrak{A}_\tau$, $\mathfrak{B}_\sigma$ be chains for $\tau \in T$, $\sigma \in S$ and $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$, $\mathfrak{B} = \prod_{\delta \in S} \mathfrak{B}_\sigma$.*

Each weak homomorphism of the distributive lattice $\mathfrak{A}$ into the distributive lattice $\mathfrak{B}$ is either a homomorphism or a dual one.

This follows directly from the definition 1 and corollary 3. From the theorem 1 and the lemma B we can easily obtain the following

Corollary 5. *Let $\mathfrak{A}_\tau$, $\mathfrak{B}_\tau$ be chains and h_τ be a weak homomorphism of $\mathfrak{A}_\tau$ into $\mathfrak{B}_\tau$ for each $\tau \in T$. Then $h = \prod_{\tau \in T} h_\tau$ is a weak homomorphism of the distributive lattice $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ into $\mathfrak{B} = \prod_{\tau \in T} \mathfrak{B}_\tau$ if all h_τ are either homomorphisms or all h_τ are dual homomorphisms.*

4. Algebras with zero. Denote Λ a class of algebras with *zero element* 0 and a binary operation $\oplus$ such that:

(i) $a \oplus 0 = 0 \oplus a = a$,

(ii) $f(00 \dots 0) = 0$,

for arbitrary $\mathfrak{A} = (A, F) \in \Lambda$, $a \in A$, $f \in \mathrm{A}$ n-ary for $n \geqq 1$. From (i), (ii) it follows that $\{0\}$ is a one-element subalgebra of each $\mathfrak{A} \in \Lambda$.

Let $\mathfrak{A}_\tau \in \Lambda$ for $\tau \in T$ and $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$. Evidently, $\mathfrak{A} \in \Lambda$. By the symbol $\overline{\mathfrak{A}}_\tau$ (or $\overline{\prod_{\tau \in T'} \mathfrak{A}_\tau}$ for $T' \subseteqq T$) is denoted a subalgebra of $\mathfrak{A}$ such that

$$pr_\tau \overline{\mathfrak{A}}_\tau = \mathfrak{A}_\tau, \qquad pr_{\tau'} \overline{\mathfrak{A}}_\tau = \{0\} \qquad \text{for } \tau' \neq \tau$$

(or $pr_\tau \overline{\prod_{\tau \in T'} \mathfrak{A}_\tau} = \mathfrak{A}_\tau$ for $\tau \in T'$ and $pr_{\tau'} \overline{\prod_{\tau \in T'} \mathfrak{A}_\tau} = \{0\}$ for $\tau' \in T - T'$, respectively).

Evidently, $\overline{\mathfrak{A}}_\tau$ is a subalgebra of $\mathfrak{A}$ isomorphic with $\mathfrak{A}_\tau$, $\overline{\prod_{\tau \in T'} \mathfrak{A}_\tau}$ is a subalgebra of $\mathfrak{A}$ isomorphic with $\prod_{\tau \in T'} \mathfrak{A}_\tau$ and $\overline{\prod_{\tau \in T} \mathfrak{A}_\tau} = \prod_{\tau \in T} \mathfrak{A}_\tau$ for $T' = T$.

Let $\mathfrak{A} \in \Lambda$, $\mathfrak{A} = (A, \mathrm{F})$. An operation $f \in \mathrm{A}$ is called *regular on* $\mathfrak{A}$ if the arity of f is greater than 1 and

(iii) $f(a_1 a_2 \dots a_n) = 0$ if and only if $a_i = 0$ for at least one $i \in \{1, \dots, n\}$, where $a_1, \dots, a_n \in A$.

Lemma C. *Let $\mathfrak{A}_\tau \in \Lambda$ for $\tau \in T$. If $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$ and $f \in \mathrm{A}$ is an n-ary regular operation on each $\mathfrak{A}_\tau$ (f need not be regular on $\mathfrak{A}$), then $a_1, \dots, a_n \in A$, $a_i = 0$ for at least one i, implies $f(a_1 a_2 \dots a_n) = 0$, where A is the support of $\mathfrak{A}$.*

Proof. Let $a_i = 0$, then for each $\tau \in T$ is $pr_\tau a_i = 0$ and $pr_\tau f(a_1 \dots a_i \dots a_n) = f[(pr_\tau a_1) \dots (pr_\tau a_i) \dots (pr_\tau a_n)] = f[(pr_\tau a_1) \dots 0 \dots (pr_\tau a_n)] = 0$, then $f(a_1 \dots 0 \dots a_n) = 0$.

Lemma D. *Let $\mathfrak{A}, \mathfrak{B} \in \Lambda$ and h be a weak isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $h(0) = 0$. If f is regular on $\mathfrak{A}$ and $f R_h f^*$, then f^* is regular on $\mathfrak{B}$.*

Proof. Let f is regular n-ary operation on $\mathfrak{A}$ and $fR_h{}^*$ for weak isomorphism h fulfilling $h(0) = 0$. Let $b_1, \ldots, b_n \in B$ and $a_1, \ldots, a_n \in A$ be elements such that $h(a_i) = b_i$ for each i; A, B are supports of $\mathfrak{A}, \mathfrak{B}$ respectively. If $b_i = 0$, then $a_i = 0$ because $h(0) = 0$ and h is a one-to-one mapping, thus $h(f(a_1 \ldots a_i \ldots a_n)) = h(0) = 0$ and $f^*(b_1 \ldots 0 \ldots b_n) = h(f(a_1 \ldots 0 \ldots a_n)) = 0$. If $b_i \neq 0$ for all $i = 1, \ldots, n$, then $a_i \neq 0$ and $f(a_1 \ldots a_n) \neq 0$, thus $0 \neq h(f(a_1 \ldots a_n)) = f^*(b_1 \ldots b_n)$, i.e. f^* is regular on $\mathfrak{B}$.

We say that $f \in \mathrm{A}$ *fulfils* (P) *on* $\mathfrak{A} = (\mathrm{A}, \mathrm{F})$ if for arbitrary $a_1, \ldots, a_n \in A$, $a_i \neq 0$ for each i, is true $f(a_1 \ldots a_n) \neq 0$.

Lemma E. *Let* $\mathfrak{A} = (A, \mathrm{F})$ *and each n-ary fundamental* $f \in \mathrm{F}$ *for* $n \geqq 1$ *fulfils* (P) *on* $\mathfrak{A}$. *Then also each n-ary algebraic operation of* $\mathfrak{A}$ *fulfils* (P) *on* $\mathfrak{A}$ *for* $n \geqq 1$.

Proof. Evidently, each trivial operation fulfils (P) on $\mathfrak{A}$ and each superposition of operations fulfilling (P) is an operation fulfilling (P).

5. Direct decompositions. In this paragraph, we shall not make any difference between algebra and its support to simplify notation. Now, we summarize assumptions which will be used in the formulation of the subsequent theorems.

Assumptions.

(1) $\mathfrak{A}_\tau = (A_\tau, \mathrm{F})$, $\mathfrak{B}_\sigma = (B_\sigma, \mathrm{G})$ are at least two-element algebras from Λ for each $\tau \in T$, $\sigma \in S$.

(2) T, S are finite index sets.

(3) $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$, $\mathfrak{B} = \prod_{\sigma \in S} \mathfrak{B}_\sigma$.

(4) h is a weak homomorphism of $\mathfrak{A}$ *onto* $\mathfrak{B}$ such that $h(0) = 0$.

(5) there exists at least one operation from A which is regular on each $\mathfrak{A}_\tau$.

(6) if $f \in \mathrm{A}$ is regular on each $\mathfrak{A}_\tau$, then at least one $f^* \in \mathrm{B}$ fulfilling fR_hf^* is regular on each $\mathfrak{B}_\sigma$.

(7) if $f \in \mathrm{A}$ is regular on each $\mathfrak{A}_\tau$, then at least one $f^* \in \mathrm{B}$ fulfilling fR_hf^* fulfils (P) on each $\mathfrak{B}_\sigma$.

By the symbols A, B are denoted sets of algebraic operations of $\mathfrak{A}_\tau$ and $\mathfrak{A}$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}$, respectively, as it is introduced above. It is clear that (6) implies (7) but not vice versa.

Lemma F. *Let the assumptions* (1), (3), (4), (5), (7) *be true. Then* $h(\overline{\mathfrak{A}}_{\tau_1}) \cap (h\overline{\mathfrak{A}}_{\tau_2}) \cap \overline{\mathfrak{B}}_\sigma = \{0\}$ *for each* $\tau_1, \tau_2 \in T$, $\sigma \in S$, $\tau_1 \neq \tau_2$.

Proof. Let $h(\overline{\mathfrak{A}}_{\tau_1}) \cap h(\overline{\mathfrak{A}}_{\tau_2}) \cap \overline{\mathfrak{B}}_\sigma \neq \{0\}$ for $\tau_1 \neq \tau_2$. Then there exist $a_1 \in \overline{\mathfrak{A}}_{\tau_1}$, $a_2 \in \overline{\mathfrak{A}}\tau_2$, $b_1, b_2 \in h(\overline{\mathfrak{A}}_{\tau_1}) \cap h(\overline{\mathfrak{A}}_{\tau_2}) \cap \overline{\mathfrak{B}}_\sigma$ (we admit $b_1 = b_2$) such that $b_1 \neq 0 \neq b_2$ and $h(a_1) = b_1$, $h(a_2) = b_2$. Let f be regular on each $\mathfrak{A}_\tau$, then $f(a_1 a_2 \ldots a_2) = 0$ because $pr_\tau a_1 = 0$ for $\tau \neq \tau_1$, $pr_\tau a_2 = 0$ for $\tau \neq \tau_2$ and $\tau_1 \neq \tau_2$. If fR_hf^*, $h(0) = 0$ and f^* fulfils (P) on each $\mathfrak{B}_\sigma$, then

$$f^*(b_1 b_2 \ldots b_2) \neq 0 \qquad \text{because} \qquad b_1, b_2 \in \overline{\mathfrak{B}}_\sigma, b_1 \neq 0 \neq b_2,$$

but $f^*(b_1 b_2 \ldots b_2) = h(f(a_1 a_2 \ldots a_2)) = h(0) = 0$, which is a contradiction.

Theorem 6. *Let the assumptions* (1), (2), (3), (4), (5), (6) *be true. Then for each* $\sigma \in S$ *there exists uniquely* $\tau_\sigma \in T$ *such that* $\overline{\mathfrak{B}}_\sigma \subseteq h(\overline{\mathfrak{A}}_{\tau_\sigma})$.

Proof. Let assumptions of the theorem be true and the assertion itself not. Then there exists at least one $\sigma \in S$ such that for any $\tau \in T$ is not $\overline{\mathfrak{B}}_\sigma \subseteq h(\overline{\mathfrak{A}}_\tau)$. Let T' be an arbitrary subset of T such that $h(\prod_{\tau \in T'} \mathfrak{A}_\tau) \supseteq \overline{\mathfrak{B}}_\sigma$. Such T' exists, for example $T' = T$. Evidently, *card* $T' > 1$. If the assertion is not true, then *card* $T' \geqq 2$ for each T, of this property. Thus two following cases are possible only:

(a) there exist $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$ and $a_1 \in \overline{\mathfrak{A}}_{\tau_1}$, $a_2 \in \overline{\mathfrak{A}}_{\tau_2}$ such that $h(a_1), h(a_2) \in \overline{\mathfrak{B}}_\sigma$ and $h(a_1) \neq 0 \neq h(a_2)$. Then contradiction can be obtained by the lemma F.

(b) there exists $b_0 \in \overline{\mathfrak{B}}_\sigma$, $b_0 \neq 0$ such that for each $a \in h^{-1}(b_0)$ is *card* $T_a \geqq 2$, where $T_a = \{\tau;\ \tau \in T,\ pr_\tau a \neq 0\}$. From (2) follows the finiteness of T_a. Then each $a \in \mathfrak{A}$ can be written in the following form:

$$a = \bar{a}_1 \oplus \ldots \oplus \bar{a}_n, \qquad \text{where} \qquad \bar{a}_i \in \overline{\mathfrak{A}}_i,\ T_a = \{1, \ldots, n\},\ pr_i a = pr_i \bar{a}_i.$$

From (i) it follows that this expression does not depend on any bracketing because in each projection all elements except one are equal to zero and the operation $\oplus$ is performed component by component. Let $\circ$ be a binary algebraic operation from B such that $\oplus R_h \circ$. Then $b_0 = h(a) = h(\bar{a}_1) \circ \ldots \circ h(\bar{a}_n)$. It evidently also does not depend on any bracketing. If $h(\bar{a}_1) \in \overline{\mathfrak{B}}_\sigma$ for each $i \in T_a$, then (by (*a*)) it is $h(\bar{a}_i) = 0$ for $i \neq \tau_0 \in T_a$, thus $b = h(a) = h(\bar{a}_{\tau_0})$, i.e. $a' \in h^{-1}(b_0)$, *card* $T_{a'} = 1$ for $a' = \bar{a}_{\tau_0}$ which is a contradiction with assumption (b). Let for $j \in T_a$ be $h(\bar{a}_j) \notin \overline{\mathfrak{B}}_\sigma$. If f is regular on each $\mathfrak{A}_\tau$ (it exists by (5)), then $f(\bar{a}_j \bar{a}_i \ldots \bar{a}_i) = 0$ for $i \in T_a$, $i \neq j$. If $f R_h f^*$ and f^* is regular on each $\mathfrak{B}_\sigma$ (it exists by (6)), then $0 = h(0) = h(f(\bar{a}_j \bar{a}_i \ldots \bar{a}_i)) = f^*(h(\bar{a}_j)\, h(\bar{a}_i) \ldots h(\bar{a}_i))$, thus for each $\sigma \in S$ there exists at least one $i \in T_a$ such that $pr_\sigma h(a_i) \neq 0$. If $h(\bar{a}_j) \notin \overline{\mathfrak{B}}_\sigma$, then there exists $\sigma' \neq \sigma$ such that $pr_{\sigma'} h(\bar{a}_j) \neq 0$ and, by the above mentioned consideration, $pr_{\sigma'} h(\bar{a}_1) = 0$ for each $i \neq j$. Thus $pr_{\sigma'} h(a) = pr_{\sigma'}(h(\bar{a}_1) \circ \ldots \circ h(\bar{a}_n)) = [pr_{\sigma'} h(\bar{a}_1)] \circ \ldots \circ [pr_{\sigma'} h(\bar{a}_1)] = 0 \circ \ldots \circ [pr_{\sigma'} h(\bar{a}_j)] \circ \ldots \circ 0 = pr_{\sigma'} h(\bar{a}_j) \neq 0$ (because $0 \oplus a = a = a \oplus 0 \Rightarrow 0 \circ h(a) = h(a) = h(a) \circ 0$ for $h(0) = 0$), however, $h(a) = b_0 \in \overline{\mathfrak{B}}_\sigma$, i.e. $pr_{\sigma'} h(a) = 0$ for $\sigma' \neq \sigma$, which is also a contradiction.

The uniqueness follows directly from the lemma F. g.e.d.

For weak isomorphisms the condition (6) can be replaced by the weaker condition (7) and the condition (2) of the finiteness of index sets can be omitted.

Theorem 7. *Let the assumptions* (1), (3), (4), (5), (7) *be true and h be a weak isomorphism. Then for each* $\sigma \in S$ *there exists just one* $\tau_\sigma \in T$ *such that* $\overline{\mathfrak{B}}_\sigma \subseteq h(\overline{\mathfrak{A}}_{\tau_\sigma})$.

Proof. Let the assumption of the theorem be true and the assertion not. Consider the same cases as in the proof of the theorem 6 and use the notation introduced there. In the case (a) we obtain contradiction by the same way as in the proof of the theorem 6 because we need the assumptions (1), (3), (4), (5) for it only. Let us consider the case (b). There exists $b_0 \in \overline{\mathfrak{B}}_\sigma$, $b_0 \neq 0$ such that for each $a \in h^{-1}(b_0)$ is *card* $T_a \geqq 2$.

Let $\tau \in T_a$, then $a = c \oplus \bar{a}_\tau$, where $c \neq 0$, $\bar{a}_\tau \neq 0$, $\bar{a}_\tau \in \mathfrak{A}_\tau$, $pr_\tau \bar{a}_\tau = pr_\tau a$, $pr_\tau c = 0$ and $pr_{\tau'} c = pr_{\tau'} a$ for $\tau' \neq \tau$. If f is regular on each $\mathfrak{A}_\tau$, then

$$f(\bar{a}_\tau c \dots c) = 0 .$$

Let $\oplus R_h \circ$, then $b = h(a) = h(c) \circ h(\bar{a}_\tau)$. Because $h(0) = 0$ and h is a one-to-one mapping, it is $h(c) \neq 0$, $h(\bar{a}_\tau) \neq 0$. If $h(c) \in \overline{\mathfrak{B}}_\sigma$, $h(\bar{a}_\tau) \in \overline{\mathfrak{B}}_\sigma$, $f R_h g$ and g fulfils (P) on each $\overline{\mathfrak{B}}_\sigma$ (such g exists by (7)), then

$$g\big(h(\bar{a}_\tau)\, h(c) \dots h(c)\big) \neq 0$$

but $g\big(h(\bar{a}_\tau)\, h(c) \dots h(c)\big) = h\big(f(\bar{a}_\tau c \dots c)\big) = h(0) = 0$ which is a contradiction. If $h(\bar{a}_\tau)$ (or $h(c)$) does not belong to $\overline{\mathfrak{B}}_\sigma$, then there exists $\sigma' \neq \sigma$ such that $pr_{\sigma'} h(\bar{a}_\tau) \neq 0$ (or $pr_{\sigma'} h(c) \neq 0$); then from $g\big(h(\bar{a}_\tau)\, h(c) \dots h(c)\big) = h\big(f(\bar{a}_\tau c \dots c)\big) = h(0) = 0$ it follows $pr_{\sigma'} h(c) = 0$ (or $pr_{\sigma'} h(\bar{a}_\tau) = 0$), then $pr_{\sigma'} b_0 = pr_{\sigma'} h(a) = pr_{\sigma'}\big(h(c) \circ h(\bar{a}_\tau)\big) = [pr_{\sigma'} h(c)] \circ [pr_{\sigma'} h(\bar{a}_\tau)] = pr_{\sigma'} h(\bar{a}_\tau) \neq 0$ (or $pr_{\sigma'} b_0 = [pr_{\sigma'} h(c)] \circ [pr_{\sigma'} h(\bar{a}_\tau)] = pr_{\sigma'} h(c) \neq 0$, respectively) which is a contradiction with $b_0 \in \overline{\mathfrak{B}}_\sigma$ for $\sigma \neq \sigma'$.

Corollary 8. *Let the assumption* (1), (3), (4), (5), (6) *be true and h be a weak isomorphism. Then card S = card T and each $\mathfrak{A}_\tau$ is weak-isomorphic with some $\mathfrak{B}_\sigma$.*

Proof. It follows directly from the theorem 7 because h^{-1} is also a weak isomorphism such that $h^{-1}(0) = 0$ and for $f^* \in \mathrm{B}$ which is regular on each $\mathfrak{B}_\sigma$ there exists $f \in \mathrm{F}$ such that $f R_h f^*$ (or $f^* R_{h^{-1}} f$), where f is regular on each $\mathfrak{A}_\tau$, further (6) $\Rightarrow$ (7) and the restriction of weak isomorphism onto a subalgebra is also a weak isomorphism.

Corollary 9. *Let the assumptions* (1), (3), (4), (5), (6) *be true and h be a weak isomorphism. Then there exist a bijection π of the index set S onto T and an isomorphism* i *of $\prod_{\sigma \in S} \mathfrak{B}_\sigma$ onto $\prod_{\sigma \in S} \mathfrak{B}_{\sigma(\sigma)} = \prod_{\tau \in T} \mathfrak{B}_\tau$ (which permutes the direct factors only) that* $\mathrm{i} \,.\, h = \prod_{\tau \in T} h_\tau$, *where h_τ is a weak isomorphism of $\mathfrak{A}_\tau$ onto $\mathfrak{B}_\tau = \mathfrak{B}_{\pi(\sigma)}$.*

Proof. By the corollary 8 and theorem 6, it is *card S = card T*, and we can permute S such that $\mathfrak{A}_\tau$ is weak-isomorphic with $\mathfrak{B}_\tau = \mathfrak{B}_{\pi(\sigma)}$ and $\pi(\sigma) = \tau_\sigma$. Denote by h_τ this weak isomorphism of $\mathfrak{A}_\tau$ onto $\mathfrak{B}_\tau$, then $\mathrm{i} \cdot h$ is a weak isomorphism of $\prod_{\tau \in T} \mathfrak{A}_\tau$ onto $\prod_{\tau \in T} \mathfrak{B}_\tau$, $\mathrm{i} \cdot h(\overline{\mathfrak{A}}_\tau) = \overline{\mathfrak{B}}_\tau$, thus

$$pr_\tau\big(h(a)\big) = h_\tau\big(pr_\tau(a)\big) \qquad \text{for each} \qquad \tau \in T,\ a \in A$$

and, by the definition 2, it is $\mathrm{i} \,.\, h = \prod_{\tau \in T} h_\tau$. q.e.d.

For the weak homomorphisms the situation is rather complicated.

Definition 3. Let $\mathfrak{A} = (A, \mathrm{F})$ be an algebra with the set of algebraic operations A. An operation $f \in \mathrm{A}$ is called *strong idempotent* if f is regular on $\mathfrak{A}$ and for arbitrary $a_1, \dots, a_n \in A$ there exists $i \in \{1, \dots, n\}$ such that $f(a_1, \dots, a_n) = a_i$.

It is clear that if f is strong idempotent on $\mathfrak{A}$, then each element $a \in A$ is idempotent with respect to f (but not vice versa in the general case). Further, each strong idempotent operation on $\mathfrak{A}$ fulfils (P) on $\mathfrak{A}$.

Theorem 10. *Let the assumptions* (1), (2), (3), (4) *be true and, furthermore, there exist at least one n-ary algebraic operation which is strong idempotent on each* $\mathfrak{A}_\tau$. *Let there exist at least one algebraic operation g such that* fR_hg *and g is strong idempotent on each* $\mathfrak{B}_\sigma$. *If card* S = *card* T, *then there exists a bijection* π *of S onto T such that* $\pi(\sigma) = \tau_\sigma$, *where* τ_σ *corresponds to* σ *by the theorem 6, and* $\mathrm{i} \cdot h = \prod_{\tau \in T} h_\tau$, *where* i *is the isomorphism of* $\prod_{\sigma \in S} \mathfrak{B}_\sigma$ *onto* $\prod_{\sigma \in S} \mathfrak{B}_{\pi(\sigma)} = \prod_{\tau \in T} \mathfrak{B}_\tau$ *which permutes the direct factors only and* h_τ *is a weak homomorphism of* $\mathfrak{A}_\tau$ *onto* $\mathfrak{B}_\tau$ *for each* $\tau \in T$.

Proof. By the theorem 6, there exists just one $\tau_\sigma \in T$ for each $\sigma \in S$ such that $\overline{\mathfrak{B}}_\sigma \subseteq h(\overline{\mathfrak{A}}_{\tau_\sigma})$.

(a) Let for $\sigma_1, \sigma_2 \in S$, $\sigma_1 \neq \sigma_2$ is $\tau_{\sigma_1} \neq \tau_{\sigma_2}$. If there exists $\sigma \in S$ such that $\overline{\mathfrak{B}}_\sigma \subset h(\overline{\mathfrak{A}}_{\tau_\sigma})$, $\overline{\mathfrak{B}}_\sigma \neq h(\overline{\mathfrak{A}}_{\tau_\sigma})$, then there exists an element $c \in h(\overline{\mathfrak{A}}_{\tau_\sigma}) - \overline{\mathfrak{B}}_\sigma$, $c \neq 0$ and $\sigma_1 \neq \sigma$ such that $pr_{\sigma_1}c = c_1 \neq 0$. Denote $d \in \overline{\mathfrak{A}}_{\tau_\sigma}$ such element that $h(d) = c$ and $d_1 \in \overline{\mathfrak{A}}_{\tau_{\sigma 1}}$ such that $h(d_1) = \bar{c}_1$, where $c_1 = pr_{\sigma_1}\bar{c}_1$ and $pr_\sigma \bar{c}_1 = 0$ for $\sigma \neq \sigma_1$. By the theorem 6, such elements exist. Let f, g be strong idempotent on each $\mathfrak{A}_\tau$, $\mathfrak{B}_\sigma$, respectively, and fR_hg. Then $f(dd_1 \dots d_1) = 0$ because for $\sigma_1 \neq \sigma$ is $\tau_{\sigma_1} \neq \tau_\sigma$ by the assumption (a) of the proof and for $\tau_{\sigma_1} \neq \tau_\sigma$ is $\overline{\mathfrak{A}}_{\tau_{\sigma 1}} \cap \overline{\mathfrak{A}}_{\tau_\sigma} = \{0\}$. However, $pr_{\sigma_1}[h(f(dd_1 \dots d_1))] = pr_{\sigma_1}[g(c\bar{c}_1 \dots \bar{c}_1)] = g[(pr_{\sigma_1}c)(pr_{\sigma_1}\bar{c}_1) \dots (pr_{\sigma_1}\bar{c}_1)] = g(c_1c_1 \dots c_1) = c_1 \neq 0$, which is a contradiction with $h(0) = 0$. Accordingly, we obtain $\overline{\mathfrak{B}}_\sigma = h(\overline{\mathfrak{A}}_{\tau_\sigma})$ for each $\tau \in T$.

(b) Let there exist $\sigma_1, \sigma_2 \in S$, $\sigma_1 \neq \sigma_2$ such that $\tau_{\sigma_1} = \tau_{\sigma_2}$. Denote $\tau_{\sigma_1} = \tau_{\sigma_2} = \tau_0$. Then $\overline{\mathfrak{B}}_{\sigma_1} \subset h(\overline{\mathfrak{A}}_{\tau_0})$, $\overline{\mathfrak{B}}_{\sigma_2} \subset h(\overline{\mathfrak{A}}_{\tau_0})$, i.e. there exist $b_1 \in \overline{\mathfrak{B}}_{\sigma_1}$, $b_2 \in \overline{\mathfrak{A}}_{\sigma_2}$, $b_1 \neq 0 \neq b_2$, a_1, $a_2 \in \overline{\mathfrak{A}}_{\tau_0}$, $h(a_1) = b_1$, $h(a_2) = b_2$. Then $f(a_1a_2 \dots a_2) = a_i$, but $a_i \neq 0$ for $i = 1$ or 2. However, $0 = g(b_1b_2 \dots b_2)$ because g is regular and $g(b_1b_2 \dots b_2) = h(f(a_1a_2 \dots a_2)) = h(a_i) = b_i \neq 0$ which is a contradiction.

Summary: there exists just one $\tau_\sigma \in T$ for each $\sigma \in S$ such that $\overline{\mathfrak{B}}_\sigma = h(\overline{\mathfrak{A}}_{\tau_\sigma})$, *card* S = *card* T, i.e. the mapping $\pi : \sigma - \tau_\sigma$ is a bijection. Denote $\pi(\sigma) = \tau_\sigma$. Then $h(\overline{\mathfrak{A}}_\tau) = \overline{\mathfrak{B}}_\tau$. If j_τ is the insertion of $\mathfrak{A}_\tau$ onto $\overline{\mathfrak{A}}_\tau$ and pr_τ is the projection of $\overline{\mathfrak{B}}_\tau$ onto $\mathfrak{B}_\tau$, then, evidently, $\mathrm{i} \cdot h = \prod_{\tau \in T} h_\tau$, where $h_\tau = pr_\tau \cdot h \cdot j_\tau$ and i is the above mentioned isomorphism.

Theorem 11. *Let* $\mathfrak{A}_\tau$, $\mathfrak{B}_\tau$ *be chains with the least and the greatest elements and* $\mathfrak{A} = \prod_{\tau \in T} \mathfrak{A}_\tau$, $\mathfrak{B} = \prod_{\tau \in T} \mathfrak{B}_\tau$ *for finite index set T. If h is a weak homomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, *then there exists a permutation* π *of T and an isomorphism* i *of* $\prod_{\tau \in T} \mathfrak{B}_\tau$ *onto* $\prod_{\tau \in T} \mathfrak{B}_{\pi(\tau)}$ *permuting direct factors only such that* $\mathrm{i} \cdot h = \prod_{\tau \in T} h_\tau$, *where* h_τ *is a weak homomorphism of* $\mathfrak{A}_\tau$ *onto* $\mathfrak{B}_{\pi(\tau)}$ *for each* $\tau \in T$.

Proof. By the corollary 5, h is either a homomorphism or a dual one. Let 0 is the least element on $\mathfrak{A}$, $\oplus = \vee$, $f = \wedge$. Then f is a strong idempotent on each $\mathfrak{A}_\tau$. If h is a homomorphism "onto", then $h(0)$ is the least element of $\mathfrak{B}$ and the assertion follows directly from the theorem 10, because corresponding g such that $fR_h g$ is $g = \wedge$, which is strong idempotent on each $\mathfrak{B}_\tau$. If h is a dual homomorphism, then $h(0)$ is the greatest element of $\mathfrak{B}$, it is zero element in the sense of algebra from Λ, where $\oplus = \wedge$ and $g = \vee$ and the assertion follows also from the theorem 10.

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