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EXISTENCE THEOREMS FOR A PARTIAL DIFFERENTIAL EQUATION OF THE FOURTH ORDER

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In the paper [1] there are derived very general existence theorems and uniqueness criteria for the characteristic initial problem of the partial differential equation of the second order.

For the partial differential equation of the fourth order $u_{xxtt} = f(t, x, u, u_{xx}, u_{tt})$ the existence and uniqueness theorems are derived only on the supposition that the function f fulfils the Lipschitz's condition [3]. In what follows there is mentioned the existence theorem for an other class of functions f, which is similarly general as that in [1].

Let

$$D = \{t, x: 0 \leqq t \leqq T, \ 0 \leqq x \leqq L, \ 0 < T < \infty, \ 0 < \mathrm{L} < \infty\},$$
$$I_1 = \{x: 0 \leqq x \leqq L, 0 < L < \infty\}, \quad I_2 = \{t: 0 \leqq t \leqq T, 0 < T < \infty\}$$

and let $f(t, x, u, v, w)$ be a continuous function in $D \times R^3$ and let $\varphi_0(x), \varphi_1(x) \in C^2(I_1)$, $\psi_0(t), \psi_1(t) \in C^2(I_2)$.
In the domain D consider the partial differential equation

$$u_{xxtt} = f(t, x, u, u_{xx}, u_{tt}) \tag{1}$$

and the characteristic initial conditions

$$\begin{aligned} u(0, x) &= \varphi_0(x), & u_t(0, x) &= \varphi_1(x), & x &\in I_1 \\ u(t, 0) &= \psi_0(t), & u_x(t, 0) &= \psi_1(t), & t &\in I_2 \end{aligned} \tag{2}$$

where $\varphi_0(0) = \psi_0(0)$, $\varphi_0'(0) = \psi_1(0)$, $\varphi_1(0) = \dot{\psi_0}(0)$, $\varphi_1'(0) = \dot{\psi_1}(0)$.

Definition 1. We say that the function $u(t, x)$ is of the class $C^*(D)$ if $u(t, x)$, $u_{xx}(t, x)$, $u_{tt}(t, x)$, $u_{xxtt}(t, x)$ are continuous in D.

Definition 2. The function $u(t, x) \in C^*(D)$ fulfilling (1), (2) is said to be the solution of the problem (1), (2).
The function $u(t, x) \in C^*(D)$ fulfilling in D the inequalities

$$u_{xxtt}(t, x) > f(t, x, u, u_{xx}, u_{tt}), \tag{3}$$

$$(4)\qquad \begin{aligned} u(0, x) &\geqq \varphi_0(x), & u_t(0, x) &\geqq \varphi_1(x), \\ u(t, 0) &\geqq \psi_0(t), & u_x(t, 0) &\geqq \psi_1(t) \end{aligned}$$

is called the upper function of the problem (1), (2). The function $u(t, x) \in C^*(D)$ fulfilling in D the inequalities

$$(5)\qquad u_{xxtt}(t, x) < f(t, x, u, u_{xx}, u_{tt}),$$

$$(6)\qquad \begin{aligned} u(0, x) &\leqq \varphi_0(x), & u_t(0, x) &\leqq \varphi_1(x), \\ u(t, 0) &\leqq \psi_0(t), & u_x(t, 0) &\leqq \psi_1(t) \end{aligned}$$

is called the lower function of the problem (1), (2).

Definition 3. The function $g(x, z)$ is said to be of the class $G[0, a]$ if the following holds:

1. $g(x, z)$ is continuous if $0 \leqq x \leqq a$, $z \geqq 0$;
2. $g(x, z) \geqq 0$, $g(x, 0) = 0$;
3. $g(x, z) \leqq g(x, \bar{z})$ for every $z \leqq \bar{z}$;
4. if for $\alpha > 0$, $\gamma > 0$ the function $z(x) = z(x, \alpha, \gamma)$ is the solution of

$$z(x) = \gamma \frac{x^2}{2} + \int_0^x \int_0^{x_1} g[x_2, z(x_2 - \alpha)] \, dx_2 \, dx_1, \qquad 0 \leqq x \leqq a,$$

such that

$$z(x, \alpha, \gamma) = 0 \quad \text{if} \quad x \leqq 0,$$
$$z_x(x, \alpha, \gamma) = 0 \quad \text{if} \quad x \leqq 0,$$

then there holds

$$\lim z(x, \alpha, \gamma) = 0 \quad \text{as} \quad \gamma \to 0$$

uniformly in x and α.

Denoting $u_{xx}(t, x) = v(t, x)$, $u_{tt}(t, x) = w(t, x)$ for $(t, x) \in D$, the system

$$(7)\qquad \begin{aligned} u(t, x) = {} & \varphi_0(x) + \psi_0(t) - \psi_0(0) + t[\varphi_1(x) - \varphi_1(0)] + \\ & + x[\psi_1(t) - \psi_1(0)] - tx\dot{\psi}_1(0) + \\ & + \int_0^t \int_0^{t_1} \int_0^x \int_0^{x_1} f(t_2, x_2, u, v, w) \, dx_2 \, dx_1 \, dt_2 \, dt_1, \end{aligned}$$

$$(8)\qquad v(t, x) = \varphi_0''(x) + t\varphi_1''(x) + \int_0^t \int_0^{t_1} f(t_2, x, u, v, w) \, dt_2 \, dt_1,$$

$$(9)\qquad w(t, x) = \ddot{\psi}_0(t) + x\ddot{\psi}_1(t) + \int_0^x \int_0^{x_1} f(t, x_2, u, v, w) \, dx_2 \, dx_1$$

is equivalent with the problem (1), (2) in the domain D.

Theorem 1. *Let* $\varphi_0(x)$, $\varphi_1(x) \in C^2(I_1)$, $\psi_0(t)$, $\psi_1(t) \in C^2(I_2)$ *and* $g_1(t, z) \in G[0, T]$, $g_2(x, z) \in G[0, L]$.

Let for arbitrary $0 < K < \infty$ *the function* $f(t, x, u, v, w)$ *be uniformly continuous for* $(t, x) \in D$ *and* $-K < u, v, w < K$ *and let for* $(t, x) \in D$ *and* $-\infty < u, v, w, \bar{v}, \bar{w} < \infty$ *hold*

$$(10 \qquad |f(t, x, u, v, w) - f(t, x, u, \bar{v}, \bar{w})| \leqq g_1(t, |v - \bar{v}|) + g_2(x, |w - \bar{w}|).$$

Then there exists at least one solution $u(t, x) \in C^*(D)$ *of the characteristic initial problem* [1], [2].

Proof. Denote

$$u_0(t, x) = \varphi_0(x) + \psi_0(t) - \psi_0(0) + t[\varphi_1(x) - \varphi_1(0)] + \\ + x[\psi_1(t) - \psi_1(0)] - tx\psi_1(0),$$

$$(11) \qquad \begin{aligned} v_0(t, x) &= \varphi_0(x) + t\varphi_1(x), \\ w_0(t, x) &= \psi_0(t) + x\psi_1(t). \end{aligned}$$

Let $\alpha > 0$ be arbitrary; for $(t, x) \in D$ we define the functions $u_\alpha(t, x)$, $v_\alpha(t, x)$, $w_\alpha(t, x)$ as follows:

$$u_\alpha(t, x) = u_0(t, x) + \int_0^t \int_0^{t_1} \int_0^x \int_0^{x_1} f[t_2, x_2, u_\alpha(t_2 - \alpha, x_2), v_\alpha(t_2 - \alpha, x_2), w_\alpha(t_2, x_2 - \alpha)] \times \\ \times \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t_2 \, \mathrm{d}t_1,$$

$$(12) \qquad v_\alpha(t, x) = v_0(t, x) + \\ + \int_0^t \int_0^{t_1} f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x), w_\alpha(t_2, x - \alpha)] \, \mathrm{d}t_2 \, \mathrm{d}t_1,$$

$$w_\alpha(t, x) = w_0(t, x) + \\ + \int_0^x \int_0^{x_1} f[t, x_2, u_\alpha(t - \alpha, x_2), v_\alpha(t - \alpha, x_2), w_\alpha(t, x_2 - \alpha)] \, \mathrm{d}x_2 \, \mathrm{d}x_1,$$

where

$$(13) \qquad \begin{aligned} &\text{for } t \leqq 0,\ 0 \leqq x \leqq L \text{ there is } u_\alpha(t, x) = \varphi_0(x), \\ &\text{for } t \leqq 0,\ 0 \leqq x \leqq L \text{ there is } v_\alpha(t, x) = \varphi_0''(x) + t\varphi_1''(x), \\ \text{and} \quad &\text{for } 0 \leqq t \leqq T,\ x \leqq 0 \text{ there is } w_\alpha(t, x) = \ddot{\psi}_0(t) + x\ddot{\psi}_1(t). \end{aligned}$$

In this way the functions $u_\alpha(t, x)$, $v_\alpha(t, x)$, $w_\alpha(t, x)$ are given in all domain D, since by (11), (12), (13) they are given for $0 \leqq x \leqq \alpha$, $0 \leqq t \leqq \alpha$, then for $0 \leqq x \leqq \alpha$, $\alpha \leqq t \leqq 2\alpha$, … and so on up to cover all the domain D. These functions $u_\alpha(t, x)$, $v_\alpha(t, x)$, $w_\alpha(t, x)$ are continuous in D. For a sequence $\{\alpha_n\}$, where $\alpha_n > 0$, $\lim \alpha_n = 0$ as $n \to \infty$, we can define in such a way sequences of functions $\{u_{\alpha_n}(t, x)\}$, $\{v_{\alpha_n}(t, x)\}$,

$\{w_{\alpha_n}(t, x)\}$. If we prove that the functions in each of these sequences are equicontinuous, then from these sequences we can choice by the Arzela–Ascoli theorem uniformly convergent sequences, the limits of which, as $\alpha_n \to 0$, satisfy the system (7), (8), (9).

From assumptions on the functions $\varphi_0, \varphi_1, \psi_0, \psi_1$ and from the uniform continuity of the function f we obtain:
for $x \in I_1$ and $t \in I_2$ there holds

$$(14) \qquad |\varphi_1(x)| \leqq M_1, \quad |\psi_1(t)| \leqq M_2, \quad |\varphi_1''(x)| \leqq M_3, \quad |\ddot{\psi}_1(t)| \leqq M_4$$

and for $(t, x) \in D$, $-\infty < u, v, w < \infty$ there is

$$(15) \qquad |f(t, x, u, v, w)| \leqq M.$$

Let $\omega(\delta)$ be the continuity modulus, for which $\omega(0) = 0$, $\omega(\delta_1) \leqq \omega(\delta_2)$ if $\delta_1 \leqq \delta_2$, such that for arbitrary $x, x + h \in I_1$ and $t, t + k \in I_2$ there holds:

$$(16) \qquad \begin{aligned} &|\varphi_0(x + h) - \varphi_0(x)| \leqq \omega(h), &&|\varphi_0''(x + h) - \varphi_0''(x)| \leqq \omega(h), \\ &|\varphi_1(x + h) - \varphi_0(x)| \leqq \omega(h), &&|\varphi_1''(x + h) - \varphi_1''(x)| \leqq \omega(h), \\ &|\psi_0(t + k) - \psi_0(t)| \leqq \omega(k), &&|\ddot{\psi}_0(t + k) - \ddot{\psi}_0(t)| \leqq \omega(k), \\ &|\psi_1(t + k) - \psi_1(t)| \leqq \omega(k), &&|\ddot{\psi}_1(t + k) - \ddot{\psi}_1(t)| \leqq \omega(k), \end{aligned}$$

if $0 < K < \infty$ is such a constant that

$$(17) \qquad |u_\alpha(t, x)| < K, \qquad |v_\alpha(t, x)| < K, \qquad |w_\alpha(t, x)| < K$$

is valid for $(t, x) \in D$ and arbitrary $\alpha > 0$, then from uniform continuity of f we have

$$(18) \qquad |f(t, x, u, v, w) - f(\bar{t}, \bar{x}, \bar{u}, \bar{v}, \bar{w})| \leqq$$
$$\leqq \omega(|t - \bar{t}|) + \omega(|x - \bar{x}|) + \omega(|u - \bar{u}|) + \omega(|v - \bar{v}|) + \omega(|w - \bar{w}|)$$

for $(t, x) \in D$, $(\bar{t}, \bar{x}) \in D$ and $-K < u, v, w, \bar{u}, \bar{v}, \bar{w} < K$.
Hence from (12), (14), (15), (16) we obtain

$$(19) \qquad |u_\alpha(t + k, x + h) - u_\alpha(t, x)| \leqq$$
$$\leqq (1 + T)\,\omega(h) + (1 + L)\,\omega(k) + |k|\,[M_1 - \varphi_1(0) + L\,|\dot{\psi}_1(0)|] +$$
$$+ |h|\,[M_2 - \psi_1(0) + T\,|\dot{\psi}_1(0)|] + MLT\left(\frac{1}{2}|h|\,T + \frac{1}{2}|k|\,L + |h|\,|k|\right)$$

for $x, x + h \in I_1$, $t, t + k \in I_2$,

$$(20) \qquad |v_\alpha(t + k, x) - v_\alpha(t, x)| \leqq |k|\,(M_3 + MT)$$

for $x \in I_1$, $t, t + k \in I_2$
and

$$(21) \qquad |w_\alpha(t, x + h) - w_\alpha(t, x)| \leqq |h|\,(M_4 + ML)$$

for $x, x + h \in I_1$, $t \in I_2$.

From (19), (20), (21) it follows that the functions $u_{\alpha_n}(t, x)$ are equicontinuous in D for each α_n, the functions $v_{\alpha_n}(t, x)$ are equicontinuous according to the variable t and the functions $w_{\alpha_n}(t, x)$ are equicontinuous according to the variable x.

Since there is

$$| v_\alpha(t + k, x + h) - v_\alpha(t, x) | \leqq | v_\alpha(t + k, x + h) - v_\alpha(t, x + h) | +$$
$$+ | \varphi_0''(x + h) - \varphi_0''(x) | + | t | | \varphi_1''(x + h) - \varphi_1''(x) | +$$
$$+ v_\alpha(t, x + h) - v_\alpha(t, x) - \{\varphi_0''(x + h) - \varphi_0''(x) + t[\varphi_1''(x + h) - \varphi_1''(x)]\} |$$

and for $x + h \in I_1$ from (20), (16) we have

$$| v_\alpha(t + k, x + h) - v_\alpha(t, x + h) | \leqq | k | (M_3 + MT),$$
$$| \varphi_0''(x + h) - \varphi_0''(x) | + | t | | \varphi_1''(x + h) - \varphi_1''(x) | \leqq (1 + T) \omega(h),$$

it suffices to prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that there is

$$(22) \qquad | v_\alpha(t, x + h) - v_\alpha(t, x) -$$
$$- \{\varphi_0''(x + h) - \varphi_0''(x) + t[\varphi_1''(x + h) - \varphi_1''(x)]\} | < \varepsilon, \qquad \text{if } | h | < \delta,$$

for all $\alpha > 0$, $0 \leqq x \leqq L$, $0 \leqq x + h \leqq L$, $0 \leqq t \leqq T$.

Similarly it must hold for the function $w_\alpha(t, x)$ that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that there is

$$(23) \qquad | w_\alpha(t + k, x) - w_\alpha(t, x) -$$
$$- \{\ddot\psi_0(t + k) - \ddot\psi_0(t) + x[\ddot\psi_1(t + k) - \ddot\psi_1(t)]\} | < \varepsilon, \qquad \text{if } | k | < \delta,$$

for all $\alpha > 0$, $0 \leqq x \leqq L$, $0 \leqq t \leqq T$, $0 \leqq t + k \leqq T$.

We prove only the (22) because the proof of (23) is the same one. Denoting

$$(24) \qquad V_\alpha(t, x, h) =$$
$$= | v_\alpha(t, x + h) - v_\alpha(t, x) - \{\varphi_0''(x + h) - \varphi_0''(x) + t[\varphi_1''(x + h) - \varphi_1''(x)]\} |$$

by (13) there is $V_\alpha(t, x, h) = 0$, if $t \leqq 0$, $0 \leqq x \leqq L$, $0 \leqq x + h \leqq L$ and by (12) there is

$$(25) \qquad V_\alpha(t, x, h) \leqq \int_0^t \int_0^{t_1} | f[t_2, x + h, u_\alpha(t_2 - \alpha, x + h), v_\alpha(t_2 - \alpha, x + h),$$
$$w_\alpha(t_2, x + h - \alpha)] - f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x), w_\alpha(t_2, x - \alpha)] | \, dt_2 dt_1$$

for $0 \leqq t \leqq T$, $0 \leqq x \leqq L$, $0 \leqq x + h \leqq L$.

It is:

$$|f[t_2, x + h, u_\alpha(t_2 - \alpha, x + h), v_\alpha(t_2 - \alpha, x + h), w_\alpha(t_2, x + h - \alpha)] -$$
$$- f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x), w_\alpha(t_2, x - \alpha)] | \leqq$$
$$\leqq |f[t_2, x + h, u_\alpha(t_2 - \alpha, x + h), v_\alpha(t_2 - \alpha, x + h), w_\alpha(t_2, x + h - \alpha)] -$$

$$-f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x + h), w_\alpha(t_2, x - \alpha)] | +$$
$$+ |f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x + h), w_\alpha(t_2, x - \alpha)] -$$
$$-f[t_2, x, u_\alpha(t_2 - \alpha, x), v_\alpha(t_2 - \alpha, x), w_\alpha(t_2, x - \alpha)] |.$$

For the first term on the right there holds

$$\leqq \omega(h) + \omega[| u_\alpha(t_2 - \alpha, x + h) - u_\alpha(t_2 - \alpha, x) |] + \tag{26}$$
$$+ \omega[| w_\alpha(t_2, x + h - \alpha) - w_\alpha(t_2, x - \alpha) |] \leqq$$
$$\leqq \omega(h) + \omega[(1 + T)\,\omega(h) + | h | (M_2 - \psi_1(0) + T | \psi_1'(0) |) + \frac{1}{2} MLT^2 | h |] +$$
$$+ \omega[| h | (M_4 + ML)].$$

With regard to (10) and (16) the second term on the right can be writen in the form

$$\leqq |f[t_2, x, u_\alpha, v_\alpha(t_2 - \alpha, x + h), w_\alpha] - f[t_2, x, u_\alpha, v_\alpha(t_2 - \alpha, x + h) - \varphi_0''(x + h) +$$
$$+ \varphi_0''(x) - (t_2 - \alpha)(\varphi_1''(x + h) - \varphi_1''(x)), w_\alpha] | +$$
$$+ |f[t_2, x, u_\alpha, v_\alpha(t_2 - \alpha, x + h) - \varphi_0''(x + h) + \varphi_0''(x) - (t_2 - \alpha)(\varphi_1''(x + h) - \varphi_1''(x)), w_\alpha] -$$
$$- f[t_2, x, u_\alpha, v_\alpha(t_2 - \alpha, x), w_\alpha] |$$
$$\leqq \omega[| \varphi_0''(x + h) - \varphi_0''(x) + (t_2 - \alpha)(\varphi_1''(x + h) - \varphi_1''(x))] +$$
$$+ g_1[t_2, | v_\alpha(t_2 - \alpha, x + h) - v_\alpha(t_2 - \alpha, x) -$$
$$- \{\varphi_0''(x + h) - \varphi_0''(x) + (t_2 - \alpha)[\varphi_1''(x + h) - \varphi_1''(x)]\} |]$$
$$\leqq \omega[(1 + T)\,\omega(h)] + g_1[t_2, V_\alpha(t_2 - \alpha, x, h)].$$

Hence

$$V_\alpha(t, x, h) \leqq \int_0^t \int_0^{t_1} \{\omega(h) + \omega[(1 + T)\,\omega(h) + | h | (M_2 - \psi_1(0) + T | \psi_1'(0) |) +$$
$$+ \frac{1}{2} MLT^2 | h |] + \omega[| \quad | (M_4 + ML)] + \omega[(1 + T)\,\omega(h) + g_1[t_2, V_\alpha(t_2 - \alpha, x, h)]\} \times$$
$$\times \mathrm{d}t_2\, \mathrm{d}t_1 .$$

Denoting

$$\omega_1(h) = \omega(h) + \omega[(1 + T)\,\omega(h) + | h | (M_2 - \psi_1(0) + T | \psi_1'(0) |) + \frac{1}{2} MLT^2 | h |] +$$
$$+ \omega[| h | (M_4 + ML)] + \omega[(1 + T)\,\omega(h)]$$

there is

$$V_\alpha(t, x, h) \leqq \frac{1}{2} t^2 \omega_1(h) + \int_0^t \int_0^{t_1} g_1[t_2, V_\alpha(t_2 - \alpha, x, h)]\, \mathrm{d}t_2\, \mathrm{d}t_1 \tag{27}$$

for $0 \leqq x \leqq L$, $0 \leqq x + h \leqq L$, $0 \leqq t \leqq T$ and every $\alpha > 0$.

For $\alpha > 0$, $\delta > 0$ let us define the function

(28) $v(t) = v(t, \alpha, \delta) = \sup \{V_\alpha(t, x, h) : 0 \leqq x \leqq L,\ 0 \leqq x + h \leqq L, |h| < \delta\}$.

It is $v(t) = 0$ if $t \leqq 0$.
From assumptions about the function $g_1(t, z)$ and from (27), (28) it follows, that there is

$$V_\alpha(t, x, h) \leqq \frac{1}{2} t^2 \omega_1(\delta) + \int_0^t \int_0^{t_1} g_1[t_2, v(t_2 - \alpha, \alpha, \delta)] \, dt_2 \, dt_1$$

hence

$$v(t) \leqq \frac{1}{2} t^2 \omega_1(\delta) + \int_0^t \int_0^{t_1} g_1[t_2, v(t_2 - \alpha)] \, dt_2 \, dt_1 .$$

Let $z(t, \alpha, \gamma)$ be a solution of the equation

$$z(t) = \frac{1}{2} \gamma t^2 + \int_0^t \int_0^{t_1} g_1[t_2, z(t_2 - \alpha)] \, dt_2 \, dt_1 ,$$

for which $z(t, \alpha, \gamma) = z_t(t, \alpha, \gamma) = 0$ if $t \leqq 0$.
There is $z(t) = v(t) = 0$ if $t \leqq 0$.
We can easy prove that for $\omega_1(\delta) < \gamma$ and $0 \leqq t \leqq T$ there is $v(t, \alpha, \delta) < z(t, \alpha, \gamma)$.

Since $z(t, \alpha, \gamma) \to 0$ uniformly as $\gamma \to 0$, there is also $\lim v(t, \alpha, \delta) = 0$ as $\delta \to 0$ and hence $V_\alpha(t, x, h) \to 0$ as $\delta \to 0$ uniformly in α, t, x. Therefore the functions $v_\alpha(t, x)$ are equicontinuous in D. Analogously we prove this for the functions $w_\alpha(t, x)$.

Note 1. If there is $g_1(t, z) = g_2(x, z) = Lz$, where L is a constant, then we obtain the classic Lipschitz's condition for the existence of solution of the characteristic initial problem (1), (2).

Note 2. In the case the function f is Lipschitz continuous in D also the Picard's sequence of functions converge to the solution of (1), (2), if we take $u_0(t, x)$ as the initial function [3]. This result may be generalized.

In what follows we denote $\quad D_{ik} = \dfrac{\partial^{i+k}}{\partial x^i \, \partial t^k}$.

Theorem 2. *Consider the problem* (1), (2). *For* $(t, x) \in D$ *and* $-\infty < u, v, w < \infty$ *let it hold*:

1.

$$|f(t, x, u, v, w)| \leqq K < \infty$$

(29) $|f(t, x, u, v, w) - f(t, x, \bar{u}, \bar{v}, \bar{w})| \leqq L_0(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|)$

2.

(30) $f(t, x, u, v, w) \leqq f(t, x, \bar{u}, \bar{v}, \bar{w})$ if $u \leqq \bar{u},\ v \leqq \bar{v},\ w \leqq \bar{w}$.

Let $h_0(t, x)$ be an arbitrary upper function of the problem (1), (2), *for which* (2) *is valid. Then the sequence of functions $h_n(t, x)$ defined by*

(31)
$$h_n(t, x) = u_0(t, x) + \int_0^t\int_0^{t_1}\int_0^x\int_0^{x_1} f(t_2, x_2, h_{n-1}, D_{20}h_{n-1}, D_{02}h_{n-1})\, dx_2\, dx_1\, dt_2\, dt_1,$$

where

(32) $u_0(t, x) = \varphi_0(x) + \psi_0(t) - \psi_0(0) + t[\varphi_1(x) - \varphi_1(0)] + x[\psi_1(t) - \psi_1(0)] - tx\psi_1^{\cdot}(0)$

with the initial function $h_0(t, x)$ is a sequence of upper functions which for $(t, x) \in D$ converges to the solution of (1), (2).

Proof. First we prove that the (31) is a convergent sequence of upper functions. By the assumption there is

$$D_{22}h_0(t, x) > f(t, x, h_0, D_{20}h_0, D_{02}h_0)$$
$$h_0(0, x) = \varphi_0(x), \qquad D_{01}h_0(0, x) = \varphi_1(x)$$
$$h_0(t, 0) = \psi_0(t), \qquad D_{10}h_0(t, 0) = \psi_1(t).$$

Hence

$$h_0(t, x) \geqq u_0(t, x) + \int_0^t\int_0^{t_1}\int_0^x\int_0^{x_1} f(t_2, x_2, h_0, D_{20}h_0, D_{02}h_0)\, dx_2\, dx_1\, dt_2\, dt_1 = h_1(t, x),$$

$$D_{20}h_0(t, x) \geqq D_{20}u_0(t, x) + \int_0^t\int_0^{t_1} f(t_2, x, h_0, D_{20}h_0, D_{02}h_0)\, dt_2\, dt_1 = D_{20}h_1(t, x),$$

$$D_{02}h_0(t, x) \geqq D_{02}u_0(t, x) + \int_0^x\int_0^{x_1} f(t, x_2, h_0, D_{20}h_0, D_{02}h_0)\, dx_2\, dx_1 = D_{02}h_1(t, x).$$

Let $h_n \leqq h_{n-1}$, $D_{20}h_n \leqq D_{20}h_{n-1}$, $D_{02}h_n \leqq D_{02}h_{n-1}$, then from (30) it follows immediately

$$h_{n+1} \leqq h_n, \qquad D_{20}h_{n+1} \leqq D_{20}h_n, \qquad D_{02}h_{n+1} \leqq D_{02}h_n.$$

Hence for every natural n it is:

$$h_n \leqq h_{n-1}, \quad D_{20}h_n \leqq D_{20}h_{n-1}, \quad D_{02}h_n \leqq D_{02}h_{n-1},$$
$$f(t, x, h_n, D_{20}h_n, D_{02}h_n) \leqq f(t, x, h_{n-1}, D_{20}h_{n-1}, D_{02}h_{n-1}) = D_{22}h_n,$$
$$h_n(0, x) = u_0(0, x) = \varphi_0(x), \qquad D_{01}h_n(0, x) = D_{01}u_0(0, x) = \varphi_1(x),$$
$$h_n(t, 0) = u_0(t, 0) = \psi_0(t), \qquad D_{10}h_n(t, 0) = D_{10}u_0(t, 0) = \varphi_1(t),$$

so that in D the functions $h_n(t, x) \in C^*(D)$ form a decreasing bounded sequence of upper functions. In a compact domain a decreasing bounded sequence is convergent. For the same reason there are convergent in D the sequences $\{D_{20}h_n(t, x)\}$, $\{D_{02}h_n(t, x)\}$.

Prove that the uniform convergence of these sequences follows from the Lipschitz continuity.

From the continuity of functions $f, \varphi_0, \varphi_1, \psi_0, \psi_1$ for $(t, x) \in D$ it follows

$$|f| \leqq A, \quad |u_0 - h_0| \leqq M, \quad |D_{20}u_0 - D_{20}h_0| \leqq M, \quad |D_{02}u_0 - D_{02}h_0| \leqq M,$$
$$|h_0| \leqq H, \quad |D_{20}h_0| \leqq H, \quad |D_{02}h_0| \leqq H, \qquad \text{if } (t, x) \in D$$

where $A > 0$, $H > 0$, $M > 0$ are available constants.

If we define $2K = \max\{2, (L + T)^2\}$ and if the L_0 is the Lipschitz constant of the (29) it is easy to prove that for every natural $n \geqq 1$ and for $(t, x) \in D$ there holds

$$|h_n(t, x) - h_{n-1}(t, x)| \leqq M\frac{(3L_0K)^{n-1}}{(2n-2)!}(t + x)^{2n-2} + \frac{A}{3L_0}\frac{(3L_0K)^n}{(2n)!}(t + x)^{2n},$$

$$|D_{20}h_n(t, x) - D_{20}h_{n-1}(t, x)| \leqq M\frac{(3L_0K)^{n-1}}{(2n-2)!}(t + x)^{2n-2} + \frac{A}{3L_0}\frac{(3L_0K)^n}{(2n)!}(t + x)^{2n},$$

$$|D_{02}h_n(t, x) - D_{02}h_{n-1}(t, x)| \leqq M\frac{(3L_0K)^{n-1}}{(2n-2)!}(t + x)^{2n-2} + \frac{A}{3L_0}\frac{(3L_0K)^n}{(2n)!}(t + x)^{2n}.$$

Hence it follows that sequences $\{h_n\}$, $\{D_{20}h_n\}$, $\{D_{02}h_n\}$ are uniformly convergent in D. Since the functions h_n, $D_{20}h_n$, $D_{02}h_n$ are continuous in D, their limits are also continuous in D so that the limit of the sequence $\{h_n\}$ is the solution of (1), (2).

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