Bohumil Šmarda

Connectivity in tl-groups

Archivum Mathematicum, Vol. 12 (1976), No. 1, 1--7

Persistent URL: http://dml.cz/dmlcz/106920

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XII: 1—8, 1976

CONNECTIVITY IN TL-GROUPS

BOHUMIL ŠMARDA, Brno

(Received May 31, 1974)

In this paper the algebraic structure of topological l-groups in relation to their topological connectivity is investigated. By a topological l-group (tl-group) we mean a lattice-ordered group (l-group) G with a topology being a T_0-space, a topological group and a topological lattice at the same time. With regard to the homogeneity of topological group, its topology $\tau(\Sigma)$ is defined by a neighbourhood basis Σ of zero in G. This tl-group is denoted by (G, Σ). The group operation has the additivity notation and the lattice operations are $\wedge$ and $\vee$. By [6] 1.5 a T_0-space of a topological group is a Kuratowski space and thus a complete regular space as well.

The paper is divided into two parts. In the first part there are results concerning mostly fully ordered tl-groups. Namely, a topological component of fully ordered tl-group is l-isomorphic with the fully ordered additive group of real numbers, in case of being different from zero.

The results from the first part are applied to the case of non fully ordered tl-groups to which the second part of the paper is devoted.

In this paper the terminology and the notation usual in the theory of l-groups and topological groups are used—see References. The fully ordered additive group of real numbers is denoted by $\boldsymbol{R}$.

1.

Remark. 1. A clopen set is a closed and open set at the same time.

2. If the elements x, y are incomparable, we write $x \parallel y$.

1.1. *Let (G, Σ) be a tl-group. Then the next assertions are equivalent:*

(i) *For any element $g \in G$ $\{x \in G: x < g\}$, $\{x \in G: x > g\}$ are open sets.*

(ii) *For any element $g \in G$ $N_g = \{x \in G: x \parallel g\}$ is a clopen set.*

Proof. (i) $\Rightarrow$ (ii): Let $y \in \bar{N}_g \setminus N_g$. Then $y - g$ non $\parallel 0$, $y - g \in \bar{N}_0 \setminus N_0$ and in the open set $\{x \in G: |y - g| > x > -|y - g|\}$ there exists an element x such that $(y - g) - x \parallel 0$. It means $y - g \parallel x$, which is a contradiction. Further $N_g = \{x \in R: x \text{ } non \geqq g\} \cap \{x \in G: x \text{ } non \leqq g\}$ is open (see [2]). (ii) $\Rightarrow$ (i): The set $\{x \in G: x \text{ } non \geqq g\} \cap (G \setminus N_g)$ is open.

Definition. Let (G, Σ) be a tl-group, $\tau(\Sigma)$ its topology, where the sets $\{x \in G: x > g\}$, $\{x \in G: x < g\}$ are open for any $g \in G$. Then $\tau(\Sigma)$ is called *a semiinterval topology (si-topology).*

1.2. *If a tl-group (G, Σ) is fully ordered, then its topology $\tau(\Sigma)$ is a si-topology.*

Proof follows from 1.1.

1.3. *A convex subgroup $H \neq \{0\}$ in a tl-group G with a si-topology is a clopen set.*

Proof. For any $0 \neq h \in H$ $\{x \in G: h > x > -h\}$ is an open set in H.

1.4. *Any connected subset $M \neq \{0\}$ in a tl-group (G, Σ) with a si-topology is convex.*

Proof. Let $h, k \in M$, $g \in G \setminus M$, $h > g > k$. $L_g = \{x \in G: x > g\}$, $U_g = \{x \in G: x < g\}$ are open. Then the sets $A = (L_g \cup N_g) \cap M$, $B = U_g \cap M$ are open in M and $A \cup B = H$, $A \cap B = \Phi$, $A \neq \Phi \neq B$ hold, which is a contradiction.

Definition. *A component K of a tl-group* (G, Σ) is a maximal connected subset containing zero in G.

1.5. *Let (G, Σ) be a non-fully ordered tl-group with a si-topology. Then (G, Σ) is a totally disconnected topological space.*

Proof. There exists a system $\{Q_i : i \in I\}$ of convex prime l-subgroups in G with a property $\cap \{Q_i : i \in I\} = \{0\}$. All $Q_i (i \in I)$ are clopen sets (see 1.3) and thus for the component K in G it is $K \subseteq \cap \{Q_i : i \in I\} = \{0\}$.

1.6. *Let (G, Σ) be a tl-group with a si-topology and K be a component in G. Then K has the following properties*:

1. *K is a clopen convex normal subgroup in G.*
2. *For any $g \in G$, $g \neq 0$ and any $k \in K$ there exists a positive integer n such that $n|g| > |k|$.*
3. *If $K \neq \{0\}$, then K is the only subset in G fulfilling properties 1. and 2.*

Proof. The property 1. follows from the properties of a component of a topological group and 1.3, 1.4.

2. If $0 \neq g \in G$, then $U_g = \{x \in G: |g| > x > -|g|\}$ is open and thus a convex subgroup $[U_g]$ in G generated by U_g has the form $[U_g] = \{x \in G: n|g| > x > -n|g|\}$ for a suitable positive integer n. The set $[U_g]$ is clopen and thus $K \subset [U_g]$.
3. If $K \neq \{0\}$, then according to 1.5, G is fully ordered. Further, if $L \subset G$ is a set fulfilling 1. and 2., then for any $l \in L$, $k \in K$, $l \neq 0 \neq k$ there exist positive integers n, m such that $n|l| > |k|$ and $m|k| > |l|$. Finally $K = L$.

1.7. Theorem. *Let (G, Σ) be a tl-group with a non-discrete si-topology and $L \neq \{0\}$ be a subgroup in G. Then the following assertions are equivalent*:

1. *L is a connected set,*
2. *L is a component in G,*

3. $L \cong \boldsymbol{R}$ *and* $\tau(\Sigma)$ *is not a totally disconnected space,*
4. $L \cong \boldsymbol{R}$, $\tau(\Sigma_L)$ *is the interval topology and* L *is a clopen set in* $\tau(\Sigma)$, *where* $\Sigma_L = \{U \cap L : U \in \Sigma\}$,
5. $L \cong \boldsymbol{R}$.

Remark. $L \cong \boldsymbol{R}$ means that L is l-isomorphic with a fully ordered additive group $\boldsymbol{R}$ of real numbers.

Proof of Theorem 1.7. $1 \Rightarrow 3$: If $L \neq \{0\}$ is a connected subfroup in G, then G is fully ordered (see 1.5) and L is archimedean (see 1.6). Thus there exists an l-isomorphism $\varphi : L \to \boldsymbol{R}$. Now we consider the interval topology s on $\boldsymbol{R}$. $(\boldsymbol{R}, s)$ is a tl-group and φ is a continuous mapping. $\varphi(L)$ is connected, convex (see 1.4) and it means that $\varphi(L) = \boldsymbol{R}$.

$3 \Rightarrow 5$ immediately. $5 \Rightarrow 4$: With regard to [8], 2.13 the topology $\tau(\Sigma_L)$ is the interval topology. From the fact that L is a clopen set (see 1.3, 1.4), there follows the locally compactness of (G, Σ).

$4 \Rightarrow 2$: The set L is a connected set in $\tau(\Sigma)$ and therefore $L \subseteq K$, where K is a component in G. For any $l \in L$, $l \neq 0$ and any $k \in K$ there exists a positive integer n such that $n|l| > |k|$ (see 1.6). L is a convex subfroup in G (see 1.3) and $L \supseteq K$. Together $L = K$.

$2 \Rightarrow 1$ is trivial.

1.8. Corollary. *If* (G, Σ) *is a tl-group with a si-topology* $\tau(\Sigma)$ *and* $K \neq \{0\}$ *is its component, then* G *is fully ordered and* $\tau(\Sigma)$ *is locally compact.*

Proof follows from the proof of Theorem 1.7.

1.9. Example. Let P be a connected tl-group with the interval topology, Q be a fully ordered tl-group with the discrete topology. Then the topological product $G = P \times Q$ with the lexicographical order is a fully ordered tl-group with the component $K = P \times \{0\}$, $\{0\} \neq K \neq G$.

1.10. *Let* (G, Σ) *be a tl-group with a si-topology,* $K \neq \{0\}$ *be a component in* G. *Then* G *is l-isomorphic with the fully ordered additive group* $\boldsymbol{R}$ *of real numbers if and only if* G *is a connected space.*

Proof follows from 1.7.

Definition. An element $g \in G$, $g \neq 0$ is called *archimedean* if for any $h \in G$, $h \neq 0$ there exists a positive integer n such that $n|h|$ *non* $\leqq |g|$.

1.11. Lemma. *If* (G, Σ) *is a fully ordered tl-group and* A *is the set of all archimedean elements of* G, $A \neq \Phi$, *then* $A \cup \{0\}$ *is a clopen l-ideal in* G.

Proof. The case $G = A \cup \{0\}$ is trivial. If $A \cup \{0\} \neq G$, $g, g' \in A$, then there exist positive integers n, n' such that $n|h| > |g|$, $n'|h| > |g'|$ for any $h \in G$, $h \neq 0$. It is $|-g| < n|h|$, $-g \in A$ and $|g + g'| \leqq |g| + |g'| + |g| < (2n + n')|h|$, $g + g' \in A$. If $z \in R$, $|z| < |g|$, then $|z| < |g| < n|h|$ and $z \in A$.

Further for any $z \in G$, $g \in A$, $h \neq 0$ there exists a positive integer n such that $n \mid z + h - z \mid > \mid g \mid$ and thus $\mid -z + g + z \mid = -z + \mid g \mid + z < -z + n \mid z + h - z \mid + z = n(-z + \mid z + h - z \mid + z) = n \mid -z + z + h - z + z \mid = n \mid h \mid$, i.e., $-z + g + z \in A$. The rest follows from 1.2 and 1.3.

1.12. *A component K of a tl-group (G, Σ) with a non-discretc si-topology in G that is not totally disconnected is the greatest subgroup in G l-isomorphic with $\mathbf{R}$ and $K = A \cup \{0\}$, where A is a set of all archimedean elements of G.*

Proof. follows from 1.6, 1.7 and 1.11.

1.13. *Let (G, Σ) be a tl-group with a non-discrete si-topology nad A be a set of all archimedean elements of G. Then (G, Σ) is totally disconnected space if and only if $(A \cup \{0\}, \Sigma_A)$ is totally disconnected space, where $\Sigma_A = \{U \cap (A \cup \{0\}) : U \in \Sigma\}$.*

Proof. follows from 1.12.

Definition. We say that an l-group G is totally non-archimedean if for any $g \in G, g \neq 0$ there exists an element $h \in G$, $h \neq 0$ such that $\mid g \mid > n \mid h \mid$, for any positive integer n.

1.14. *If (G,Σ) is a tl-group with a si-topology and G is totally non-archimedean, then (G, Σ) is a totally disconnected space.*

Proof. follows from 1.12.

1.15. *Let (G, Σ) be a fully ordered tl-group with a non discrete topology. Then (G, Σ) is a locally compact space if and only if (G, Σ) is not a totally disconnected space.*

Proof. 1. If $K \neq \{0\}$ is a component in G, then (G, Σ) is a locally compact space (see 1.8).

2. If (G, Σ) is locally compact and totally disconnected, then according to [3], p. 139, Th. 6 there exists a clopen compact subgroup H in G which is a tl-group. From [6], 2.5 it follows that $\{0\} \neq H \subseteq \cap \{U : U \in \Sigma\} = \{0\}$, which is a contradiction.

2.

Now, we shall investigate the connectivity in non fully ordered tl-groups.

2.1. Theorem. *If (G, Σ) is a tl-group and K is its component, then $K \subset P + B$, where P is a closed convex prime l-subgroup in G, B is a convex l-subgroup in G, B non $\subseteq P$ and P or B is an l-ideal in G.*

Proof. Let π be a restriction of the lattice homomorphism $\pi : G \to G/P$, where $G/P = \{P + g : g \in G\}$ on B. From [1], L. 11 it follows that $B\pi$ is an open set in the topological space G/P and we prove that $B\pi$ is also closed in G/P: If $P + g$ *non* $\in B\pi$, then $B\pi + (P + g)$ is a neighbourhood of $P + g$ in G/P and $[B\pi + (P + g)] \cap B\pi = \Phi$. On the contrary the elements $b_1, b_2 \in B$ exist with the property $(P + b_1) + (P + g) = P + b_2$ and $b_1 + g \in P + b_2$, $b_1 + g = p + b_2$ for a suitable element $p \in P$. Further P or B is normal and thus $b_1 + g = p + b_2 = b_3 + p_1$, where

$b_3 \in B$, $p_1 \in P$ are suitable elements. From there $g = (-b_1 + b_3) + p_1 = p_2 + b_4$, where $b_4 \in B$, $p_2 \in P$ are suitable elements and $P + g = P + p_2 + b_4 \subset P + B$, $P + g \in B\pi$, which is a contradiction. From this $B\pi$ is a clopen set in G/P and $K_{G/P} \subset \subset B\pi$, where $K_{G/P}$ is a component of zero in G/P. It means that $K\pi \subseteq K_{G/P} \subseteq B\pi$ and $P + K \subset P + B$, $K \subset P + B$.

2.2. Lemma. *If (G, Σ) is a tl-group with a T_0-topology, then any polar P in G is a closed set in the topology $\tau(\Sigma)$.*

Proof. With regard to [1], Prop. 4 $\bar{g}'$ is a convex l-subgroup in G, for any $g \in G$, and we shall prove that $\overline{(g')^+} \subset g'$. It holds $\{|g|\} \wedge \overline{(g')^+}\ \overline{\{|g|\}} \wedge \overline{(g')^+} \subset \overline{\{|g|\} \wedge \wedge (g')^+} = \overline{\{0\}} = \{0\}$ (see the proof of Prop. 4, [1]). Further, for any polar P in G there is $P = Q'$ for a suitable set Q in G and $P = \cap \{g' : g \in Q\}$, $\bar{P} = P$.

Definition. Let G be a non fully ordered l-group. We say that a *system* $\{P_i : i \in I\}$ *of minimal convex prime l-subgroups in G has the property* (α) *if it is* $\cap \{P_i : i \in I\} = \{0\}$, $\cap \{P_i : i \in I \setminus \{j\}$ *non* $\subseteq P_j$ for any $j \in I$.

2.3. *If (G, Σ) is a non fully ordered tl-group with a T_0-topology and a system $\{P_i : i \in I\}$ has the property (α), then P_i is a closed set for any $i \in I$.*

Proof. We choose arbitrary elements $a_i \in \cap \{P_i : i \in I \setminus \{j\}\} \setminus P_j$ for any $j \in I$. Then $a'_j \subseteq P_j$ and for any $x \in P_j$ it is $|x| \wedge |a_j| \in P_j \cap \cap \{P_i : i \in I \setminus \{j\}\} = \{0\}$. From this $x \in a_j$ and $P_j = a'_j$ is a closed set (see 2.2).

2.4. Theorem. *Let (G, Σ) be a non fully ordered tl-group, $\{P_i : i \in I\}$ be a system of closed prime l-ideals in G, $\cap \{P_i : i \in I\} = \{0\}$. Let (G, Σ) be a connected topological space. Then G is l-isomorphic with a subdirect product of fully ordered additive group of real numbers.*

Proof. $\tau(\Sigma)$ is a T_0-topology. Then G/P_j is a connected fully ordered tl-group for any $j \in I$. From 1.7 it follows that G/P_j is l-isomorphic with $\boldsymbol{R}$. Finally, G is l-isomorphic with a subdirect product of groups of the type $\boldsymbol{R}$.

2.5. Theorem. *Let (G, Σ) be a non fully ordered tl-group, $\{P_i : i \in I\}$ be a system with the property (α). If (G, Σ) is a disconnected topological space, then (G, Σ) is a totally disconnected space.*

Proof. If K is a component in (G, Σ), then with regard to 2.1 and 2.3 it holds $K \subseteq \cap \{B_j + P_j : j \in I\}$, where $B_j = \cap \{P_i : i \in I \setminus \{j\}\}$. It means that for any $j \in I$ and any $k \in K$ there exist elements $p_j \in P_j$, $b_j \in B_j$ with the property $k = b_j + p_j$. Now, we choose a fixed element $j_0 \in I$. Then $k = b_{j0} + p_{j0} = b_1 + p_1$ for suitable elements $b_{j0} \in B_{j0}$, $p_{j0} \in P_{j0}$, $b_1 \in B_1$, $p_1 \in P_1$ and any $l \in I$. Evidently $|b_j| \wedge \wedge |p_j| = 0$, $b_j + p_j = p_j + b_j$ for any $b_j \in B_j$, $p_j \in P_j$. From there for any $l \neq j_0$ we have $p_{j0} - b_1 = -b_{j0} + p_1$, $p_{j0} - b_1 \in P_{j0}$, $-b_{j0} + p_1 \in P_1$ and $p_{j0} - b_1 =$

$= -b_{j0} + p_1 \in P_{j0} \cap B_{j0} = \{0\}$. It means that $p_{j0} = b_1$, $p_1 = b_{j0}$ and $p_{j0} \in \cap \cap \{P_j : j \neq 1, j \in I\}$, $l \in I \setminus \{j_0\}$. Together $p_{j0} \in P_i$ for any $i \in I$ and $p_{j0} = 0$ and because $j_0 \in I$ was chosen arbitrarily it is $k = b_{j0}$ for any $k \in K$, i.e., $k = 0$, $K = \{0\}$.

2.6. *Let (G, Σ) be a non-fully ordered tl-group with a non discrete topology, $\{P_i : i \in I\}$ be a system of closed prime l-ideals in G, $\cap \{P_i : i \in I\} = \{0\}$, K be a componen: in G. Then $K = \{0\}$ or K is the greatest subgroup in G l-isomorphic with a subdirect product $\Pi\, R_i$, where R_i is a fully ordered additive group of real numbers, in case K non $\subseteq P_i$ and $R_i = 0$, in case $K \subseteq P_i$, $i \in I$.*

Further $K \subset A \cup \{0\}$, where A is the set of all archimedean elements in G. Moreover if G is totally non archimedean, then G is a totally disconnected space.

Proof. With regard to 2.1 it is $K \subset P_i + \{g \in G : nx \geqq g \geqq -nx\}$ for any $0 < x \notin P_i$, $i \in I$. $(K + P_i)/P_i$ is a fully ordered tl-subgroup in a tl-factorgroup G/P_i that is connected and in case K *non* $\subseteq P_i$, it is the greatest subgroup in G/P_i l-isomorphic with a fully ordered additive group of real numbers for $i \in I$ (see 1.12). Further G has a realisation $\pi : G \to \prod_{i \in I} G/P_i$, π is an l-isomorphism and thus the first part of the proposition is proved.

If $K \neq \{0\}$, then K is l-isomorphic with $\prod_{i \in I} R_i$. If there exists an element $0 \neq k \in K$, $k \notin A$, then there exists an element $0 \neq h \in G$ such that $n \mid h \mid < \mid k \mid$ for any positive integer n. Further there exists $i \in I$ such that $h \notin P_i$ and thus $k \notin P_i$. Then $k + P_i \in \in G/P_i$, $k + P_i \in (K + P_i)/P_i \subset K_{G/Pi}$, where $K_{G/Pi}$ is a component in the fully ordered tl-group G/P_i, $k + P_i$ is no archimedean element in G/P_i, which is in a contradiction with 1.12.

Definition. A topological product $\prod^*_{i \in I}(R_i, \Sigma_i)$ of tl-groups (G_i, Σ_i) is a topological product of topological groups (G_i, Σ_i), $i \in I$ and a direct product of l-groups G_i, $i \in I$ at the same time.

2.7. *A topological product $\prod^*_{i \in I}(G_i, \Sigma_i)$ of tl-groups (G_i, Σ_i), $i \in I$, is a tl-group.*

If K_i is a component of (G_i, Σ_i), $i \in I$, then a topological product $\prod^*_{i \in I} K_i$ of topological groups K_i is a component in $\prod^*_{i \in I}(G_i, \Sigma_i)$.

Proof. If we denote $\prod^*_{i \in I}(G_i, \Sigma_i) = (G, \Sigma)$, then (G, Σ) is a topological group. If $U \in \Sigma$ is an arbitrary neighbourhood and $g \in G$ is an arbitrary element, then $U = \prod^*_{i \in I} U_i$, where $U_i \in \Sigma_i$ for $i \in J \subset I$, card $J < \chi_0$, $U_i = G_i$ for $i \in I \setminus J$, $g = (g_i)\, i \in I$, $g_i \in G_i$. Then for $i \in J$ there exists $V_i \subset U_i$, $V_i \in \Sigma_i$, $-g_i^+ \vee (V_i + g_i^-) \subset U_i$ because (G_i, Σ_i) are tl-groups (see [6],1.1). If we choose $V_i = G_i$ for $i \in I \setminus J$, then for $V = \prod^*_{i \in I} V_i$ it holds $V \in \Sigma$, $-g^+ \vee (V + g^-) \subset U$ and thus (G, Σ) is a tl-group (see [6],1.1). The rest follows from [7], p. 151, Th. 8.

2.8. Example. If $\boldsymbol{R}_i$ is a fully ordered additive group of real numbers with the interval topology for $i \in J \subset I$ and $\boldsymbol{R}_i$ is a fully ordered additive group of real numbers with the discrete topology for $i \in I \setminus J$, then the (topological) component K of the topological product $\prod\limits_{i \in I}^{*} \boldsymbol{R}_i$ is l-isomorphic with $\prod\limits_{i \in I}^{*} \boldsymbol{R}_i$.

REFERENCES

[1] R. L. Madell: *Embeddings of tl-groups*, Trans. Amer. Math. Soc. 146 (1969), 447—455.

[2] R. L. Madell: *Chains which are coset spaces of tl-groups*, Proc. Amer. Math. Soc. 25 (1970), 755—759.

[3] L. S. Pontrjagin: *Nepreryvnyje gruppy*, Moskva 1954.

[4] F. Šik: *Struktur und Realisierung von Verbandsgruppen*, Memorias Fac. Cie. Univ. Habana, III. vol. 1, no 4 (1966), 1—20.

[5] B. Šmarda: *Topologies in l-groups*, Archivum Mathematicum (Brno), T.3 (1967), 69—81.

[6] B. Šmarda: *Some types of topological l-groups*, Publ. Fac. Sci. Univ. J. E. Purkyně (Brno), 507 (1969), 341—352.

[7] K. Kuratowski: *Topologija, II.*, Moskva 1969.

[8] B. Šmarda: *The lattice of topologies of topological l-groups* (to appear).

B. Šmarda
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia