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PROJECTIVE AND INJECTIVE COMPLETE LATTICES

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This paper is concerned with solving the problem of existence of projective and injective objects, projective and injective retracts, respectively, in the category whose objects are the complete lattices, and morphisms are the complete lattice homomorphisms. From the category point of view, it is quite evident that while problems of injective objects of this category are trivial, the situation with projective objects and even with projective retracts is much more complicated. In the present paper there are mentioned some necessary conditions for the given lattice to be projective object, projective retract, respectively, and there are given necessary and sufficient conditions e.g., for chains.

There are used the following symbols:

Categories are denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, objects of categories by capitals $A, B, C, \ldots$, morphisms by letters $f, g, h, \ldots$. If A, B are objects of the category $\mathfrak{A}$, then symbols $H_{\mathfrak{A}}(A, B)$ or briefly $H(A, B)$ denote the set of all morphisms from A to B. Identity morphism from A to A is denoted by symbol id_A. For $f \in H_{\mathfrak{A}}(A, B)$, $g \in H_{\mathfrak{A}}(B, C)$, the composition of given morphisms is denoted by $g \circ h$ or gh.

If X, Y are sets, the symbol $f: X \to Y$ denotes the mapping of the set X into the set Y.

If $f: X \to Y$, we put $f(X) = \{f(t) \mid t \in X\}$ and for $y \in Y$, $f^{-1}(y) = \{x \mid x \in X, f(x) = y\}$. For $U \subseteqq X$, $f \mid U$ denotes the restriction of the mapping f on the set U.

We say the mapping $f: X \to Y$ is an *injection* if $f(x_1) \neq f(x_2)$ holds for any two elements $x_1, x_2 \in X$, $x_1 \neq x_2$, and a *surjection* in case $f(X) = Y$. If f is both an injection and a surjection, then it is called a *bijection*.

By *ordered set* we mean a partially ordered set i.e., a set with reflexive, antisymmetrical and transitive relation. If A is an ordered set, $\emptyset \neq X \subseteqq A$, the least upper bound of the subset X in the set A—if it exists—is denoted by the symbol $\sup_A X$ or only $\sup X$. Analogously, the greatest lower bound of a subset X in the set A is denoted by $\inf_A X$ or $\inf X$. $x \vee y$ is also used instead of $\sup \{x, y\}$ and $x \wedge y$ instead of $\inf \{x, y\}$. $X - Y$ denotes the difference of sets X, Y.

If A is an ordered set, $x, y \in A$, the symbol $x \parallel y$ means that the elements x, y are *incomparable* i.e., neither $x \leqq y$ nor $y \leqq x$ holds. For $x, y \in A$, $x \leqq y$, $\langle x, y\rangle$

denotes a closed interval $\{t \mid t \in A,\ x \leqq t \leqq y\}$, (x, y) denotes the corresponding open interval $\{t \mid t \in A,\ x < t < y\}$. We say the element y *covers* the element x if $y > x$, $\langle x, y\rangle = \{x, y\}$. If there exists the smallest element in the ordered set A, it is denoted by 0_A, the greatest element—if it exists—by 1_A.

An isomorphism of ordered sets is denoted by the symbol $\cong$.

If A, B are ordered sets, $A + B$ denote their cardinal sum, $A \oplus B$ their ordinal sum, and $A \times B$ their cardinal product (see [1], pp. 55 and 198). If $a \in A$, $b \in B$, then $[a, b] \in A \times B$.

Let $\mathfrak{A}$ be a category. An object $A \in \mathfrak{A}$ is called *projective* if for arbitrary $B, C \in \mathfrak{A}$, for arbitrary epimorphism $g \in H_{\mathfrak{A}}(B, C)$ and for arbitrary morphism $f \in H_{\mathfrak{A}}(A, C)$ we have $h \in H_{\mathfrak{A}}(A, B)$ so that $gh = f$.

$A \in \mathfrak{A}$ is called the *projective retract* if for every $B \in \mathfrak{A}$ and for arbitrary epimorphism $g \in H_{\mathfrak{A}}(B, A)$ we have $h \in H_{\mathfrak{A}}(A, B)$ such that $gh = id_A$.

$A \in \mathfrak{A}$ is called the *injective* object if for arbitrary $B, C \in \mathfrak{A}$, arbitrary monomorphism $g \in H_{\mathfrak{A}}(C, B)$ and arbitrary morphism $f \in H_{\mathfrak{A}}(C, A)$ there exists a morphism $h \in H_{\mathfrak{A}}(B, A)$ such that $hg = f$.

$A \in \mathfrak{A}$ is called the *injective retract* if for arbitrary $B \in \mathfrak{A}$, and arbitrary monomorphism $g : A \to B$, there exists $h \in H_{\mathfrak{A}}(B, A)$ such that $hg = id_A$.

It is evident from the above definitions that *every projective object is a projective retract*, and *every injective object is an injective retract*.

§ 1. Basic properties of the category of complete lattices

1.1. Definition. Let A, B be complete lattices. The mapping $f: A \to B$ is called the *complete homomorphism* if

$$f(\sup_A X) = \sup_B \{f(X)\} \qquad f(\inf_A X) = \inf_B \{f(X)\}$$

holds for arbitrary subset $\emptyset \neq X \subseteqq A$.

1.2. Definition: Let A be a complete lattice. A subset $\emptyset \neq X \subseteqq A$ is called the *closed sublattice* of the lattice A if

$$\sup_A Y \in X, \qquad \inf_A Y \in X$$

holds for every subset $\emptyset \neq Y \subseteqq X$.

Thus it is obvious that a *closed sublattice of a complete lattice is a complete lattice.*

It may be easily proved

1.3. Lemma: *Let A, B be complete lattices, $f: A \to B$ a complete homomorphism. Then $f(A)$ is a closed sublattice of the lattice B.*

1.4. Notation: The category whose objects are complete lattices and morphisms complete homomorphisms is denoted by $\mathfrak{L}$.

1.5. Lemma: *The morphism $f \in \mathfrak{L}$ is a monomorphism if and only if it is an injection and an epimorphism if and only if it is a surjection.*

Proof: It is evident that if $f \in \mathfrak{L}$ is an injection, then it is a monomorphism and if it is a surjection, then it is an epimorphism. Now the reverse statements will be proved.

I. Let $f \in H_{\mathfrak{L}}(A, B)$ be no injection. Then there exist elements $a, b \in A$, $a \neq b$ so that $f(a) = f(b)$. Let C be an arbitrary one-element lattice. Let us define $\alpha : C \to A$, $\beta : C \to A$ in the following way: $\alpha(x) = a$, $\beta(x) = b$. Then $\alpha \in H_{\mathfrak{L}}(C, A)$, $\beta \in H_{\mathfrak{L}}(C, A)$, $f\alpha = f\beta$, but $\alpha \neq \beta$ so that f is not a monomorphism.

II. Let $f \in H_{\mathfrak{L}}(A, B)$ be no surjection. It will be shown that f is not an epimorphism. By assumption $f(A) = B_1 \neq B$. Let us denote $B_2 = B - B_1$. Thus $B_2 \neq \emptyset$, and by Lemma 1.3, B_1 is a closed sublattice of the complete lattice B.

Let us now consider the following three cases:

1. Let $B_2 = \{1_B\}$. Let us denote $f(1_A) = b_0$. Then $b_0 \in B_1$, b_0 is the greatest element in B_1, and 1_B covers b_0. Now let us define morphisms α, $\beta \in H_{\mathfrak{L}}(B, B)$ as follows:

$$\alpha = id_B, \qquad \beta(x) = \begin{cases} x & \text{for } x \in B, x \neq 1_B, \\ b_0 & \text{for } x = 1_B. \end{cases}$$

Then $\alpha \neq \beta$ and $\alpha f = \beta f$. Thus f is no epimorphism.

2. The assertion can be analogously proved if $B_2 = \{0_B\}$ or $B_2 = \{0_B, 1_B\}$.

3. Let $x \in B_2$, $0_B \neq x \neq 1_B$ exist. Now let us define the mapping $\varphi^* : B_2 \to B_1 \cup \{1_B\}$ as follows:

$$\varphi^*(t) = \begin{cases} \inf_B \{b \mid b \in B_1, b > t\} & \text{if such } b \in B_1 \text{ exists} \\ 1_B \text{ if } b \in B_1, b > t \text{ does not exist.} \end{cases}$$

Then we define dually $\varphi_* : B_2 \to B_1 \cup \{0_B\}$.

Let us denote $B_2' = B_2 - \{0_B, 1_B\}$. We have $B_2' \subseteq B_2 \subseteq B$, and by assumption $B_2' \neq \emptyset$. Let B_2'' be an arbitrary ordered set isomorphic with B_2', where $B \cap B_2'' = \emptyset$. Let $\varphi : B_2' \to B_2''$ be a corresponding isomorphism.

Let us put $C = B_2'' \cup B$, and let us define the relation $\leqq_C$ on the set C as follows:

For $x_1, x_2 \in B$, there holds $x_1 \leqq_C x_2$ if and only if $x_1 \leqq x_2$ in B.

For $x_1, x_2 \in B_2''$, there holds $x_1 \leqq_C x_2$ if and only if $x_1 \leqq x_2$ in B_2''.

For $x \in B$, $\varphi(t) \in B_2''$, there holds $x \leqq_C \varphi(t)$ if and only if $x \leqq \varphi_*(t)$ in B, and $\varphi(t) \leqq_C x$ if and only if $\varphi^*(t) \leqq x$ in B.

It is easily seen that the relation $\leqq_C$ is an ordering on the set C and that $(C, \leqq_C)$ is a complete lattice.

Let us define the mapping $\psi : B \to C$ in the following way:

$$\psi(x) = \begin{cases} x & \text{for } x \in B - B'_2, \\ \varphi(x) & \text{for } x \in B'_2. \end{cases}$$

Then it is obvious that ψ is an isomorphism from B into C.

Let now $\emptyset \neq X \subseteqq C$ be arbitrary. The following three possibilities may occur:

(1) $X \subseteqq B$. But then necessarily $\sup_C X = \sup_B X$.

(2) $X \subseteqq B''_2$. Then obviously $\sup_C X = \psi\{\sup_B \{\varphi^{-1}(X)\}\}$.

(3) $X \cap B \neq \emptyset$, $X \cap B''_2 \neq \emptyset$. We shall prove that $\sup_C X$ exists also in this case.

Denote $X \cap B = Y$, $X \cap B''_2 = Z$. By (1) and (2) there exist $\sup_C Y = y$, $\sup_C Z = z$. If we prove now that there exists $\sup_C \{y, z\}$, then there will be $\sup_C \{y, z\} = \sup_C X$. If $z \in B$ or $z \in C - B$, but z comparable with y, it is nothing to be proved. Thus, let $z \in C - B$, $y \parallel z$. Then we have $z = \varphi(t)$ for a suitable element $t \in B$ and there holds obviously

$$\sup_C \{y, z\} = \sup_C \{y, \varphi(t)\} = \sup_B \{y, \varphi^*(t)\}.$$

So there exists $\sup_C X$ also in this case.

The existence of $\inf_C X$ may be proved analogously.

Hence C is a complete lattice and obviously $\psi \in H_{\mathfrak{L}}(B, C)$.

Now let us define $\alpha : B \to C$ as follows: $\alpha(x) = x$ for every $x \in B$. Then $\alpha \in H_{\mathfrak{L}}(B, C)$, $\alpha \neq \psi$, but $\alpha f = \psi f$ so that f is no epimorphism. By this Lemma 1.5 is proved.

§ 2. Reducible elements

2.1. Definition: Let A be an ordered set, $B \subseteqq A$. We say an element $x \in A$ is *J°-reducible in respect of B* if there exists a set $\emptyset \neq X \subseteqq B$ such that $x \notin X$, $x = \sup_A X$. We say an element $x \in A$ is *M°-reducible in respect of the set B* if there exists a set $\emptyset \neq X \subseteqq B$ such that $x \notin X$, $x = \inf_A X$.

Let us denote

$J^\circ(A; B) = \{x \mid x \in A,\ x \text{ is } J^\circ\text{-reducible in respect of } B\}$,

$M^\circ(A; B) = \{x \mid x \in A,\ x \text{ is } M^\circ\text{-reducible in respect of } B\}$.

Further we put

$$R(A; B) = J^\circ(A; B) \cup M^\circ(A; B),$$
$$JM(A; B) = J^\circ(A; B) \cap M^\circ(A; B).$$

Elements of the set $R(A; B)$ are called *reducible elements of the set A in respect of the set B*, elements of the set $JM(A; B)$ are called *JM-reducible in respect of the set B*.

Instead of $J^\circ(A; A)$, we write briefly $J^\circ(A)$ and in analogy with it also $M^\circ(A)$, $R(A)$, $JM(A)$. Elements of the set $R(A)$ are called *reducible elements of the set A*, and elements of $JM(A)$ *JM-reducible elements of the set A*, etc.

2.2. Theorem: *Let A be a complete lattice. Then the set $R(A)$ of its reducible elements is either empty or it forms a closed sublattice in A.*

Proof: Let $R(A) \neq \emptyset$. Let $\emptyset \neq X \subseteqq R(A)$ be arbitrary. Then $\sup_A X \in R(A)$, since if $x_0 = \sup_A X \notin R(A)$, it would hold $x_0 \in J(A) \subseteqq R(A)$, which is a contradiction. Analogously $\inf_A X \in R(A)$. Thus $R(A)$ is a closed sublattice in A.

2.3. Definition: Let A be a complete lattice, such that $JM(A) = 0$ and let all chains in the set $R(A)$ be finite. Let the sets $J^\circ(A), \ldots, J^k(A), M^\circ(A), \ldots, M^k(A)$ be already defined. Then we say that the element $x \in M^\circ(A)$ is *$M^{k+1}(y)$-reducible in A* (or more concisely *M^{k+1}-reducible in A*) if $y \in J^k(A), y > x$ is such that $x \in M^\circ(A; A - \langle x, y\rangle)$. We shall denote the set of all M^{k+1}-reducible elements in A by $M^{k+1}(A)$.

Analogously we say that the element $u \in J^\circ(A)$ is *$J^{k+1}(v)$-reducible in A* (or more concisely *J^{k+1}-reducible in A*) if $v \in M^k(A), v < u$ exists such that $u \in J^\circ(A; A - \langle v, u\rangle)$. We shall denote the set of all J^{k+1}-reducible elements in A by $J^{k+1}(A)$.

Further we put

$$SJ^\infty(A) = \bigcap_{i=0}^{\infty} J^i(A), \qquad SM^\infty(A) = \bigcap_{i=0}^{\infty} M^i(A).$$

Finally for $k = 0, 1, 2, \ldots$ we denote

$$SJ^k(A) = \{x \mid x \in J^k(A) \text{ and for every } i > k \text{ is } x \notin J^i(A)\},$$
$$SM^k(A) = \{x \mid x \in M^k(A) \text{ and for every } i > k \text{ is } x \notin M^i(A)\}.$$

If $x \in SJ^k(A) \cup SM^k(A)$, k is called the characteristic of the element x ($k = 0, 1, 2, \ldots, \infty$.).

2.4. Remark: It is clear that if $x \in J^{i+1}(A)$ or $x \in M^{i+1}(A)$ ($i = 0, 1, 2, \ldots$), then $x \in J^i(A)$ or $x \in M^i(A)$, respectively. At the same time it is clear that to every $x \in R(A)$ there exists exactly one k such that $x \in SJ^k(A) \cup SM^k(A)$.

2.5. Example: Let A be a complete lattice on Figure 1. Then

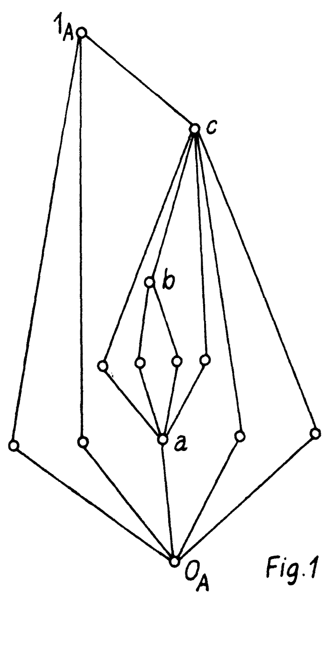


Fig. 1

$$\begin{aligned}
J^\circ(A) &= \{b, c, 1_A\},\\
M^\circ(A) &= \{0_A, a\},\\
JM(A) &= \emptyset,\\
R(A) &= \{0_A, a, b, c, 1_A\},\\
J^1(A) &= \{c, 1_A\},\\
M^1(A) &= \{0_A, a\},\\
J^2(A) &= \{c, 1_A\},\\
M^2(A) &= \{0_A\},\\
J^3(A) &= J^4(A) = \ldots = \emptyset,\\
M^3(A) &= \{0_A\},\\
SM^2(A) &= \emptyset.
\end{aligned}$$

2.6. Definition: Let $A \neq \emptyset$ be an ordered set with the least element 0_A and the greatest element 1_A. We say that A is *below ramified* (*above ramified*) if the elements $x, y \in A$, $x \neq 0_A \neq y$ ($x \neq 1_A \neq y$) exist such that $\inf_A \{x, y\} = 0_A$ ($\sup_A \{x, y\} = = 1_A$).

We say that A is *ramified* if it is below and above ramified.

2.7. Definition: Let A be a ramified complete lattice such that $JM(A) = \emptyset$ and all chains in the set $R(A)$ are finite. Let $SJ^k(A) \neq \emptyset$. Then we denote:

$SJ_1^k(A) = \{x \mid x$ is a minimal element of the set $SJ^k(A)\}$. After having defined the set $SJ_i^k(A)$, we put

$SJ_{i+1}^k(A) = \{x \mid x$ is minimal element of the set $SJ^k(A) - \bigcup_{j=1}^{i} SJ_j^k(A)\}$. Let us define the sets $SM_i^k(A)$ ($i = 1, 2, 3, \ldots$) in case $SM^k(A) \neq 0$ as follows:

(a) If $SJ^k(A) \neq \emptyset$, we put

$SM_1^k(A) = \{x \mid x \in SM^k(A),\ x < t$ for some $t \in SJ_1^k(A)\}$ and for $i = 2, 3, \ldots$ we put

$$SM_i^k(A) = \{x \mid x \in SM^k(A) - \bigcup_{j=1}^{i-1} SM_j^k(A),\ x < t \text{ for some } t \in SJ_i^k(A)\}.$$

(b) If $SJ^k(A) \mp \emptyset$, we put

$SM_1^k(A) = \{x \mid x$ is a minimal element of the set $SM^k(A)\}$ and after having defined the set $SM_i^k(A)$, we put

$SM_{i+1}^k(A) = \{x \mid x$ is a minimal element of the set $SM^k(A) - \bigcup_{j=1}^{i} SM_j^k(A)\}$.

The set $SJ_i^k(A) \cup SM_i^k(A)$ will be called *i-th layer* of the set $SJ^k(A) \cup SM^k(A)$.

2.8. Example: In the complete lattice A on Figure 2 we have $J^\circ(A) = \{f, g, j, p, o, 1_A\}$, $M^\circ(A) = \{0_A, e, i, m, n\}$, $J^1(A) = \{g, o\}$ because the element g is $J^1(e)$-reducible and the element o is $J^1(m)$-reducible. $M^1(A) = \{0_A\}$ because 0_A is, e.g., $M^1(f)$-reducible (but also $M^1(g)$-, $M^1(j)$- or $M^1(o)$-reducible and similarly.)

Then, e.g.,

$SJ^\circ(A) = \{f, j, p, 1_A\}$,
$SM^\circ(A) = \{e, i, m, n\}$,

and thus

$SJ_1^0(A) = \{f, j\}$,
$SJ_2^0(A) = \{p\}$,
$SJ_3^0(A) = \{1_A\}$,

$SM_1^0(A) = \{e\}$,
$SM_2^0(A) = \{i, m, n\}$,
$SM_3^0(A) = \emptyset$.

2.9. Definition: Let A be a complete lattice, let $x, y \in A$. We say the elements x, y form an *aprojective couple* in A if it holds:

1°: $x < y$,
2°: $x \in M^{\circ}(A; A - \langle x, y \rangle)$,
3°: $y \in J^{\circ}(A; A - \langle x, y \rangle)$.

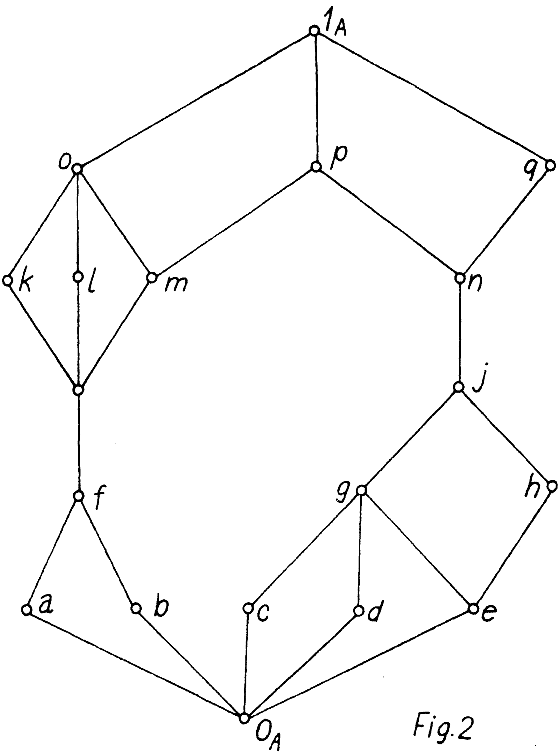


Fig. 2

2.10. Lemma: *Let A be a complete lattice such that $JM(A) = \emptyset$ and all chains in $R(A)$ are finite. Let the elements $x < y$ form in A an aprojective couple. Then $x \in SM^{\infty}(A)$, $y \in SJ^{\infty}(A)$.*

Proof: From definition 2.3 it follows immediately that $x \in M^1(A)$. Then certainly $y \in J^2(A)$, $x \in M^3(A)$ etc. Now the assertion is clear.

2.11. Example: In the lattices shown in Figs. 3a–3d, the elements d, e form an aprojective couple.

2.12. Remark: It is easy to see that every complete lattice containing an aprojective couple has at least 8 elements. All four lattices demonstrated in Example 2.11 are eight-element lattices. It will be easily shown that there does not exist any eight-element lattice containing an aprojective couple and being not isomorphic with some of them.

By 2.12, the following lemma can be easily proved:

2.13. Lemma: *Let A be a complete lattice in which all chains are finite. Then A contains an aprojective couple if and only if it contains eight-element subset isomorphic with some of lattices in Figs. 3a–3d, namely, such that its subset which is isomorphic with the set $\{a, b, c, d, e\}$ forms a join-subsemilattice, and its subset which is isomorphic with the set $\{d, e, f, g, h\}$ forms a meetsubsemilattice of the complete lattice A.*

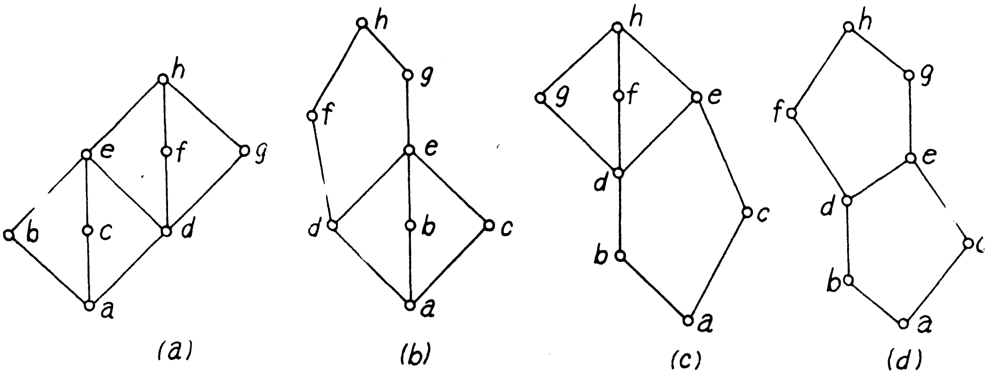


Fig. 3.

2.14. Theorem: *Let A be a complete lattice such that $JM(A) = \emptyset$ and the set $R(A)$ is finite. If the set $SJ^{\infty}(A) \cup SM^{\infty}(A)$ is nonempty, then an aprojective couple exists in A.*

Proof: Let, e.g., exist $x_0 \in SM^{\infty}(A)$, i.e. $x_0 \in M^k(A)$ for every $k = 0, 1, 2, \ldots$. Since there exist finitely many elements $y \in J^{\circ}(A)$ such that $x_0 < y$, there must exist among them an element of the characteristic ∞, i.e. $y_0 \in SJ^{\infty}(A)$ such that $x_0 \in$ $\in M^{\circ}(A; A - \langle x_0, y_0 \rangle)$. If the elements x_0, y_0 do not form an aprojective couple, i.e. if $y_0 \notin J^{\circ}(A; A - \langle x_0, y_0 \rangle)$, there must exist analogously the element $x_1 \in SM^{\infty}(A)$ such that $y_0 \in J^{\circ}(A; A - \langle x_1, y_0 \rangle)$. If, moreover, the elements x_1, y_0 do not form an aprojective couple, i.e. if $x_1 \notin M^{\circ}(A; A - \langle x_1, y_0 \rangle)$ holds, there must exist analogously the element $y_1 \in SJ^{\infty}(A)$ such that $x_1 \in M^{\circ}(A; A - \langle x_1, y_1 \rangle)$ etc. Thus we shall construct the sequences $\{x_n\}$ in $SM^{\infty}(A)$ and $\{y_n\}$ in $SJ^{\infty}(A)$. Since, however, we suppose the set $R(A)$ to be finite, each of these sequences contains only finitely many elements mutually different. Then some elements x_i, y_j form clearly an aprojective couple.

2.15. Remark: It is evident, that there exists a complete lattice A such that $JM(A) = \emptyset$, A does not contain an aprojective couple, all chains in A are finite and at the same time $SJ^{\infty}(A) \cup SM^{\infty}(A) \neq \emptyset$. Then, however, A must clearly contain an infinite antichain.

§ 3. Complete congruences.

3.1. Definition: Let A be a complete lattice, Θ an equivalence relation on A. For $X, Y \subseteq A$, we say $X\Theta Y$ if there exists a bijection $f: X \to Y$ such that $x\Theta f(x)$ for every $x \in X$.

We say the equivalence relation Θ on A is a complete congruence if, for arbitrary $X \subseteq A$ and arbitrary $Y \subseteq A$ such that $X\Theta Y$, there holds

$$(\sup_A X)\,\Theta(\sup_A Y), \qquad (\inf_A X)\,\Theta(\inf_A Y).$$

3.2. Notation: Let A be a complete lattice, Θ a complete congruence relation on A. For every $a \in A$, let us denote $\bar{a} = \{x \mid x \in A,\ x\Theta a\}$. Further we put $A/\Theta = \{\bar{a} \mid a \in A\}$.

It is known (see [3]) that the elements of the set A/Θ are closed intervals in A. It is obvious that these closed intervals are closed sublattices in A.

Further we use the following notation: For $a \in A$, let a° denote the greatest element, a_0 the smallest element in $\bar{a}$ i.e., $\bar{a} = \langle a_0, a^\circ\rangle$. And finally, let us denote

$$A^\circ = \{a^\circ \mid a \in A\}, \qquad A_0 = \{a_0 \mid a \in A\}.$$

3.3. Definition: Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Let us define the relation $\leqq$ on the set $\bar{A}$ as follows:

$$\bar{x} \leqq \bar{y} \text{ if and only if there exist } t_1 \in \bar{x},\ t_2 \in \bar{y},\ t_1 \leqq t_2 \text{ in } A.$$

3.4. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Then the relation $\leqq$ defined in 3.3 is an order relation of the set $\bar{A}$.*

Proof: (a) Reflexivity and transitivity of the relation $\leqq$ is obvious.

(b) Let $\bar{x} \leqq \bar{y}$, $\bar{y} \leqq \bar{x}$. Then $t_1, t_2 \in \bar{x}$, $t_3, t_4 \in \bar{y}$ so that $t_1 \leqq t_3$, $t_2 \geqq t_4$. It holds $t_1 \wedge t_3 = t_1 \in \bar{x}$, $t_2 \wedge t_4 = t_4 \in \bar{y}$. But we have $t_1\Theta t_2$, $t_3\Theta t_4$ such that $t_1\Theta t_4$ since $(t_1 \wedge t_3)\,\Theta(t_2 \wedge t_4)$. Thus $\bar{x} = \bar{t}_1 = \bar{t}_4 = \bar{y}$, and the relation $\leqq$ is also antisymmetric.

3.5. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Let $a, b \in A$. Then the following is equivalent:*

(1) $\bar{a} \leqq \bar{b}$ *in* $\bar{A}$,
(2) $a^\circ \leqq b^\circ$ *in* A,
(3) $a_0 \leqq b_0$ *in* A.

Proof: I. From the definition of the ordering on A, it follows that (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1).

II. Let (1) hold. Then $t_1 \in \bar{a}$, $t_2 \in \bar{b}$, $t_1 \leqq t_2$. We have $(a_0 \wedge b_0)\,\Theta(t_1 \wedge t_2)$ and $t_1 \wedge t_2 = t_1$, i.e., $(a_0 \wedge b_0) \in \bar{a}$, and thus $a_0 \wedge b_0 = a_0$, so that (3) holds.

Analogously (1) implies (2).

From Lemma 3.5 it follows directly

3.6. Corollary: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Then $\bar{A} \cong A^\circ \cong A_0$.*

3.7. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Then $\bar{A}$ is a complete lattice.*

Proof: Let $\emptyset \neq \bar{X} \subseteq \bar{A}$ be arbitrary subset in $\bar{A}$. Let us denote $X = \{x \mid x \in A, \bar{x} \in \bar{X}\}$. Then $\emptyset \neq X \subseteq A$. From the definition of the complete congruence relation, it follows that $\bar{X}$ has in $\bar{A}$ both the least upper bound and the greatest lower bound since $\inf_{\bar{A}} \bar{X} = \overline{\inf_A X}$, $\sup_{\bar{A}} \bar{X} = \overline{\sup_A X}$.

3.8. Remark: It follows from 3.6 and 3.7 that A_0 and A° are also complete lattices. But generally they are not closed sublattices of the complete lattice A. A° is obviously closed in respect to the greatest lower bounds and A_0 in respect to the least upper bounds in A.

3.9. Definition: Let A, B be sets, $f: A \to B$. Let us define the relation Kerf on the set A as follows:

$$[x, y] \in Kerf \qquad \text{if and only if} \qquad f(x) = f(y).$$

It is evident that there holds

3.10. Lemma: *Let A, B be complete lattices, f: $A \to B$ a complete homomorphism. Then the relation Kerf is a complete congruence on A.*

3.11. Lemma: *Let A, B be complete lattices, $f: A \to B$ a complete homomorphism. Then*

$$A/Kerf \cong f(A).$$

Proof: Let us denote $\bar{A} = A/Kerf$, and let us define the mapping $\bar{f}: \bar{A} \to f(A)$ as follows:

For $\bar{x} \in \bar{A}$ let us choose an arbitrary $x \in \bar{x}$. We put $\bar{f}(\bar{x}) = f(x)$. Then obviously $\bar{f}$ is a bijection $\bar{A}$ on $f(A)$. At the same time f is evidently a complete homomorphism, and hence

$$\bar{x} \leqq \bar{y} \text{ in } \bar{A} \qquad \text{if and only if} \qquad \bar{f}(\bar{x}) \leqq \bar{f}(\bar{y}) \text{ in } f(A).$$

Thus $\bar{f}$ is an isomorphism A/Kerf on $f(A)$.

3.12. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Then it holds:*

(a) $\bar{a} \in J^\circ(A)$ if and only if $a_0 \in J^\circ(A)$,
(b) $\bar{a} \in M^\circ(\bar{A})$ if and only if $a^\circ \in M^\circ(A)$.

Proof: We shall prove, for instance, the statement (a). The proof of the statement (b) is analogous.

I. Let $\bar{a} \in J^\circ(\bar{A})$, i.e., there exists a set $\emptyset \neq \bar{X} \subseteqq \bar{A}$ such that $\bar{a} \notin \bar{X}$, $\bar{a} = \sup_{\bar{A}} \bar{X}$. Let us denote $X_0 = \{x_0 \mid x \in \bar{x},\ \bar{x} \in \bar{X}\}$. Then $a_0 \notin X_0$, $a_0 = \sup_A X_0$, so that $a_0 \in J^\circ(A)$.

II. Let $\bar{a} = \langle a_0, a^\circ \rangle \in \bar{A}$, $a_0 \in J^\circ(A)$. Then there exists a set $\emptyset \neq X \subseteqq A$ such that $a_0 \notin X$, $a_0 = \sup_A X$. But then obviously $\bar{a} = \sup_{\bar{A}} \bar{X}$, $\bar{a} \notin \bar{X}$, where $\bar{X} = \{\bar{x} \mid x \in X\}$. Hence $\bar{a} \in J^\circ(\bar{A})$.

3.13. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Let $\bar{a}, \bar{b} \in \bar{A}$, $\bar{a} < \bar{b}$. Let us choose arbitrary, $a \in \bar{a}$, $b \in \bar{b}$ such that $a < b$. Then for every $\bar{c} \in \langle \bar{a}, \bar{b} \rangle$, we have $\langle a, b \rangle \cap \bar{c} \neq \emptyset$.*

Proof: Let $\bar{c} = \langle c_0, c^\circ \rangle \in \bar{A}$ be arbitrary such that $\bar{a} < \bar{c} < \bar{b}$. Let $a \in \bar{a}$, $b \in \bar{b}$ be arbitrary such that $a < b$. Since Θ is a complete congruence, it holds $a \vee c_0 \in \bar{c}$, where $a \vee c_0 > a$. By 3.5 it holds further $b_0 > c_0$ so that $b > a \vee c_0$ for $b > a$, $b \geqq b_0 > c_0$. Then $a \vee c_0 \in \langle a, b \rangle$ so that $\langle a, b \rangle \cap \bar{c} \neq \emptyset$.

3.14. Definition: Let A be an ordered set with the smallest element 0_A, and with the greatest element 1_A. Let K be a component of the set $A - \{0_A, 1_A\}$. The set $K \cup \{0_A, 1_A\}$ is called a *$(0, 1)$-component* of the set A. Further we say the set A is *$(0, 1)$-connected* if it contains only one $(0, 1)$-component. In case A is not $(0, 1)$-connected, we say it is *$(0, 1)$-disconnected.* (Obviously the set A is $(0, 1)$-connected if and only if the set $A - \{0_A, 1_A\}$ is connected.

3.15. Lemma: *Let A be a $(0, 1)$-disconnected complete lattice with more than two $(0, 1)$-components. Let Θ be a complete congruence on A, $\bar{A} = A/\Theta$. Then there occurs one and only one of the following possibilities:*

(1) $\bar{0}_A = \{0_A\}$, $\bar{1}_A = \{1_A\}$ *and every further element $\bar{a} \in \bar{A}$ is a subset of one and only one $(0, 1)$-component of the set A.*

(2) $\bar{A} = \{A\}$,

Proof: 1. Let $\{0_A\}$, $\{1_A\}$ be the elements of the lattice $\bar{A}$. Let $\bar{a} \in \bar{A}$, $\{0_A\} \neq \bar{a} \neq \{1_A\}$. Let K, L be $(0, 1)$-components in A such that $\bar{a} \cap K \neq \emptyset$, $\bar{a} \cap L \neq \emptyset$. For arbitrary $x \in \bar{a} \cap K$, $y \in \bar{a} \cap L$, we have $x \vee y = 1_A$ provided $K \neq L$. But since $\bar{a}$ is a closed sublattice in A, then $1_A \in \bar{a}$ and thus $\bar{a} = \{1_A\}$ which is not possible. Hence $K = L$ and every element $\bar{a} \in \bar{A}$ is a subset of one and only one $(0, 1)$-component.

2. Let e.g., $\overline{1_A} \neq \{1_A\}$. Then $x_0 \in A$, $x_0 \neq 1_A$, $x_0 \Theta 1_A$. Let $K \subseteq A$ be the $(0, 1)$-component for which $x_0 \in K$ is valid. Let L, M be different $(0, 1)$-components in A, $L \neq K \neq M$. (By our supposition, such components exist.) Let $y \in L$, $z \in M$ be arbitrary such that $y \parallel x_0$, $z \parallel x_0$. Then $y \wedge x_0 = 0_A = z \wedge x_0$, $(y \wedge x_0)\, \Theta (y \wedge 1_A)$ i.e., $(y \wedge x_0)\, \Theta y$, and finally $(z \wedge x_0)\, \Theta (z \wedge 1_A)$ i.e., $(z \wedge x_0)\, \Theta z$. Hence $y \Theta 0_A$, $z \Theta 0_A$. But then $(y \vee z)\, \Theta (0_A \vee 0_A)$ i.e., $1_A \Theta 0_A$ and thus $\bar{A} = \{A\}$.

3.16. Lemma: *Let A be a $(0, 1)$-disconnected complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. If A has two and only two $(0, 1)$-components, then there may occur, besides* (1), (2) *of Lemma 3.15, the possibility*

(3) *$\bar{A}$ is a two-element chain $\overline{O_A} < \overline{I_A}$.*

Proof: Let the complete lattice A have two $(0, 1)$-components K, L. Let us admit that there have occured no one of two cases (1), (2) in 3.15. First, let us realize that it is not possible that only one of the sets $\overline{O_A}$, $\overline{I_A}$ is one-element set. If, namely, for example $\overline{O_A} = \{0_A\}$, $\overline{I_A} \neq \{1_A\}$, the elements $x, y \in A$, $x \parallel y$ would exist such that $x\Theta 1_A$, $y\Theta 1_A$, $x \wedge y = 0_A$ (for, with regard to the validity of Theorem 3.5, $\overline{I_A}$ cannot be clearly a subset of a unique $(0, 1)$-component) so that it would hold $(x \wedge 1_A)\, \Theta 0_A$ as well, i.e. $x \in \overline{O_A}$, which is a contradiction.

Now let $\bar{a} = \overline{I_A} \neq \{1_A\}$, $A \neq \bar{a}$. If $\bar{a} \cap K \neq \emptyset$, $\bar{a} \cap L \neq \emptyset$ were true, it would be obviously hold $\bar{a} = A$ which, by assumption, is not possible. Thus $\bar{a}$ is a subset of some $(0, 1)$-component e.g., K. First we shall show that $L - \{1_A\} \in \bar{A}$. Let $x \in$ $\in L - \{1_A\}$ be arbitrary. Since $\bar{x} < \bar{a}$ holds (because of $x < 1_A$), thus $x_0 < a_0$. But $\bar{x} \in L$ and a unique element $t \in L$ such that $t < a_0$ is an element 0_A. Hence $x_0 = 0_A$ i.e., $\bar{x} = L - \{1_A\}$. It can be analogously proved that $\bar{a} = K - \{0_A\}$, thus $\bar{A} = \{K - \{0_A\}, L - \{1_A\}\}$.

From Lemmas 3.15 and 3.16, it follows

3.17. Corollary: *Let A be a $(0, 1)$-disconnected complete lattice, Θ a complete congruence on A. If the complete lattice A/Θ is $(0,1)$-connected, then it is either $A/\Theta = \{A\}$, or it is a two element chain.*

§ 4. Injective objects of the category $\mathfrak{L}$.

4.1. Lemma: *Let A be a complete lattice, card $A > 1$. Let B be an arbitrary set such that $B \cap A = \emptyset$, card $B >$ card A. Let us put $C = A \cup B$ and let us define the relation $\leqq_C$ on the set C as follows:*

$x \leqq_C y$ if and only if (a) $x \leqq y$ in A, or
(b) $x = 0_A$, $y \in B$, or
(c) $x \in B$, $y = 1_A$.

Then $\leqq_C$ is an ordering on C, and $(C, \leqq_C)$ is a complete lattice. If $g : C \to A$ is a complete homomorphism, then g must be a constant mapping.

Proof: It is evident that $\leqq_C$ is an ordering on C and $(C, \leqq_C)$ is a complete lattice. Let $g : C \to A$ be a complete homomorphism. Then by 1.3, $g(C) = A_1$ is a closed sublattice in A. Let b_1 be the greatest element in A_1, a_1 the smallest element in A_1, and let us admit that $a_1 \neq b_1$. Let $x, y \in B$ be arbitrary, $x \neq y$. Then $g(x) \neq g(y)$ since $g(x) \vee g(y) = g(x \vee y) = g(1_C) = b_1$, $g(x) \wedge g(y) = g(x \wedge y) = g(0_A) = a_1$. Thus $g \mid B : B \to A_1$ is an injection which is, however, not possible, since card $B >$ card $A \geqq$ card A_1. Thus $a_1 = b_1$ and hence $g(C)$ is one-element set.

4.2. Theorem: *The complete lattice A is an injective retract in the category $\mathfrak{L}$ if and only if card $A = 1$.*

Proof: I. If card $A = 1$, then obviously A is an injective retract.

II. Let card $A > 1$. Let us form a complete lattice C in the same way as in Lemma 4.1. Let us define the complete homomorphism $f : A \to C$ as follows:

$$f(x) = x \qquad \text{for every } x \in A.$$

Then there does not exist the complete homomorphism $h : C \to A$ such that $hf = id_A$, since by 4.1, every $h \in H_{\mathfrak{L}}(C, A)$ is constant.

Since it is evident that one-element lattice is an injective object in $\mathfrak{L}$ and every injective object is an injective retract, there follows from Theorem 4.2

4.3. Corollary: *The complete lattice A is an injective object in the category $\mathfrak{L}$ if and only if card $A = 1$.*

§ 5. Projective retracts and projective objects.

5.1. Definition: Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. A *projective selection* in A is a set $P \subseteqq A$ with:

(P1) P is a closed sublattice in A.

(P2) $P \cap \bar{a}$ is a one-element set for every $\bar{a} \in \bar{A}$.

5.2. Lemma: *Let P be a projective selection in $\bar{A} = A/\Theta$. Then $P \cong \bar{A}$.*

Proof: Let us denote $P \cap \bar{t} = \{\bar{t}_p\}$ for $\bar{t} \in \bar{A}$. Since P is a closed sublattice in A, it is sufficient to prove that for arbitrary $\bar{x}, \bar{y} \in \bar{A}$ such that $\bar{x} \leqq \bar{y}$, there holds $\bar{x}_p \leqq \bar{y}_p$. By 3.3 and 3.4, $\bar{x}_p > \bar{y}_p$ cannot hold. Let us admit that $\bar{x}_p \parallel \bar{y}_p$. Then we have $\bar{x}_p \vee \bar{y}_p > \bar{x}_p$, $\bar{x}_p \vee \bar{y}_p > \bar{y}_p$. But $(\bar{x}_p \vee \bar{y}_p)\, \Theta \bar{y}_p$ holds, and since P is a closed sublattice, there must hold $\bar{x}_p \vee \bar{y}_p \in P$, which is in contradiction with the supposition (P2). Thus $\bar{x}_p \leqq \bar{y}_p$. The implication $\bar{x}_p \leqq \bar{y}_p \Rightarrow \bar{x} \leqq \bar{y}$ follows from 3.3.

By this Lemma is proved.

5.3. Lemma: *Let A be a complete lattice, Θ a complete congruence on A, $\bar{A} = A/\Theta$. Let P be a projective selection in $\bar{A}$, B a closed sublattice in A. Denote $\bar{B} = \{\bar{x} \mid \bar{x} \in \bar{A}$, $\bar{x} \cap B \neq \emptyset\}$. Then $P \cap \{x \mid x \in \bar{b},\ \bar{b} \in \bar{B}\}$ is a projective selection in $\bar{B}$.*

The proof is evident.

5.4. Theorem: *A complete lattice A is a projective retract in $\mathfrak{L}$ if and only if for arbitrary complete lattice B, and for arbitrary complete congruence Θ on B such that $A \cong B/\Theta$, there exists a projective selection in B/Θ.*

Proof: I. Let A be a projective retract, B a complete lattice, and $g : B \to A$ an epimorphism. By suppositions, there exists a complete homomorphism $h : A \to B$ such that $gh = id_A$. By Lemma 3.11, we have $B \mid \mathrm{Ker} f \cong A$, $h(A) \subseteqq B$ and it is obvious that $h(A)$ is a projective selection in $B \mid \mathrm{Ker} f$.

II. Let A, B be complete lattices, let $g : B \to A$ be an epimorphism. Let the projective selection P exist in $\bar{B} = B \mid \mathrm{Ker} f$. Let us denote $p(\bar{b}) = P \cap \bar{b}$ for $\bar{b} \in \bar{B}$. Let us define the mapping $h : A \to B$ as follows:

$$h(x) = p[g^{-1}(x)] \qquad \text{for } x \in A.$$

Then obviously h is a complete homomorphism and $gh = id_A$ is valid, so that A is a projective retract.

5.5. Theorem: *A complete lattice A is a projective object in $\mathfrak{L}$ if and only if any of its $(0, 1)$-component is a projective object in $\mathfrak{L}$.*

Proof: I. Let A be a projective complete lattice, K its $(0, 1)$-component. Let $A \neq K$. Let B, C be complete lattices, $g : B \to C$ an epimorphism, and $f : K \to C$ a complete homomorphism. Let us denote $f(K) = C_1$, $g^{-1}(C_1) = B_1$. Then $f : K \to C_1$ is an epimorphism, C_1 a closed sublattice in C and by Lemma 3.11, there holds $K \mid \mathrm{Ker} f \cong$ $\cong C_1 \cong B_1 \mid \mathrm{Ker}\, g_1$, where $g_1 = g \mid B_1$. Let us denote $H = A - K$. By assumption, we have $H \neq \emptyset$ and obviously $H \cup \{0_A, 1_A\}$ is a closed sublattice in A. Let H_1 be an arbitrary ordered set such that $H_1 \cong H$, $H_1 \cap (B \cup C) = \emptyset$, and let us denote φ an isomorphism of the set H on the set H_1. Let us put $B_2 = B \cup H_1$, $C_2 = C \cup H_1$, and let us define the relation $\leqq$ on B_2 (and analogously on C_2) as follows:

For $x, y \in B_2$, we have $x \leqq y$ if and only if:

$x \in B$, $y \in B$, $x \leqq y$ in B, or
$x \in H_1$, $y \in H_1$, $x \leqq y$ in H_1, or
$x \in B$, $y \in H_1$, $x \leqq 0_{B_1}$ in B, or
$x \in H_1$, $y \in B$, $y \geqq 1_{B_1}$ in B.

The sets B_2, C_2 with the ordering defined in this way are obviously complete lattices, B and B_1 are closed sublattices in B_2, C and C_1 are closed sublattices in C_2.

Let us define the mapping $g_2 : B_2 \to C_2$ in the following way:

$$g_2(x) = \begin{cases} g(x) & \text{for } x \in B, \\ x & \text{for } x \in H_1, \end{cases}$$

and the mapping $f_2 : A \to C_2$ as follows:

$$f_2(x) = \begin{cases} f(x) & \text{for } x \in K, \\ \varphi(x) & \text{for } x \in H. \end{cases}$$

Then g_2, f_2 are obviously complete homomorphisms and moreover g_2 is an epimorphism. Since A is projective, there exists a complete homomorphism $h_2 : A \to B_2$ such that $f_2 = g_2 \circ h_2$. But then $f = f_2 \mid K$, $g = g_2 \mid B$ and $h = h_2 \mid K : K \to B$ is a complete homomorphism such that $f = gh$, so that K is projective.

II. Let A be a complete lattice whose each $(0, 1)$-component is projective. Let B, C be arbitrary complete lattices, let $g : B \to C$ be an epimorphism and $f : A \to C$ a complete homomorphism. Let us denote by $\mathscr{K}$ the system of all (0, 1)-components in A, let $K \in \mathscr{K}$ be arbitrary. Let us denote $f_K = f \mid K$, $C_K = f(K)$, $B_K = g^{-1}(C_K)$, $g_K = g \mid C_K$. Then $g_K : B_K \to C_K$ is an epimorphism and since the complete lattice K is projective, there exists $h_K : K \to B_K$ such that $g_K \circ h_K = f_K$. With regard to the fact that for $K, L \in \mathscr{K}$, $K \neq L$, there holds $K \cap L = \{0_A, 1_A\}$, the mapping $h : A \to B$, defined by the relation

$$h(x) = h_K(x) \qquad \text{for } x \in A \cap K, K \in \mathscr{K},$$

will be a complete homomorphism, if it is possible to select a mapping h_K such that for arbitrary K, $L \in \mathscr{K}$, there may hold $h_K(1_A) = h_L(1_A)$, $h_K(0_A) = h_L(0_A)$. By 3.13, it is possible to put $h_K(0_A) = p$, $h_K(1_A) = q$ for every $K \in \mathscr{K}$, where p is the greatest element of the closed sublattice $g^{-1}[f(0_A)]$ of the lattice B and $q = \inf_B \{t \mid t \in g^{-1}[f(1_A)], t \geqq p\}$. But then $h : A \to B$ is a complete homomorphism and $gh = f$ is valid, so that A is a projective object in $\mathfrak{L}$.

We shall prove now an analogous statement also for projective retracts.

5.6. Theorem: *A complete lattice is a projective retract if and only if each of its $(0, 1)$-component is a projective retract.*

Proof: I. Let a complete lattice A be a projective retract. Let K be its arbitrary $(0, 1)$-component. Let B be an arbitrary complete lattice and Θ an arbitrary complete congruence on B such that $K \cong B/\Theta$. By Theorem 5.4, it is sufficient to prove that in $B/\Theta = \bar{B}$ there exists a projective selection.

Let us denote $H = A - K$. If $H = \emptyset$, there is nothing to be proved. Thus let $H = \emptyset$. Let p be the greatest element in $\overline{O_B}$, $q = \inf_B \{t \mid t \in \overline{1_B}, t \geqq p\}$, where $\overline{O_B}$ is the smallest element and $\overline{I_B}$ the greatest element of the set $\bar{B} = B/\Theta$. Furthermore let H_1 be an arbitrary ordered set such that $H_1 \cong H$, $H_1 \cap B = \emptyset$. Let us put $B_1 = \{p\} \oplus [\{p, q\} + H_1] \oplus \{q\}$. Then B_1 is obviously a complete lattice. Now let us define the relation ϱ on B_1 as follows:

$x \varrho y$ if and only if (a) $x \Theta y$, or
(b) $x \in H_1, y \in H_1, x = y$.

Then ϱ is a complete congruence on B_1 and obviously $B_1/\varrho \cong A$ holds. Since A is a projective retract, there exists a projective selection P_1 in B_1/ϱ. But $\langle p, q\rangle \cap B$ is obviously a closed sublattice in B, and by 5.3, and 5.13 $P_1 \cap B$ is a projective selection in $\bar{B}$. It means that K is a projective retract.

II. Let A be a complete lattice whose each $(0, 1)$-component is a projective retract. Let $\mathscr{K}$ be a system of all $(0, 1)$-components in A. Furthermore let B an arbitrary complete lattice, Θ a complete congruence on B such that $A \cong B/\Theta$. Let $\varphi : A \to B/\Theta$ be a given isomorphism. To complete the proof that A is a projective retract, it is sufficient to prove by Theorem 5.4 that in $\bar{B} = B/\Theta$ there exists a projective selection.

Let us denote $\varphi_K = \varphi \mid K$, $B_K = \varphi(K)$, $\bar{B}_K = \{\bar{b} \mid \bar{b} \in \bar{B}, \bar{b} \cap B_K \neq \emptyset\}$ for $K \in \mathscr{K}$. B_K is a closed sublattice in B and by assumption, there exists a projective selection in $\bar{B}_K$ denoted by P_K. By 3.13 the projective selection P_K may be chosen so that $P_K \subseteq \subseteq \langle p, q \rangle$, where p is the greatest element in $\overline{O_B}$ and $q = \inf_B \{t \mid t \in \overline{1_B}, t \geq p\}$. But then $\bigcup_{K \in \mathscr{K}} P_K$ is evidently a projective selection in $\bar{B}$.

5.7. Theorem: *If a complete lattice is a projective retract, it does not contain JM-reducible element.*

Proof: Let A be a complete lattice, $a \in JM(A)$ JM-reducible element. Let us put $B = A \cup \{a_1\}$, where $a_1 \notin A$, and let us define the ordering $\leq_B$ of the set B as follows:

For $x \in A$, $y \in A$, we have $x \leq_B y$ if and only if $x \leq y$ in A.

For $x \in A$, there holds: $x \leq_B a_1$ if and only if $x \leq a$ in A,

$$a_1 \leq_B x \text{ if and only if } a < x \text{ in } A.$$

It is easy to see that $(B, \leq_B)$ is a complete lattice. Let us define now the mapping $f: B \to A$ as follows:

$$f(x) = \begin{cases} x & \text{for } x \in B, x \neq a_1, \\ a & \text{for } x = a_1. \end{cases}$$

Then evidently $f \in H_{\mathfrak{L}}(B, A)$ and it is an epimorphism. In A, by assumption, there exist sets $X \neq \emptyset \neq Y$ such that $a \notin X \cup Y$, $a = \sup_B X = \inf_A Y$. But then a projective selection does not exist in $B \mid Ker f$ though $A \cong B \mid Ker f$, since $\sup_B X = a$, $\inf_B Y = a_1$ and thus it is not possible to select in a suitable manner a unique element in the interval $\langle a, a_1 \rangle \in B \mid Ker f$.

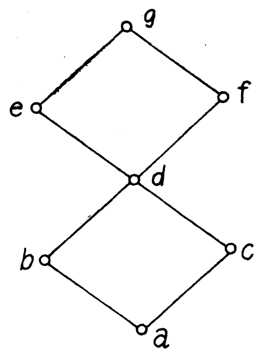
From Theorem 5.7, there directly follows

5.8. Corollary: *If a complete lattice contains a JM-reducible element, then it is not a projective object of the category $\mathfrak{L}$.*

5.9. Example: (a) Let A be the lattice shown in Fig. 4a. The element d is JM-reducible element in A. If we form the lattice B in the same manner as in the proof of Theorem 5.7, we get the lattice in Fig. 4b. Now it is obvious that if we define $f: B \to A$, so that $f(d_1) = d$, $f(x) = x$ for the other $x \in B$, there does not exist a projective selection in $B \mid Ker f$. Thus A is not a projective retract.

(b) Let A be a closed interval $\langle 0, 1 \rangle$ of real numbers. Then $JM(A) = (0, 1)$ and if we double an arbitrary element $x \in (0, 1)$ analogously to the example (a), it is again evident that A is not a projective retract.

It will be seen later that it is not possible to reverse in general the statement of Theorem 5.7 and its Corollary respectively. But it will be shown now that the mentioned necessary condition is a sufficient condition for chains.



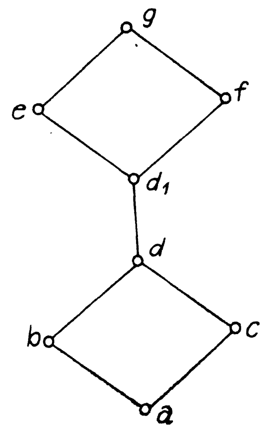


Fig. 4

5.10. Theorem: *A chain $A \in \mathfrak{L}$ is a projective object of the category $\mathfrak{L}$ if and only if it does not contain JM-reducible element.*

Proof: I. If $JM(A) \neq \emptyset$ holds, A is not projective by Lemma 5.7.

II. Let a chain $A \in \mathfrak{L}$ contain no JM-reducible element, i.e., $JM(A) = 0$. Let $B, C \in \mathfrak{L}$ be arbitrary, let $f: A \to B$ be a complete homomorphism, let $g: C \to B$ be an epimorphism. Let us denote $\bar{A} = A \mid Ker f$, $\bar{C} = C \mid \mathrm{Ker} g$, $C_1 = g^{-1}\{f(A)\}$, $g_1 = g \mid C_1$. It is clear that C_1 is a closed sublattice in C and $\bar{C}_1 = C_1 \mid \mathrm{Ker} g_1$ is a closed sublattice in $\bar{C}$. Let us now construct the mapping $\bar{h}: \bar{A} \to \bar{C}$ as follows:

For $\bar{x} \in \bar{A}$, we have $\bar{h}(\bar{x}) = g^{-1}[f(x)]$, where x is an arbitrary element of $\bar{x}$. By Lemma 3.11, $\bar{h}$ is an isomorphism $\bar{A}$ in $\bar{C}$ (since $\bar{h}: \bar{A} \to \bar{C}_1$ is an isomorphism). Now let us construct $h: A \to C$ in the following way:

Let us denote $i_x = \sup_C \bar{h}(\bar{x})$ for every element $x \in A$. There exists the element i_x for every $x \in A$. Now for every element $x \in A$ there occurs one of the two following possibilities:

(α) $\bar{x} \notin J(\bar{A})$,
(β) $\bar{x} \in J(\bar{A})$ (i.e., $\bar{x}$ is J-reducible in $\bar{A}$).

Now let us define

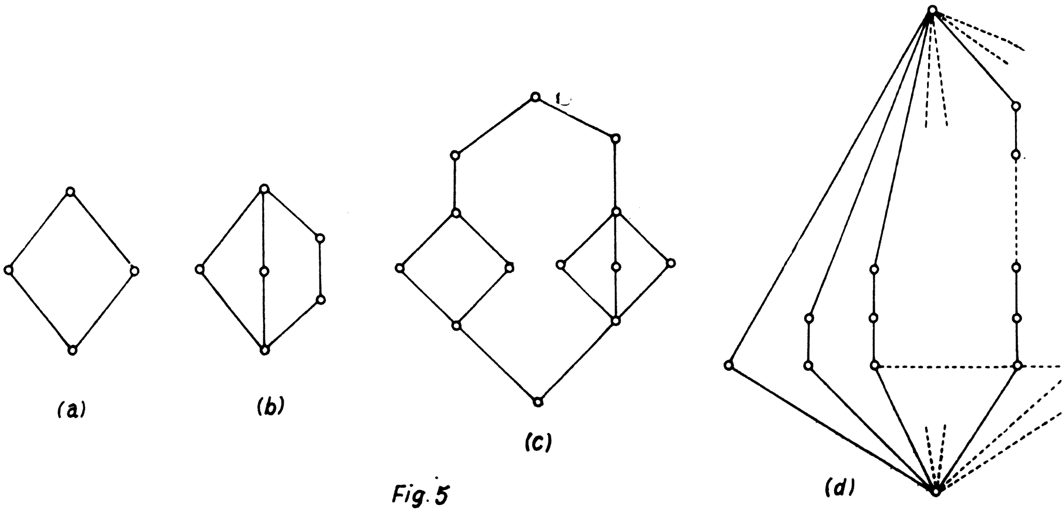
$$(\alpha)\ h(x) = i_x; \qquad (\beta)\ h(x) = \begin{cases} i_x & \text{for every } x \neq x_0 (= 0_p), \\ \sup_C \{i_y \mid y \in A, \bar{y} < \bar{x}\} & \text{for } x = x_0 . \end{cases}$$

It is evident that $h : A \to C$ is a complete homomorphism and $gh = f$ is valid. Thus A is a projective object in $\mathfrak{L}$.

Further we get

5.11. Theorem: *Let A be a complete lattice such that $JM(A) = \emptyset$. Let $a \in R(A)$ be the smallest and $b \in R(A)$ the greatest element of the closed sublattice $R(A)$. If $\langle a, b\rangle$ is a projective object (resp. projective retract) in $\mathfrak{L}$, then A is a projective object (resp. projective retract) in $\mathfrak{L}$.*

Proof: There evidently holds $A = K \oplus \langle a, b\rangle \oplus L$, where of the sets K, L is either empty, or it is a chain without reducible elements. The statement of the theorem is now quite evident.



(a) (b) (c) (d)

Fig. 5

5.12. Examples: It follows from 5.5, 5.10, and from 5.11 that e.g., the lattices in Figs. 5a – 5d are projective objects of the category $\mathfrak{L}$.

Now another necessary condition will be stated for a complete lattice to be a projective object or a projective retract in $\mathfrak{L}$, respectively. But first let us mention two auxiliary statements.

5.13. Lemma: *Let A, B be complete lattices. Then the cardinal product $A \times B$ is a complete lattice.*

Proof is evident.

5.14. Lemma: *Let A be a complete lattice, let $a, b \in A$ be arbitrary such that $a \leqq b$. Let $B = \{t_1, t_2\}$, $t_1 < t_2$ be two-element chain such that*

$$(A - \langle a, b\rangle) \cap (\langle a, b\rangle \times B) = \emptyset.$$

Let us define the relation $\leqq_C$ on the set

$$C = (A - \langle a, b\rangle) \cup (\langle a, b\rangle \times B)$$

as follows:

For $x, y \in C$, there holds $x \leqq_C y$ if and only if there occurs one of the following possibilities:

(*a*) $x, y \in A - \langle a, b\rangle, x \leqq y$ *in* A,
(*b*) $x, y \in \langle a, b\rangle \times B, x \leqq y$ *in* $\langle a, b\rangle \times B$,
(*c*) $x \in A - \langle a, b\rangle, y = [u, t_i] \in \langle a, b\rangle \times B, x < u$ *in* A,
(*d*) $x = [u, t_i] \in \langle a, b\rangle \times B, y \in A - \langle a, b\rangle, u < y$ *in* A.

Then $(C, \leqq_C)$ is a complete lattice.

Proof: It is clear that the relation $\leqq_C$ is an ordering on the set C. Now let $X \subseteqq C$, $X \neq \emptyset$ be an arbitrary set. It will be shown e.g., that there exists $\inf_C X$.

For $X \subseteqq A - \langle a, b\rangle$, there exists $\inf_A X$. If $\inf_A X \in A - \langle a, b\rangle$, then obviously $\inf_C X = \inf_A X$. If $\inf_A X \in \langle a, b\rangle$, then $\inf_C X = [\inf_A X, t_2]$.

For $X \subseteqq \langle a, b\rangle \times B$, let us denote $X_1 = \{y \mid y \in \langle a, b\rangle, [y, t_i] \in X \text{ for } i = 1, \text{ or } i = 2\}$, and $B_X = \{t_i \mid t_i \in B, [x, t_i] \in X \text{ for some } x \in \langle a, b\rangle\}$. Then there evidently holds $\inf_C X = [\inf_A X_1, \inf_B B_X]$. Finally, if $X \subseteqq C$ is such a subset that $X \cap \cap (\langle a, b\rangle \times B) \neq \emptyset$, $X \cap (A - \langle a, b\rangle) \neq \emptyset$, there holds

$$\inf_C X = \inf_C \{\inf_C (X \cap (\langle a, b\rangle \times B)),\ \inf_C (X \cap (A - \langle a, b\rangle))\},$$

where there exists this greatest lower bound.

Analogously, there may be proved the existence of $\sup_C X$.

By this Lemma 5.14 is proved.

5.15. Theorem: *If a complete lattice is a projective retract in the category $\mathfrak{L}$, it does not contain an aprojective couple.*

Proof: Let A be a complete lattice and let the elements $a, b \in A$, $a < b$ form in A an aprojective couple. Let B, C have the same meaning as in Lemma 5.14 i.e., it holds: $C = (A - \langle a, b\rangle) \cup (\langle a, b\rangle \times B)$. Let us define the mapping $f: C \to A$ as follows:

$$f(x) = \begin{cases} x & \text{for } x \in A - \langle a, b\rangle, \\ x_1 & \text{for } x = [x_1, t_i] \in \langle a, b\rangle \times B. \end{cases} \tag{*}$$

It is easy to see that f is an epimorphism, $f \in H_{\mathfrak{L}}(C, A)$. By Lemma 3.11, $A \cong C/\text{Ker}f$. It will be shown now that in $\bar{C} = C/\text{Ker}f$ there does not exist the projective selection.

It follows from the definition of the mapping f that for $\bar{c} \in \bar{C}$, there holds either $\bar{c} = \{c\}$, where $c \in A - \langle a, b\rangle$, or $\bar{c} = \{[x, t_1], [x, t_2]\} \subseteqq \langle a, b\rangle \times B$. By assumption, the elements $a, b \in A$ form an aprojective couple in A i.e., (see Definition 2.9) there exist the sets $X_1, X_2 \subseteqq A - \langle a, b\rangle$ such that $a = \inf_A X_1$, $b = \sup_A X_2$. Since

$f|(A - \langle a, b\rangle) = id_{A-\langle a,b\rangle}$, we have $f^{-1}(X_1) = X_1$, $f^{-1}(X_2) = X_2$. But then evidently

$$\inf_C X_1 = [a, t_2], \qquad \sup_C X_2 = [b, t_1].$$

From this follows that a projective selection does not exist in $\bar{C}$, because $\overline{[a, t_2]} < < \overline{[b, t_1]}$ and it is not possible to select the representants in these classes in such a way that we may get a projective selection, since $[a, t_2] \parallel [b, t_1]$.

Thus A is not a projective retract.

5.16. Corollary: *If a complete lattice is a projective object of the category $\mathfrak{L}$, it does not contain an aprojective couple.*

5.17. Examples: (a) It follows from 5.15, and 5.16 that no lattice of those in Figs. 3a–3d of Example 2.11 is a projective retract and thus no one of them is a projective retract and thus no one of them is a projective object of the category $\mathfrak{L}$.

(b) Let A be the lattice in Fig. 6a where the elements a, b form an aprojective couple. Let us form the lattice C in the way described in the proof of Lemma 5.14 (see Fig. 6b).

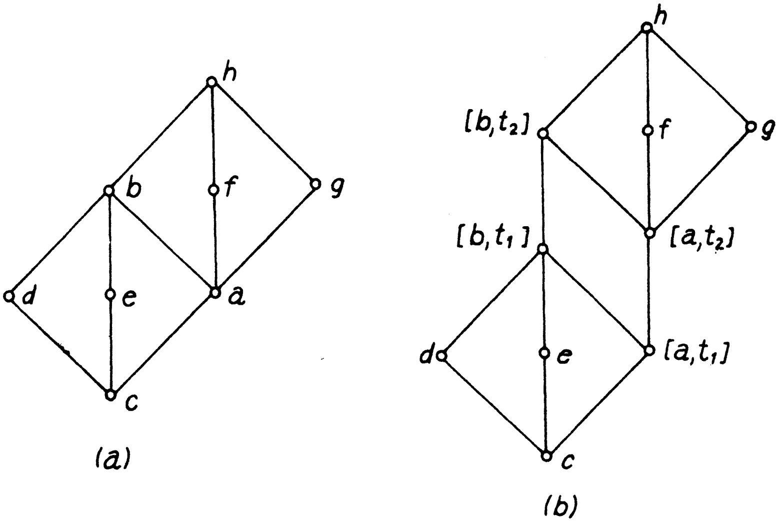


Fig. 6

Now it is obvious that in lattice C it holds: $f \wedge g = [a, t_2]$, $d \vee e = [b, t_1]$. If we define the mapping $f: C \to A$ by the relation (*) in the proof 5.15, then $C/\mathrm{Ker} f \cong A$ and in $C/\mathrm{Ker} f$ there does not exist a projective selection, since the only closed sublattice in C, containing at least one element of every class of the decomposition $C/\mathrm{Ker} f$, is obviously the lattice C itself.

5.18. Remark: The statements of Theorems 5.7, and 5.15 may be summarized in the following way: we say *the complete lattice contains an aprojective subset if it contains JM-reducible element or an aprojective couple.* Then from 5.7, and 5.15 there follows:

If a complete lattice is a projective retract in the category $\mathfrak{L}$, it does not contain an aprojective subset.

It will be shown that the mentioned necessary condition is by certain assumptions also a sufficient condition.

5.19. Theorem: *Let A be a ramified complete lattice such that the set $R(A)$ of its reducible elements is finite. Then A is a projective retract in $\mathfrak{L}$ if and only if it does not contain an aprojective subset.*

Proof: If A is a projective retract, it does not contain an aprojective subset by 5.18.

Let now A be a ramified complete lattice such that $R(A)$ is finite and A does not contain an aprojective subset. We shall prove that A is a projective retract.

Let $C \in \mathfrak{L}$ be arbitrary, $f: C \to A$ be an epimorphism. By Theorem 5.4 it suffices to prove that in $\bar{C} = C/\mathrm{Ker} f$ there exists a projective selection. A construction of this projective selection will be mentioned in conclusion of this proof. But first let us be aware of the requirements that such a construction must satisfy.

Since $\bar{C} \cong A$, then also $R(\bar{C}) \cong R(A)$ and thus $R(\bar{C})$ is a nonempty finite subset in $\bar{C}$ and at the same time $\bar{C}$ does not contain an aprojective subset. From Theorems 2.2 and 5.3 there follows further that if P is a projective selection in $\bar{C}$, then $\{x \mid x \in P, \bar{x} \in R(\bar{C})\}$ is a projective selection in $R(\bar{C})$. When constructing a projective selection in $\bar{C}$, the choice of representants in the classes of the set $R(\bar{C})$ is obviously "critical" and that is why we shall choose first the representants in these classes. To be, however, our construction correct, we must obviously choose representants of classes from $R(\bar{C})$ so as

(1) to obtain a projective selection in $R(\bar{C})$,

(2) to be able to complete this projective selection by choosing representants of classes from $\bar{C} - R(\bar{C})$.

Requirement (1) in our construction can be satisfied by choosing the greatest "admissible" elements as representants in classes from $M^\circ(\bar{C})$, and the smallest "admissible" elements as representants in classes $J^\circ(\bar{C})$. (The meaning of the word "admissible" will be cleared in the construction).

If we construct the projective selection Q in $R(\bar{C})$ in this manner, for fulfilling requirement (2) it is necessary to satisfy the following conditions: If $\bar{a}, \bar{b} \in R(\bar{C})$, $\bar{a} < \bar{b}$ are arbitrary, $a \in \bar{a} \cap Q$, $b \in \bar{b} \cap Q$, and $\bar{c} \in \bar{C}$ is arbitrary such that $\bar{a} < \bar{c} < \bar{b}$, then there must be $\bar{c} \cap \langle a, b\rangle \neq \emptyset$ for to choose a representant in the class $\bar{c}$. From Lemma 3.13, however, it follows that this set is nonempty. From the construction of the set Q it immediately follows that in the set $\bar{c} \cap \langle a, b\rangle$ any element may be chosen as a representant.

Thus, the construction of projective selection in $R(\bar{C})$ will be as follows. The choice of representants in these classes is naturally in a certain manner forced because each element $\bar{a} \in R(\bar{C})$ is subjected to certain "connections". At the same time it is not difficult to mention that the choice of a representant in $\bar{a} \in R(\bar{C})$ is the more complicated the higher is the characteristic of this element $\bar{a}$. (See Definition 2.3). Therefore we shall proceed from the elements with the highest characteristic to those with the lowest characteristic and in the sets $SM^k(\bar{C}) \cup SJ^k(\bar{C})$ we shall choose representants by layers (see Definition 2.7). The fact, that in the set $\bar{C}$ an aprojective subset does not exist, ensures that the choice of more than one representant cannot be forced in a class $\bar{a} \in R(\bar{C})$ (compare e.g. with 5.17).

And now

Construction: The following terminology will be used in the whole construction: Let a representant x be chosen in the class $\bar{x} \in R(\bar{C})$. Let $\bar{y} \in \bar{C}$, $\bar{y}$ be comparable with $\bar{x}$ in $R(\bar{C})$. Then the elements of the set $\{t \mid t \in \bar{y},\ t$ is comparable with the element x in $C\}$ will be called the admissible elements of the class $\bar{y}$. By Lemma 3.13 the set of all *admissible* elements in $\bar{y}$ is nonempty and there exists the greatest element and the least element in this set (since admissible elements in $\bar{y}$ forms obviously a complete lattice).

As a representant of the class $0_{\bar{C}}$ let us now choose its greatest element, as a representant of the class $1_{\bar{C}}$ its least admissible element (obviously $0_{\bar{C}}$, $1_{\bar{C}} \in R(\bar{C})$ since $\bar{C}$ is ramified).

Only finitely many sets $SM^k(\bar{C}) \cup SJ^k(\bar{C})$ are nonempty since the set $R(\bar{C})$ is finite. Let now k be the greatest number such that $SM^k(\bar{C}) \cup SJ^k(\bar{C}) \neq \emptyset$. Then we shall choose the representants in classes of this set in the following way:

(a) $SJ^k(\bar{C}) \neq \emptyset$: We shall choose the least admissible elements in the sets $\bar{x} \in SJ^k_1(\bar{C})$, and then the greatest admissible elements in the sets $\bar{x} \in SM^k_1(\bar{C})$. After having determined the representants in classes from $SJ^k_i(\bar{C}) \cup SM^k_i(\bar{C})$, we choose the least admissible elements in the sets $\bar{x} \in SJ^k_{i+1}(\bar{C})$ and then the greatest admissible elements in the classes from $SM^k_{i+1}(\bar{C})$, etc. Since the set $SJ^k(\bar{C}) \cup SM^k(\bar{C})$ is finite, then after finite number of steps we choose the representants in all classes from this set.

(b) $SJ^k(\bar{C}) = \emptyset$: Then necessarily $SM^k(\bar{C}) \neq \emptyset$ and representants are chosen as follows:

We choose the greatest admissible elements in classes from $SM^k_1(\bar{C})$, then the greatest admissible elements in classes from $SM^k_2(\bar{C})$ etc. After a finite number of steps a choice of representants in all classes of this set is obtained.

After having chosen representants in the classes from $SM^k(\bar{C}) \cup SJ^k(\bar{C})$, we choose quite analogously representants in the classes from $SM^i(\bar{C}) \cup SJ^i(\bar{C})$, where i is the greatest number such that $i < k$ and $SM^i(\bar{C}) \cup SJ^i(\bar{C}) \neq \emptyset$ (at the same time the "admissibility" of elements is obviously influenced by the previous choice of representants in $SM^k(\bar{C}) \cup SJ^k(\bar{C})$) etc. After a finite number of steps we choose representants in the whole set $R(\bar{C})$ and from the construction it is obvious that we have got a projective selection in $R(\bar{C})$. A construction of projective selection in the whole set $\bar{C}$ is now quite simple since in remaining classes from $\bar{C}$ (i.e. in classes $\bar{x} \in \bar{C} - R(\bar{C})$) we can choose arbitrary admissible elements as representants.

5.20. Remark: The statement of Theorem 5.19 can be extended also to complete lattices that are not ramified. Namely, it is obvious that in case $A \in \mathfrak{L}$ is not a chain, then $A = P \oplus Q \oplus G$, where the sets P, G—provided they are nonempty—are chains and Q is a ramified set. At the same time it obviously holds

5.21. Theorem: *Let $A_1, A_2, \ldots, A_n \in \mathfrak{L}$. Then the complete lattice $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ is a projective retract in $\mathfrak{L}$ if and only if every complete lattice A_i, $(i = 1, 2, \ldots, n)$ is a projective retract in $\mathfrak{L}$.*

5.20, 5.21 and 5.10 imply

5.22. Theorem: *Let A be a complete lattice. Let $JM(A) = \emptyset$ and $A = P \oplus Q \oplus R$, where P, R are empty or complete chains and Q is a ramified complete lattice such that $R(Q)$ is a finite set and Q does not contain an aprojective couple. Then A is a projective retract in $\mathfrak{L}$.*

5.23. Example: In the case when α is an ordinal number, let us denote $W(\alpha) = \{\beta \mid \beta$ is an ordinal number, $\beta < \alpha\}$. From 5.22 it follows that if α is an isolated ordinal number and C is a complete lattice from the example 5.17 (see Fig. 6b), then $W(\alpha) \oplus C$ is a projective retract in $\mathfrak{L}$.

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