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On some properties of the metric dimension

LADISLAV MIŠÍK JR., TIBOR ŽÁČIK

Abstract. In the paper two covering functions N, M defined on a given compact metric space K are studied; their binary logarithms are usually called ε-entropy and ε-capacity of this space, respectively. For a function u with suitable properties a compact countable metric space, for which the function u is the covering function, is constructed. By means of covering functions the both lower $\underline{\dim}$ and upper $\overline{\dim}$ metric dimensions of K are defined. It is shown that for a given compact metric space K and every $\alpha \in [0, \overline{\dim}\ K]$ and $\beta \in [0, \underline{\dim}\ K]$ there is a compact countable subspace X of K with the unique cluster point such that $\overline{\dim}\ X = \alpha$ and $\underline{\dim}\ X \leq \beta$. Finally, it is shown that there exist compact spaces with arbitrary small $\overline{\dim}$ which are not isometrically embeddable into $\mathbf{R}^m$ for each $m \in \mathbf{N}$.

Keywords: Covering function, metric dimension, entropy dimension, limit capacity

Classification: Primary 54D20, 54F45; Secondary 51K99

Introduction.

There are two well-known numerical characteristics of the "massiveness" of metric spaces: topological dimension td, which is a natural number in any case, and Hausdorff dimension hd, which need not be an integer. In [**PS**] a new characteristic is defined, which is in [**KT**] called a lower metric dimension $\underline{\dim}$. Hereby the upper metric dimension $\overline{\dim}$ was defined here. Both these dimensions are given by some integer-valued functions, the covering functions, defined for totally bounded subsets of a metric space. Binary logarithms of these functions are called an ε-entropy and an ε-capacity ([**KT**]) of the metric space, respectively. That is why the metric dimension ([**CS**], [**H**], [**KT**], [**V**]) is also called an entropy dimension ([**B**], [**P**], [**Y**]) or a limit·capacity ([**M**], [**PT**]). The basic relations of the different notions of the dimensions for a totally bounded metric space K, are given by the inequalities:

$$\text{td } K \leq \text{hd } K \leq \underline{\dim}\ K \leq \overline{\dim}\ K, \quad \text{(see e.g. [P], [V])}.$$

The main difference between hd and $\underline{\dim}$ consists in the fact that hd $X = 0$ for a countable set X , while $\underline{\dim}\ X$ can be positive. So $\underline{\dim}$ and $\overline{\dim}$ can better control the partition of points of the metric space. On the other hand it is proved in [**V**] and [**CS**] that there exists a perfect subset $X \subset \mathbf{R}$ with prescribed Hausdorff and metric dimensions. In the general case a perfect subset D of a complete metric space A with hd $A = 0$ and $\underline{\dim}\ D = \underline{\dim}\ A$ is constructed there. Note that in [**CS**] a slightly different notion of metric dimension is used. Some other different properties of hd and $\underline{\dim}$ can be found in [**B**] as well.

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The aim of this paper is to describe the behavior of the covering functions and some of the properties of $\underline{\dim}$ and $\overline{\dim}$ for compact metric spaces. The main result of the first section is the fact that for every covering-like integer-valued function there is a compact, countable subspace of l^∞ with the unique cluster point, covering function of which is the given function. Further, for compact subspaces of $\mathbf{R}^m$ upper bounds of the "jumps" in points of discontinuity of the covering function N are shown. It is shown in the second section that every compact metric space contains a countable subset fulfilling some requirements given in advance. Then the consequence is the existence of a subset $X \subset K$ such that $\underline{\dim}\ X = 0$ and $\overline{\dim}\ X = \overline{\dim}\ K$. The last theorem of the paper says that in spite of finiteness of upper metric dimension of a metric space K there need not exist an isometrical embedding of K to any finite dimensional Euclidean space $\mathbf{R}^m$.

1. Covering functions N and M.

Let (K, d) be a nonempty compact metric space. For $p \in K$ and $r > 0$ denote by $B(p, r)$ an open ball centered in p with radius r and by $\overline{B(p, r)}$ its closure. Let $\mathbf{N}$ be the natural numbers and $\mathbf{R}$ the reals. Define the covering function $N(., K) : \mathbf{R}^+ \to \mathbf{N}$, where $N(r, K)$ for every $r > 0$ denotes the least number of open balls with radius r covering K.

The compactness of K implies that $N(r, K)$ is finite for each $r > 0$, so the function $N(., K)$ is well defined. In general this function is defined for totally bounded spaces. In this paper we shall need one more function $M(., K) : \mathbf{R}^+ \to \mathbf{N}$:

For a set $F \subset K$ denote $\mu(F) = \inf\{d(x, y); x, y \in F, x \neq y\}$. We shall call a finite set F r-discrete, for $r > 0$, if $\mu(F) \geq r$. Now the number $M(r, K)$ means the maximal cardinality of r-discrete subsets of K.

There are some similar functions defined for totally bounded metric spaces. Their properties and mutual relations can be found in [**KT**]. Note that in [**KT**] the functions N and M are defined dually in some sense: $N(r, K)$ is defined by means of closed sets of diameter $2r$ and to get $M(r, K)$ only finite sets F with $\mu(F) > r$ are taken.

In the following Proposition 1 the basic properties of the functions N and M are summarized.

Proposition 1. *Let K be a compact metric space and $r > 0$. Then the following hold:*

(i) *Let $A \subset K$ be a compact set. Then*
$N(2r, A) \leq N(r, K), \quad M(r, A) \leq M(r, K)$.

(ii) *Let $K = K_1 \cup K_2$, where K_1, K_2 are compact subsets of K. Then*
$N(r, K) \leq N(r, K_1) + N(r, K_2), \quad M(r, K) \leq M(r, K_1) + M(r, K_2)$.

(iii) *If $K_1, K_2 \subset K$ are compact and $r \leq d(K_1, K_2)$, then*
$N(r, K_1 \cup K_2) = N(r, K_1) + N(r, K_2)$,
$M(r, K_1 \cup K_2) = M(r, K_1) + M(r, K_2)$.

(iv) *If $F \subset K$ is an r-discrete set and $0 < p \leq r$ then $N(p, F) = M(p, F) = |F|$, where $|F|$ denotes the cardinality of F.*

(v) $M(2r, K) \leq N(r, K) \leq M(r, K)$.

PROOF : The proof is left to the reader. ■

Remarks. (a) Note that if $A \subset K$ then $M(r,A) \leq M(r,K)$ is valid although the inclusion $A \subset K$ does not imply the inequality $N(r,A) \leq N(r,K)$, as the following example shows: Let $K = \{0,1,2\}$ and $A = \{0,2\}$. Then $B(1,2) \supset K$ and hence $N(2,K) = 1$, while $N(2,A) = 2$.

(b) One can show by induction that (ii) and (iii) holds for any finite number of subsets .

(c) With regard to the inequalities in (v),

$$M(2r,K) \leq N(r,K) \leq M(r,K), \tag{1}$$

we also call M the covering function.

The basic behavior of the functions $N(.,K)$ and $M(.,K)$ for a fixed compact K is given in the following:

Theorem 1. *The function $N(.,K) : \mathbf{R}^+ \to \mathbf{N}$ is piecewise constant, continuous on the left, nonincreasing and the set of all points of discontinuity of N can be arranged into a decreasing sequence $\{r_n\}$, such that $N(r,K) = 1$ for $r > r_1$. K is infinite iff $\{r_n\}$ is infinite, and then $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} N(r_n,K) = \infty$. The same is valid for the function M.*

PROOF : Let $\{B(y_i,r)\}_{i=1}^{N(r,K)}$ be a covering of K. Then for $s > r$ $\{B(y_i,s)\}_{i=1}^{N(r,K)}$ is the covering of K, too. Therefore the function N is nonincreasing and since the values of N are integers, N is piecewise constant. Denote $Y = \{y_1,\dots,y_{N(r,K)}\}$ and define a function $f : K \to \mathbf{R}_0^+$ by $f(x) = d(x,Y)$. The function f is continuous and attains on K the maximum $r^* < r$. Then for an arbitrary $\rho \in (r^*,r)$ one has $N(\rho,K) = N(r,K)$ and hence N is continuous on the left.

Let $X = \{x_1,\dots,x_{M(r,K)}\}$ be an r-discrete set of K and $s < r$. Then X is also s-discrete set and therefore $M(s,K) \geq M(r,K)$. So M is nonincreasing and piecewise constant. Let $\{s_k\}_{k=1}^{\infty}$ be an increasing sequence of real numbers tending to $r > 0$ such that $M(s_i,K) = M(s_j,K)$, $i \neq j$, and denote by X_k the corresponding s_k-discrete set of K. As 2^K is the compact metric space in Hausdorff metric h, we can choose a convergent subsequence $\{X_{k_i}\}$ in this metric tending to some set X_0. Then X_0 is a finite set with $|X_0| = |X_{k_i}|$, $i \geq 1$, and $\mu(X_0) \geq r$. So the function M is continuous on the left.

The last statement of the theorem is obvious. ■

From the foregoing statement it follows that for a given compact K the functions N, M are uniquely determined by their points of discontinuity and by values in these points. The question is, if the opposite assertion is valid, i.e. if for every function $u : \mathbf{R}^+ \to \mathbf{N}$ with the properties from Theorem 1 we can find a compact metric space K such that $u(r) = N(r,K)$ or $u(r) = M(r,K)$. The following lemma shows the existence of such a space.

Lemma 1. *Let $\{r_n\}_{n=1}^{p}$ and $\{k_n\}_{n=1}^{p}$, for $p \in N$ or $p = \infty$, be two sequences of real numbers such that*

(i) $r_n \in \mathbf{R}_0^+$, $r_{n+1} < r_n$, $r_p = 0$ *if* $p < \infty$ *and* $\lim_{n\to\infty} r_n = 0$ *if* $p = \infty$,
(ii) $k_n \in \mathbf{N}$, $k_1 = 1$, $k_{n+1} > k_n$.
Define the function $u : \mathbf{R}^+ \to \mathbf{N}$,

$$u(r) = \begin{cases} k_1, & \text{for} \quad r > r_1, \\ k_n, & \text{for} \quad r_n < r \le r_{n-1}, \quad n > 1. \end{cases}$$

Then there exists a countable, compact metric space K_u *such that* $u(r) = N(r, K_u) = M(r, K_u)$.

PROOF : Put $a_n = k_{n+1} - k_n$, for $n \in \mathbf{N}$, $n < p$. Further put $K_u = \{x_0\} \cup \bigcup\{K_n; n \in \mathbf{N}, n < p\}$, where $K_n = \{x_n^1, \ldots, x_n^{a_n}\}$ is a finite set, $x_m^i \ne x_q^j$ for $i \ne j$ or $m \ne q$, and $x_q^i \ne x_0$. Define a metric d on K_u in the following way:

$$d(x_0, x_n^i) = r_n \qquad \text{for } 1 \le i \le a_n,$$
$$d(x_n^i, x_m^j) = r_{\min\{m,n\}} \qquad \text{for } 1 \le i \le a_n,\ 1 \le j \le a_m.$$

The space (K_u, d) is compact (it is finite for finite p and countable, with unique cluster point x_0, in the case of infinite p). Take r, $r_{n+1} < r \le r_n$. The set $B(x_0, r) = \{x_0\} \cup \bigcup_{i>n} K_i$ of diameter r_{n+1} has the distance r_n from the r_n-discrete set $\bigcup_{j=1}^n K_j$, and $|\bigcup_{j=1}^n K_j| = \sum_{j=1}^n a_j = k_{n+1} - 1$. Therefore by Proposition 1 (iii) and (iv)

$$N(r, K_u) = N(r, B(x_0, r)) + N(r, \cup_{j=1}^n K_j) = 1 + (k_{n+1} - 1) = k_{n+1} = u(r),$$

and

$$M(r, K_u) = M(r, B(x_0, r)) + M(r, \cup_{j=1}^n K_j) = 1 + (k_{n+1} - 1) = k_{n+1} = u(r).$$

■

It is well known that every metric space can be isometrically embedded into some Banach space. In our case there is one Banach space into which any space K from Lemma 1 can be embedded.

Theorem 2. *Let* u *be the function from Lemma 1. Then there exists a countable, compact subspace* L_u *of the space* l^∞, *with unique cluster point* Θ, *such that* $u(r) = N(r, L_u) = M(r, L_u)$.

PROOF : Recall that l^∞ is the space of all bounded sequences of real numbers with the supremum metric ρ. Denote by ε_j a sequence from l^∞ having 1 on j-th place and 0 on i-th place for $i \ne j$; $\Theta = (0, 0, \ldots)$ is the zero element in l^∞. Taking K_u constructed in Lemma 1 it is sufficient to find an isometry $g : K_u \to l^\infty$. Define g in the following way:

$$g(x_0) = \Theta,$$
$$g(x_n^i) = r_n \cdot \varepsilon_{k_n - 1 + i}.$$

We have

$$\rho(g(x_0), g(x_n^i)) = \rho(\Theta, r_n \cdot \varepsilon_{k_n-1+i}) = r_n = d(x_0, x_n^i)$$

and

$$\rho(g(x_n^i), g(x_m^j)) = \rho(r_n \cdot \varepsilon_{k_n-1+i}, r_m \cdot \varepsilon_{k_m-1+j}) =$$
$$= \max\{r_n, r_m\} = r_{\min\{n,m\}} = d(x_n^i, x_m^j).$$

It follows that g is an isometry and $N(r, L_u) = N(r, g(K_u)) = N(r, K_u) = u(r)$, where $L_u = g(K_u)$. ∎

The compact subsets of $\mathbf{R}^m$ play an important role in mathematics. That is why it would be useful to have some criterions for a given compact metric space to be isometrically embeddable into $\mathbf{R}^m$ for some $m \geq 1$. In Theorem 3 we give only a necessary condition for it.

We shall consider the space $\mathbf{R}^m$ with an arbitrary metric derived from some norm on $\mathbf{R}^m$. This gives for any two such metrics d_1, d_2 the existence of constants m, M such that $m \cdot d_1(x,y) \leq d_2(x,y) \leq M \cdot d_1(x,y)$ for all $x, y \in \mathbf{R}^m$. Moreover, each such metric d is invariant with respect to translation and $d(\alpha x, \alpha y) = \alpha \cdot d(x,y)$ for any $\alpha \in \mathbf{R}^+$ and $x, y \in \mathbf{R}^m$. This implies, for an affine mapping $f : \mathbf{R}^m \to \mathbf{R}^m$ given by $f(x) = \alpha x + b$, $\alpha \in \mathbf{R}$, $b \in \mathbf{R}^m$, the equality

$$N(r, A) = N(\alpha r, f(A)), \tag{2}$$

whenever A is a compact subset of $\mathbf{R}^m$ and $r > 0$.

Proposition 2. *Let d_1, d_2 be metrics on $\mathbf{R}^m$, let N_1, N_2 be the corresponding covering functions, let $A \subset \mathbf{R}^m$ be a compact subset. Then*

(i) *there exists a constant $k \in \mathbf{N}$ such that $N_2(r, A) \leq k \cdot N_1(r, A)$, $r > 0$.*
(ii) *for an arbitrary $c \in \mathbf{R}^+$ there exists a constant $l_c \in \mathbf{N}$ such that for $r > 0$ $N_1(r, A) \leq l_c \cdot N_1(cr, A)$.*

PROOF : (i) Fix $r > 0$ and for $x \in \mathbf{R}^m$ and $i = 1, 2$ put $B_i(x, r) = \{z \in \mathbf{R}^m;\ d_i(x, z) < r\}$, call it a d_i-ball. Let $\{B_1(y_j, r)\}_{j=1}^{N_1(r,A)}$ be a covering of A. Since d_1 and d_2 are topologically equivalent, $\overline{B_1(x,\varepsilon)}$ is compact in d_2 for every $x \in \mathbf{R}^m$ and $\varepsilon > 0$. Denote $k = N_2(\frac{1}{2}, \overline{B_1(0,1)})$. This implies by (2) that $\overline{B_1(y_j, r)}$, $1 \leq j \leq m$, can be covered by k d-balls with radius $\frac{r}{2}$ and therefore $\overline{B_1(y_j, r)} \cap A$ can be by Proposition 1 (i) surely covered by at most k d_2-balls with radius r. Then we can cover the whole set A by $k \cdot N_1(r, A)$ d_2-balls with radius r and hence $N_2(r, A) \leq k \cdot N_1(r, A)$.

(ii) Put $d_2 = c \cdot d_1$ and apply (i). ∎

Lemma 2. *Let K be a compact metric space and let $r' < r''$ be two consecutive points of the discontinuity of the function $N(., K)$. Denote $q = N(r'', K)$. Then there exist points $y_1, \ldots, y_q$ in K such that*

$$\bigcup_{i=1}^{q} B(y_i, r') \subsetneq K \subset \bigcup_{i=1}^{q} \overline{B(y_i, r')}.$$

PROOF : Let $\{\rho_k\}$ be a sequence of points from the interval $(r', r'']$ tending to r'. For every ρ_k there exists $\{x_1^k, \ldots, x_q^k\} \subset K$ such that $K \subset \bigcup_{i=1}^q B(x_i^k, r)$ for $r \geq \rho_k$. Since K^q with the supremum metric is a compact metric space, we can choose a convergent subsequence of $\{(x_1^k, \ldots, x_q^k)\}_{k=1}^\infty$ of points of K; we can assume that the original sequence is convergent. Let $\lim_{k\to\infty}(x_1^k, \ldots, x_q^k) = (y_1, \ldots, y_q)$ and $r > r'$. Since $x_j^k \to y_j$, $j = 1, \ldots, q$, and $\rho_k \to r'$, there exist $k_j \in N$, $j = 1, \ldots, q$, such that $B(x_j^k, \rho_k) \subset B(y_j, r)$ for $k \geq k_j$. Then for $k \geq \max\{k_1, \ldots, k_q\}$ we have $K \subset \bigcup_{i=1}^q B(x_i^k, \rho_k) \subset \bigcup_{i=1}^q B(y_i, r)$. Therefore $K \subset \bigcup_{i=1}^q \overline{B(y_j, r')}$. ∎

The following theorem shows that the jumps of the function $N(., A)$ cannot be arbitrary when the set A is a compact subset of $\mathbf{R}^m$, $m \geq 1$.

Theorem 3. *For every $m \in \mathbf{N}$ there exists $k_m \in \mathbf{N}$ such that for every $r > 0$ and an arbitrary compact set $A \subset \mathbf{R}^m$ the following inequality holds:*

$$N(r, A) \leq k_m \cdot \lim_{s\to r+} N(s, A). \tag{3}$$

PROOF : If r is not a point of discontinuity of $N(., A)$ then (3) holds for $k_m = 1$. If r is a point of discontinuity, the Lemma 2 implies the existence of points $y_1, \ldots, y_q$, where $q = \lim_{s\to r+} N(s, A)$, with $A \subset \bigcup_{i=1}^q \overline{B(y_i, r)}$. Proposition 1 (i), (ii) and (2) then imply:

$$N(r, A) \leq N(\frac{r}{2}, \bigcup_{i=1}^q \overline{B(y_i, r)}) \leq \sum_{i=1}^q N(\frac{r}{2}, \overline{B(y_i, r)}) = q \cdot N(\frac{1}{2}, \overline{B(0, 1)}).$$

Denote $k = N(\frac{1}{2}, \overline{B(0, 1)})$. For such k we obtain (3). ∎

2. Metric dimensions $\underline{\dim}$ and $\overline{\dim}$.

There are more equivalent definitions of the lower ([B], [H], [V], [P]) and upper ([BT], [H], [V], [Y]) metric dimension. We will use the following definition. Let K be a compact metric space. Then we put

$$\underline{\dim}\, K = \liminf_{r\to 0+} \frac{\log N(r, K)}{-\log r},$$
$$\overline{\dim}\, K = \limsup_{r\to 0+} \frac{\log N(r, K)}{-\log r}.$$

Taking into account the inequalities (1) one can replace the function N by the function M which can be sometimes useful.

Proposition 3. *Let (K, d) be a compact metric space and let $X \subset K$ be its compact subset. Then*

(i) $\underline{\dim}\, X \leq \underline{\dim}\, K$,

(ii) $\overline{\dim}\, X \leq \overline{\dim}\, K$.

PROOF : We have, by Proposition 1 (i),

$$\underline{\dim}\ X = \liminf_{r\to 0+} \frac{\log N(r,X)}{-\log r} \leq \liminf_{r\to 0+} \frac{\log N(r,K)}{-\log r} = \underline{\dim}\ K,$$

which proves (i) . (ii) can be proved in the same way. ■

Proposition 4. *Let* $K = \bigcup_{i=1}^{n} K_i$, *where* K_i *are compact subsets of the metric space* K. *Then* $\overline{\dim}\ K = \max_{1\leq i\leq n} \{\overline{\dim}\ K_i\}$.

PROOF :

$$\overline{\dim}\ K = \limsup_{r\to 0+} \frac{\log N(r,K)}{-\log r} \leq$$

$$\leq \limsup_{r\to 0+} \frac{\log(n \cdot \max_{1\leq i\leq n} \{N(r,K_i)\})}{-\log r} = \limsup_{r\to 0+} \left(\frac{\log n}{-\log r} + \frac{\max_{1\leq i\leq n} \{\log N(r,K_i)\}}{-\log r} \right) =$$

$$= \max_{1\leq i\leq n} \left\{ \limsup_{r\to 0+} \frac{\log N(r,K_i)}{-\log r} \right\} = \max_{1\leq i\leq n} \{\overline{\dim}\ K_i\}.$$

■

Remark. A similar result is not valid for the lower metric dimension, for the counterexample see e.g. [**B**].

Corollary 1. *In each compact metric space* (K,d) *there exists a point* x_0 *with the property* $\overline{\dim}\ K = \overline{\dim}\ (K,x_0) \overset{\text{def}}{=} \inf\{\overline{\dim}\ \overline{B(x_0,r)}; r > 0\}$.

PROOF : Put $B_0 = K$ and suppose that for each $i = 1,2,\ldots,n$ we have a closed ball B_i such that the radius of B_i is less than or equal to 2^{-i}, $B_i \subset B_{i-1}$ and $\overline{\dim}\ B_i = \overline{\dim}\ K$. Let us have a finite covering of B_n with balls of radius less than or equal to $2^{-(n+1)}$. Applying Proposition 4 we can choose a ball B_{n+1}. Take x_0 to be the unique point in the intersection $\bigcap_{n=1}^{\infty} B_n$. ■

By the Lemma 1 we can prescribe the function $u(r)$ arbitrarily in spirit of Theorem 1, and we are able to find a compact metric space K with $N(r,K) = M(r,K) = u(r)$. Now we may ask: What happens if we seek such a space only as a subspace of a given compact metric space? Although we have seen in Proposition 1 (i), Proposition 2 (ii) and Theorem 3 that we are not so free in prescribing the function $N(r,K)$ or $M(r,K)$ in this case; the relative great freedom will be stated in Lemma 3 and Theorem 4 where metric dimensions are concerned.

Lemma 3. *Let* (K,d) *be an infinite compact metric space. Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *be two sequences from the interval* $[0, \overline{\dim}\ K]$ *and let* $\{\delta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of positive real numbers such that* $\delta_n \to 0$, $a_n \to 0$ *and for each* n *the inequality* $b_{n+1} < a_n < b_n$ *holds. Then there exists a compact subspace* $X \subset K$ *with the unique cluster point and a decreasing sequence* $\{\varepsilon_n\}_{n=1}^{\infty}$ *of real numbers converging to 0 such that the following hold:*

(i) $\forall n \in \mathbf{N} \quad M(\varepsilon_n, X) \in \left((1/\varepsilon_n)^{\alpha_n - \delta_n}, (1/\varepsilon_n)^{\alpha_n + \delta_n}\right)$,

(ii) $\forall \alpha > \limsup_{n\to\infty} \alpha_n \quad \exists r_0 > 0; \forall r < r_0 \quad M(r, X) < (1/r)^{\alpha}$,

(iii) *there exists a sequence* $\{k_n\}_{n=1}^{\infty}$ *of natural numbers such that* $\forall n \in \mathbf{N}$ *and* $\forall r \in (a_{k_n}, b_{k_n})$ *we have* $M(r, X) < (1/r)^{\beta_m}$.

PROOF : Let x_0 be such a point that $\overline{\dim}\, K = \overline{\dim}\,(K, x_0)$. For $r \in \mathbf{R}^+$ denote $S_n(r) = \left((1/r)^{\alpha_n - \delta_n}, (1/r)^{\alpha_n + \delta_n}\right) \cap \mathbf{N}$. Note that for fixed n the cardinality of $S_n(r)$ grows to infinity as r tends to 0. We shall construct consecutively by induction positive real numbers ρ_n, ε_n, a finite set $X_n \subset K$ and a natural number k_n in such way that the following properties will be true for each $n \in \mathbf{N}$:

(a) $a_{k_n} < b_{k_n} < \varepsilon_n \le 2\rho_n < \min\left\{a_{k_{n-1}}, \frac{d(x_0, X_{n-1})}{2}\right\}$,

(b) $|S_n(2\rho_n)| \ge 3$,

(c) $\max S_n(2\rho_n) > |X_{n-1}| + 1$

(d) $\varepsilon_n = \max\left\{r \in (0, 2\rho_n]; M(r, \overline{B(x_0, \rho_n)}) + |X_{n-1}| \ge (1/r)^{\alpha_n - \delta_n} + 1\right\}$,

(e) X_n is an ε_n-discrete set,

(f) $|X_n| \in S_n(\varepsilon_n)$ and $|X_n| + 1 \in S_n(\varepsilon_n)$,

(g) $|X_n| + 1 < (1/b_{k_n})^{\beta_n}$.

Put $X_0 = 0$ and $k_0 = 1$. Suppose that $n \in \mathbf{N}$ and ρ_i, ε_i, X_i, k_i are constructed for all $i \in \mathbf{N}$, $i < n$. Choose ρ_n fulfilling (b), (c) and, if $n > 1$, the last inequality in (a). Put $B_n = \overline{B(x_0, \rho_n)}$. For each $\eta < \overline{\dim}\, B_n$ and $d \in \mathbf{R}$ there are arbitrarily small $r > 0$ such that $M(r, B_n) > (1/r)^{\eta} + d$. Now the special form of the function M (see Theorem 1) implies that the set in (d) has the greatest element. Denote this maximum by ε_n. Note that if $0 < p \le q$ are real numbers then $|S_n(p)| \ge |S_n(q)| - 1$ and $\max S_n(p) \ge \max S_n(q)$. Using this, (b) and (c) imply the existence of the minimal nonnegative integer q_n with

$$\left(\frac{1}{\varepsilon_n}\right)^{\alpha_n - \delta_n} < q_n + |X_{n-1}| < \left(\frac{1}{\varepsilon_n}\right)^{\alpha_n + \delta_n} - 1.$$

Now (d) implies $q_n < M(\varepsilon_n, B_n)$ and therefore there exists a finite ε_n-discrete set $F_n \subset B_n$ which does not contain x_0 with $|F_n| = q_n$. The last inequality in (a) implies that F_n and X_{n-1} are disjoint. Put $X_n = X_{n-1} \cup F_n$ and we can see that (e) and (f) are fulfilled. Finally choose k_n fulfilling the conditions (a) and (g). Put $X = \bigcup_{n=1}^{\infty} X_n \cup \{x_0\}$. Using Proposition 1 we are going to prove (i)–(iii).

First note that for each $n \in \mathbf{N}$:

$$M(r,X) = M(r, X_n \cup X \setminus X_n) \le M(r, X_n) + M(r, X \setminus X_n) \le M(r, X_n) + M(r, B_{n+1})$$

and for $r \le \varepsilon_n$ we have $M(r, X_n) = |X_n|$, for $r > 2\rho_{n+1}$ $M(r, B_{n+1}) = 1$.

(i) By the choice of X_n and by (f):

$$\left(\frac{1}{\varepsilon_n}\right)^{\alpha_n - \delta_n} < |X_n| \le M(\varepsilon_n, X_n) \le M(\varepsilon_n, X) \le |X_n| + 1 < \left(\frac{1}{\varepsilon_n}\right)^{\alpha_n + \delta_n},$$

and so $M(\varepsilon_n, X) \in \left((1/\varepsilon_n)^{\alpha_n - \delta_n}, (1/\varepsilon_n)^{\alpha_n + \delta_n}\right)$.
(ii) Let $\alpha > \limsup_{n\to\infty} \alpha_n$. Choose n_0 such that $\alpha > \alpha_n + \delta$ for each $n > n_0$ and put $r_0 = \varepsilon_{n_0}$. Now let $r < r_0$. Then there is an $n > n_0$ such that $r \in (\varepsilon_{n+1}, \varepsilon_n]$. If $r > 2\rho_{n+1}$ then using (f)

$$M(r,X) \le |X_n| + 1 < \left(\frac{1}{\varepsilon_n}\right)^{\alpha_n+\delta_n} \le \left(\frac{1}{r}\right)^{\alpha_n+\delta_n} < \left(\frac{1}{r}\right)^{\alpha}.$$

On the other hand if $r \le 2\rho_{n+1}$ then using (b) and (d)

$$M(r,X) \le M(r,B_{n+1}) + |X_n| < \left(\frac{1}{r}\right)^{\alpha_{n+1}-\delta_{n+1}} + 1 < \left(\frac{1}{r}\right)^{\alpha_{n+1}+\delta_{n+1}} < \left(\frac{1}{r}\right)^{\alpha}.$$

(iii) Let $r \in [a_{k_n}, b_{k_n}]$. Then using (a) and (g)

$$M(r,X) \le |X_n| + M(r,B_{n+1}) = |X_n| + 1 < \left(\frac{1}{b_{k_n}}\right)^{\beta_n}.$$

■

Theorem 4. *Let K be an infinite compact metric space, $\alpha \in [0, \overline{\dim}\, K]$, $\beta \in [0, \underline{\dim}\, K]$, and $\alpha \ge \beta$. There exists a countable, compact set $X \subset K$ with the unique cluster point such that $\overline{\dim}\, X = \alpha$ and $\underline{\dim}\, X \le \beta$.*

PROOF : Put $\alpha_n = \alpha$, $\beta_n = \beta$, $\delta_n = \frac{1}{n}$ for each $n \in \mathbf{N}$ and a_n, b_n arbitrary with $a_n \to 0$, $b_{n+1} < a_n < b_n$, and use Lemma 3. The conditions (i) and (ii) imply $\overline{\dim}\, X = \alpha$ while (iii) implies $\underline{\dim}\, X \le \beta$. This completes the proof. ■

The following corollary says that each infinite compact metric space with positive upper metric dimension contains a simple infinite compact subspace which is "pathological" in the sense of distinction between upper and lower metric dimension.

Corollary 2. *Each infinite compact metric space K contains a countable compact subspace X with unique cluster point for which $\overline{\dim}\, X = 0$ and $\overline{\dim}\, X = \overline{\dim}\, K$.*

While the upper metric dimension for a countably compact subspace with the unique cluster point in Theorem 4 is prescribed exactly, for the lower metric dimension we have only the upper estimate. The following example shows that this restriction is substantial.

Example. There is a compact $K \subset \mathbf{R}$ with $\underline{\dim}\, K = 1$ such that $\underline{\dim}\, X = 0$ for each compact subset $X \subset K$ with unique cluster point.
PROOF : According to [H], there are subsets $F_1 \subset [0,1]$ and $F_2 \subset [2,3]$ with $\underline{\dim}\, F_1 = \underline{\dim}\, F_2 = 0$ and $\underline{\dim}\, (F_1 \cup F_2) = 1$. Then put $K = F_1 \cup F_2$. ■

One further pathological property of the metric dimension compared to the topological dimension is given in the following theorem.

Theorem 5. *For each $0 \leq c < \infty$ there exists a compact metric space K_c such that $\overline{\dim}\, K_c = c$ and K_c is not isometrically embeddable into $\mathbf{R}^m$ for any $m \in \mathbf{N}$.*

PROOF : Let $\{k_m\}_{m=1}^{\infty}$ be a sequence of constants from Theorem 3. Let $\{c_m\}_{m=1}^{\infty}$ be a decreasing sequence of real numbers with $\lim_{m\to\infty} c_m = c$. Construct the sequences $\{l_m\}_{m=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$:
$l_m = (k_1+1)\cdot(k_2+1)\cdot\ldots\cdot(k_m+1)$ and $r_m = l_m^{-1/c_m}$. Defining the function

$$u(r) = \begin{cases} 1, & \text{for } r > r_1, \\ l_m, & \text{for } r_{m+1} < r \leq r_m, \end{cases}$$

this fulfils the conditions of Lemma 1 and so there exists a compact metric space K_c (even countable with unique cluster point) such that $N(r, K_c) = u(r)$. Now $\overline{\dim}\, K_c = c$, since $u(r) \leq (1/r)^{c_m}$ for $r \leq r_m$ and $u(r_m) = (1/r_m)^{c_m} > (1/r_m)^c$. Moreover,

$$N(r_1, K_c) = k_1 + 1 > k_1 = k_1 \cdot \lim_{s\to r_1^+} N(s, K_c),$$

and for $m > 1$

$$N(r_m, K_c) = (k_m+1)\cdot N(r_{m-1}, K_c) > k_m \cdot \lim_{s\to r_m^+} N(s, K_c),$$

and so by (3) K_c cannot have an isometrical image in $\mathbf{R}^m$, $m \geq 1$. ■

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