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Abstract. A space is called dyadic if it is a Hausdorff continuous image of some power of the discrete space 2. A space X is called supercompact if it possesses an open subbase $\mathcal{S}$ such that every open cover of X consisting of members of $\mathcal{S}$ has an at most 2 subcover. We show that there is an example of a dyadic space which is not supercompact thus answering a question of E. van Douwen and J. van Mill.

Keywords: Dyadic, supercompact, subbase, continuous image

Classification: Primary 54D30; Secondary 54G20, 54B15

1. Introduction.

All spaces in this paper will be assumed to be Hausdorff. Dyadic spaces, introduced by Alexandroff [1], are continuous images of 2^κ for some cardinal κ. Supercompact spaces, introduced by De Groot [6], are spaces X which possess an open subbase $\mathcal{S}$ such that every cover of X consisting of members of $\mathcal{S}$ has a subcover of at most 2 members. For our purposes it is more elegant to work with closed subbases. A collection of sets $\mathcal{S}$ is linked if every 2 members of $\mathcal{S}$ have a non-empty intersection. A collection of sets $\mathcal{S}$ is binary if every linked subcollection of $\mathcal{S}$ has a non-empty intersection. So, X is supercompact iff X has a binary closed subbase.

Supercompact spaces are plentiful. They include all compact metric spaces, Strok & Szymanski [11], all compact groups, Mills [8], and all compact ordered spaces. Supercompactness is preserved by products, so all Cantor cubes 2^κ and all Tychonoff cubes I^κ are supercompact. Not all compact spaces are supercompact, Bell [2], since if the Stone–Čech compactification of a space X is supercompact, then X must be pseudocompact. There are even first countable compact spaces which are not supercompact. This was shown in van Douwen & van Mill [4] where the question of whether all dyadic spaces are supercompact was raised. This question also appears in the problem list of Rudin [10]. Van Mill & Mills [7] have an example of an at most 2 to 1 irreducible image of a supercompact space that is not supercompact. Bell [3] has given an example of a polyadic space that is not supercompact. In this paper, we produce a dyadic space that is not supercompact. It is of a particularly simple form, i.e. it is gotten from 2^κ by collapsing a closed subset to a point. However, for our argument to work, we require $\kappa \geq \omega_3$.

2. Preliminaries.

The following elementary facts will be used throughout this paper. A subcollection of a binary collection is again binary. If $\pi : X \to Y$ is an onto map and $\mathcal{S}$ is a binary collection of subsets of Y, then $\{\pi^{-1}[S] : S \in \mathcal{S}\}$ is a binary collection of

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subsets of X. A collection $\mathcal{S}$ of sets is said to be stable if whenever $\mathcal{R} \subseteq \mathcal{S}$, then $\bigcap \mathcal{R} \in \mathcal{S}$. The hull of a collection $\mathcal{S}$ of sets is the smallest stable collection which contains $\mathcal{S}$. If $\mathcal{S}$ is a binary collection, then the hull of $\mathcal{S}$ is also binary. So, if X is supercompact, then X has a stable binary closed subbase $\mathcal{S}$. We make use of this as follows. Let $\mathcal{S}$ be a stable closed subbase for a compact space X. Then, whenever A is closed in X, V is open in X, and $A \subseteq V$, there exists a finite $\mathcal{R} \subseteq \mathcal{S}$ such that $A \subseteq \bigcup \mathcal{R} \subseteq V$. A simplification occurs if A is clopen; we then get that there exists a finite $\mathcal{R} \subseteq \mathcal{S}$ such that $A = \bigcup \mathcal{R}$.

If S is a set, then S^c represents the complement of S taken in the largest previously defined set which contains S as a subset. If $\alpha < \gamma$ are 2 ordinals, then (α, γ) denotes $\{\beta : \alpha < \beta < \gamma\}$. We abbreviate zero-dimensional by 0-dim'l.

3. Quotients.

Throughout this section X will represent a compact space and $H(X)$ will represent the collection of all non-empty closed subsets of X. If A is a closed subset of X, then $X \operatorname{MOD} A$ represents the quotient space of X gotten by collapsing A to a point. We will reserve the symbol π henceforth for the associated quotient map from X onto $X \operatorname{MOD} A$. If $\mathcal{A} \subseteq \mathcal{B} \subseteq H(X)$, then the notation $\mathcal{A} \uparrow \mathcal{B}$ denotes the following statement: for all $B \in \mathcal{B}$ and for all open sets V of X with $B \subseteq V$, there exists a finite $\mathcal{C} \subseteq \mathcal{A}$ with $B \subseteq \bigcup \mathcal{C} \subseteq V$. If $A \in H(X)$, then put $A\hat{} = \{B \in H(X) :$ either $A \subseteq B$ or $A \cap B = \emptyset\}$.

Theorem 3.1. (The Quotient Theorem) *Let X be compact and let A be a closed subset of X. $X \operatorname{MOD} A$ is supercompact iff there exists a binary $\mathcal{S} \subseteq A\hat{}$ such that $\mathcal{S} \uparrow A\hat{}$.*

PROOF : Assume that $X \operatorname{MOD} A$ is supercompact and that $\mathcal{B}$ is a stable binary closed subbase for $X \operatorname{MOD} A$. Put $\mathcal{S} = \{\pi^{-1}[B] : b \in \mathcal{B}\}$. $\mathcal{S}$ is binary since π is onto. Clearly, $\mathcal{S} \subseteq A\hat{}$. Let $F \in A\hat{}$ and let V be open in X with $F \subseteq V$. If $A \cap F = \emptyset$ then put $U = V \cap A^c$ otherwise put $U = V$. Then $F \subseteq U \subseteq V$. So, $\pi[F] \subseteq \pi[U]$. $\pi[F]$ is closed since $\pi^{-1}[\pi[F]] = F$ and $\pi[U]$ is open since $\pi^{-1}[\pi[U]] = U$. Choose a finite $\mathcal{C} \subseteq \mathcal{B}$ such that $\pi[F] \subseteq \bigcup \mathcal{C} \subseteq \pi[U]$. Then, $F \subseteq \bigcup\{\pi^{-1}[B] : B \in \mathcal{C}\} \subseteq V$. Thus we have shown that $\mathcal{S} \uparrow A\hat{}$.

Assume that $\mathcal{S} \subseteq A\hat{}$ is binary and that $\mathcal{S} \uparrow A\hat{}$. Put $\mathcal{B} = \{\pi[S] : S \in \mathcal{S}\}$. $\mathcal{B}$ consists of closed sets of $X \operatorname{MOD} A$. Let D be closed in $X \operatorname{MOD} A$ and let V be open in $X \operatorname{MOD} A$ with $D \subseteq V$. Then, $\pi^{-1}[D] \in A\hat{}$ and $\pi^{-1}[D] \subseteq \pi^{-1}[V]$; so we can choose a finite $\mathcal{C} \subseteq \mathcal{S}$ with $\pi^{-1}[D] \subseteq \bigcup \mathcal{C} \subseteq \pi^{-1}[V]$. Thus, $D \subseteq \bigcup\{\pi[S] : S \in \mathcal{C}\} \subseteq V$. So, $\mathcal{B}$ is a closed subbase for $X \operatorname{MOD} A$. Let $\{\pi[S] : S \in \mathcal{C} \subseteq \mathcal{B}\}$ be linked. Then $\mathcal{C}$ must be linked. Choose $x \in \bigcap \mathcal{C}$. Then, $\pi(x) \in \bigcap\{\pi[S] : S \in \mathcal{C}\}$. Thus $\mathcal{B}$ is binary and therefore $X \operatorname{MOD} A$ is supercompact. ∎

If we put $\kappa = \omega_1$ and take $p \neq q$ in 2^κ, then $2^\kappa \operatorname{MOD}\{p, q\}$ is an interesting space since, by Pelczynski [**9**], this space is a 0-dim'l dyadic space which is not homeomorphic to any retract of 2^κ. We must mention that the first example of such a space was given by Engelking [**5**]. However, this space is supercompact by the next result.

Proposition 3.2. *If X is supercompact and 0-dim'l and A is a finite subset of X, then X MOD A is supercompact.*

PROOF : By induction, it suffices to prove this for a 2-element set $A = \{p, q\}$. Let $\mathcal{B}$ be a stable binary closed subbase for X. Choose a clopen set D of X such that $p \in D$ and $q \notin D$. Put $\mathcal{S} = \{B \in \mathcal{B} : B \cap A = \emptyset$ and either $B \subseteq D$ or $B \subseteq D^c\} \cup \{D \cup B : q \in B \in \mathcal{B}\} \cup \{D^c \cup B : p \in B \in \mathcal{B}\}$. $\mathcal{S}$ is seen to be binary, $\mathcal{S} \subseteq A\hat{}$, and $\mathcal{S} \uparrow A\hat{}$. By the Quotient Theorem, X MOD A is supercompact. ∎

We are unable to prove Proposition 3.2 without assuming that X is 0-dim'l. So, we ask: Does there exist a supercompact space X and a subset A of X of cardinality 2 such that X MOD A is not supercompact?

4. The example.

Let κ be an infinite cardinal, $F \subseteq \kappa$, and $A \subseteq 2^\kappa$. We say that A depends on F if $A = \pi_F^{-1}[\pi_F[A]]$ where $\pi_F : 2^\kappa \to 2^F$ is the Fth projection map. A basiç fact about 2^κ that we will use is that if A is a regular closed subspace of 2^κ, then there is a countable $F \subseteq \kappa$ such that A depends on F.

We need the following result about the cardinal $\kappa = \omega_3$. We remark that the result is no longer true if κ is either ω_2 or ω_1.

Theorem 4.1. *Put $\kappa = \omega_3$ and for each $\alpha < \lambda < \kappa$ let $F_{\alpha\lambda}$ be a countable subset of κ. Then there exists $\alpha < \lambda < \gamma < \kappa$ such that $\alpha \notin F_{\lambda\gamma}, \gamma \notin F_{\alpha\lambda}$ and $F_{\alpha\gamma} \cap F_{\lambda\gamma} \cap (\alpha, \gamma) = \emptyset$.*

PROOF : (i) Put $\gamma(0) = 0$ and for $\alpha < \omega_2$ choose

$$\gamma(\alpha) > \sup(\{\gamma(\lambda) : \lambda < \alpha\} \cup \bigcup\{F_{ab} : a < b < \gamma(\lambda) \text{ and } \lambda < \alpha\}).$$

Put $\gamma = \sup(\{\gamma(\alpha) : \alpha < \omega_2\})$. Then $\mathrm{cf}(\gamma) = \omega_2$ and for every $\alpha < \lambda < \gamma$ we have that $\gamma \notin F_{\alpha\lambda}$.

(ii) Fix γ as in (i). Put $\gamma(0) = 0$ and for $\alpha < \omega_1$ choose $\gamma(\alpha)$ such that

$$\gamma > \gamma(\alpha) > \sup(\{\gamma(\delta) : \delta < \alpha\} \cup \bigcup\{F_{\gamma(\delta)\gamma} \cap (\gamma(\delta), \gamma) : \delta < \alpha\}).$$

Now, fix λ with $\sup(\{\gamma(\delta) : \delta < \omega_1\}) < \lambda < \gamma$. By construction, if $a < b$, then $F_{\gamma(a)\gamma} \cap (\gamma(a), \gamma) \cap F_{\gamma(b)\gamma} \cap (\gamma(b), \gamma) = \emptyset$. Since F_{ab} is countable, this means that we can choose $\delta < \omega_1$ such that

$$F_{\lambda\gamma} \cap (\{\gamma(\delta)\} \cup [F_{\gamma(\delta)\gamma} \cap (\gamma(\delta), \gamma)]) = \emptyset.$$

Finally, put $\alpha = \gamma(\delta)$. ∎

Put $\kappa = \omega_3$ for the remainder of this section. For $\alpha \leq \kappa$, define $w_\alpha \in 2^\kappa$ by

$$w_\alpha(\delta) = 0 \quad \text{iff } \delta < \alpha .$$

Put $W = \{w_\alpha : \alpha \leq \kappa\}$. W is a homeomorph of the compact ordinal space $\omega_3 + 1$. For $\alpha < \lambda < \kappa$ define $p_{\alpha\lambda} \in 2^\kappa$ by

$$p_{\alpha\lambda}(\delta) = 0 \quad \text{iff } \delta < \alpha \quad \text{or } \delta = \lambda$$

and put $B_{\alpha\lambda} = \{f \in 2^\kappa : f(\alpha) = 1 \text{ and } f(\lambda) = 0\}$. Then $p_{\alpha\lambda} \in B_{\alpha\lambda}, B_{\alpha\lambda}$ is a clopen subset of 2^κ, and $W^c = \bigcup\{B_{\alpha\lambda} : \alpha < \lambda < \kappa\}$.

Theorem 4.2. $2^\kappa \operatorname{MOD} W$ *is not supercompact.*

PROOF : We will use the Quotient Theorem. Assume that $\mathcal{S} \subseteq W^\wedge$ satisfies $\mathcal{S} \uparrow W^\wedge$. We will show that $\mathcal{S}$ cannot be binary. For brevity, if $A \subseteq 2^\kappa$, then put $A^* = \mathrm{cl}(\mathrm{int}(A))$. Since $\mathcal{S} \uparrow W^\wedge$, for each $\alpha < \lambda < \kappa$, choose finite subcollections $\mathcal{P}_{\alpha\lambda}$ and $\mathcal{R}_{\alpha\lambda}$ of $\mathcal{S}$ such that $B_{\alpha\lambda} = \bigcup \mathcal{P}_{\alpha\lambda}$ and $B^c_{\alpha\lambda} = \bigcup \mathcal{R}_{\alpha\lambda}$. Since $p_{\alpha\lambda} \in B_{\alpha\lambda}$, choose $P_{\alpha\lambda} \in \mathcal{P}_{\alpha\lambda}$ such that $p_{\alpha\lambda} \in P^*_{\alpha\lambda}$. Since $w_{\lambda+1} \in B^c_{\alpha\lambda}$, choose $Q_{\alpha\lambda} \in \mathcal{R}_{\alpha\lambda}$ such that $w_{\lambda+1} \in Q^*_{\alpha\lambda}$. Since $w_\alpha \in B^c_{\alpha\lambda}$, choose $R_{\alpha\lambda} \in \mathcal{R}_{\alpha\lambda}$ such that $w_\alpha \in R^*_{\alpha\lambda}$. Let $P^*_{\alpha\lambda}, Q^*_{\alpha\lambda}$, and $R^*_{\alpha\lambda}$ depend on the countable subsets $C_{\alpha\lambda}, D_{\alpha\lambda}$, and $E_{\alpha\lambda}$ of κ respectively. For $\alpha < \lambda < \kappa$, put $F_{\alpha\lambda} = C_{\alpha\lambda} \cup D_{\alpha\lambda} \cup E_{\alpha\lambda}$. Invoke Theorem 4.1 and choose $\alpha < \lambda < \gamma < \kappa$ such that $\alpha \notin F_{\lambda\gamma}, \gamma \notin F_{\alpha\lambda}$, and $F_{\alpha\gamma} \cap F_{\lambda\gamma} \cap (\alpha, \gamma) = \emptyset$.

Claim. $\{P_{\alpha\gamma}, Q_{\lambda\gamma}, R_{\alpha\lambda}\}$ is not binary.

PROOF of Claim: $P_{\alpha\gamma} \cap Q_{\lambda\gamma} \cap R_{\alpha\lambda} \subseteq B_{\alpha\gamma} \cap B^c_{\lambda\gamma} \cap B^c_{\alpha\lambda} = \emptyset$. Since $w_{\gamma+1} \in Q_{\lambda\gamma}, Q_{\lambda\gamma} \cap W \neq \emptyset$ and so $W \subseteq Q_{\lambda\gamma}$. Since $w_\alpha \in R_{\alpha\lambda}, R_{\alpha\lambda} \cap W \neq \emptyset$ and so $W \subseteq R_{\alpha\lambda}$. Hence, $Q_{\lambda\gamma} \cap R_{\alpha\lambda} \neq \emptyset$. $P_{\alpha\gamma} \cap R_{\alpha\lambda} \neq \emptyset$ since $p_{\alpha\gamma} \in P_{\alpha\gamma} \cap R_{\alpha\lambda}$. The reason that $p_{\alpha\gamma} \in R_{\alpha\lambda}$ is that $w_\alpha \in R^*_{\alpha\lambda}, R^*_{\alpha\lambda}$ depends on $E_{\alpha\lambda}$ and $p_{\alpha\gamma} \mid E_{\alpha\lambda} = w_\alpha \mid E_{\alpha\lambda}$. For this last fact we need that $\gamma \notin E_{\alpha\lambda}$. If we define $f \in 2^\kappa$ by $f(\delta) = 1$ iff $\delta = \alpha$ or $\delta > \gamma$ or $\delta \in C_{\alpha\gamma} \cap (\alpha, \gamma)$ then $f \in P_{\alpha\gamma} \cap Q_{\lambda\gamma}$. The reason that $f \in P_{\alpha\gamma}$ is that $p_{\alpha\gamma} \in P^*_{\alpha\gamma}$, $P^*_{\alpha\gamma}$ depends on $C_{\alpha\gamma}$ and $f \mid C_{\alpha\gamma} = p_{\alpha\gamma} \mid C_{\alpha\gamma}$. The reason that $f \in Q_{\lambda\gamma}$ is that $w_{\gamma+1} \in Q^*_{\lambda\gamma}$, $Q^*_{\lambda\gamma}$ depends on $D_{\lambda\gamma}$ and $f \mid D_{\lambda\gamma} = w_{\gamma+1} \mid D_{\lambda\gamma}$. For this last fact, we need that $\alpha \notin D_{\lambda\gamma}$ and that $C_{\alpha\gamma} \cap D_{\lambda\gamma} \cap (\alpha, \gamma) = \emptyset$. ■

5. Conclusion.

An alternate description of the space $2^\kappa \operatorname{MOD} W$ is that it is the Stone space of the boolean algebra generated by $\{B_{\alpha\lambda} : \alpha < \lambda < \kappa\}$.

If we trace Theorem 4.2 back to Theorem 3.1, then we see that the nature of our proof was as follows: We showed that if $\mathcal{S}$ is a family of closed sets and if f is a function which associates, to each clopen set B, a finite subcollection $f(B)$ of $\mathcal{S}$ such that $B = \bigcup f(B)$, then, we could find 3 clopen sets A, B, C and 3 elements $R \in f(A), S \in f(B), T \in f(C)$ such that $\{R, S, T\}$ is linked and $A \cap B \cap C = \emptyset$. This nature allows one to easily extend Theorem 4.2 to the stronger statement that $2^\kappa \operatorname{MOD} W$ cannot be embedded as a neighbourhood retract of any supercompact space. We mention this because an important unsolved problem in supercompactness today is the question of van Douwen and van Mill [4] of whether supercompactness is preserved by retractions.

We would like to mention our first failed attempt in solving the problem of this paper. If, for $\alpha \leq \kappa$ we define $x_\alpha \in 2^\kappa$ by $x_\alpha(\delta) = 1$ iff $\delta = \alpha$ and put $F = \{x_\alpha : \alpha \leq \kappa\}$, then F is a homeomorph of the Alexandroff one-point compactification of the discrete space κ; however, $2^\kappa \operatorname{MOD} F$ can be shown to be supercompact.

References

[1] Alexandroff P.S., *Zur Theorie der topologischen Räume*, (Doklady) Acad. Sci. URSS **11** (1936), 55-58.

[2] Bell M.G., *Not all compact spaces are supercompact*, General Topology Appl. **8** (1978), 151–155.

[3] Bell M.G., *Polyadic spaces of arbitrary compactness numbers*, Comment. Math. Univ. Carolinae **26** (1985), 353–361.
[4] Douwen E. van, Mill J. van, *Supercompact Spaces*, Topology and its Applications **13** (1982), 21–32.
[5] Engelking R., *Cartesian products and dyadic spaces*, Fund. Math. **57** (1965), 287–304.
[6] Groot J. de, *Supercompactness and superextensions*, in Contributions to extension theory of topological structures, Symp. Berlin 1967, Deutscher Verlag Wiss., Berlin 1969, 89–90.
[7] Mill J. van, Mills C.F., *A nonsupercompact continuous image of a supercompact space*, Houston J. Math. **5** (1979), 241–247.
[8] Mills C.F., *Compact topological groups are supercompact*, Wiskundig Seminarium rapport nr. 81, Vrije Univ., Amsterdam 1978.
[9] Pelczynski A., *Linear extensions, linear averagings, and their application to linear topological classification of spaces of continuous functions*, Dissertationes Math. **58**, Warszawa 1968.
[10] Rudin M.E., *Lectures on set theoretic topology*, Regional Conf. Ser. in Math. No. 23, Amer. Math. Soc., Providence, RI, 1977.
[11] Strok M., Szymanski A., *Compact metric spaces have binary bases*, Fund. Math. **89** (1975), 81–91.

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