Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 655--660

Persistent URL: `http://dml.cz/dmlcz/106901`

Orthogonal vector measures on projection lattices in a Hilbert space

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Abstract. We characterize orthogonal vector measures on the lattice of all projections in a von Neumann algebra (without type I_2-direct summand) by means of operator-valued mappings (Theorem 3). As a corollary we obtain a lucid description of orthogonal vector measures in the finitely-dimensional case (Corollary 2). Our results contribute to the non-commutative probability theory. They also extend hitherto known results on correlation functions of orthogonal measures [4,6] and give a new insight into Gleason theorems [3], [7], [8].

Keywords: Gleason measures, projection lattices in von Neumann algebras

Classification: 81B10, 46L10, 46L30

1. Introduction and preliminaries.

Throughout the paper, let $\mathcal{A}$ be a von Neumann algebra without type I_2-direct summand which acts on a Hilbert space H. By the symbol $L(\mathcal{A})$ we denote the lattice of all orthogonal projections in $\mathcal{A}$. We shall be interested in orthogonal vector measures on $L(\mathcal{A})$. In the quantum mechanical foundations measures of this type may be viewed as generalized stochastics processes [**1**],[**9**]. They also play an important role in the description of the evolution of quantun systems [**1**],[**5**],[**9**].

We first fix some notation as we shall use it in the sequel. Throughout the paper, let K be an arbitrary Hilbert space. By $\mathcal{B}(K)$ we shall mean the C^*–algebra of all bounded operators acting on H. Let $C_2(K)$ denote the two-sided ideal of all Hilbert–Schmidt operators acting on K. As known, the space $C_2(K)$ endowed with an inner product $\langle .,. \rangle$ defined by the equality $\langle A,B\rangle = \mathrm{Tr}\ AB^*$ $(A,B \in C_2(K))$ (here Tr denotes the faithful normal semifinite trace on $\mathcal{B}(K)$) forms a Hilbert space. The space $C_2(K)$ is isomorphic to $l^2(E \times E)$, where E is an orthonormal basis of K (for details, see [**6**]).

Definition 1. A mapping $m : L(\mathcal{A}) \to K$ is said to be an orthogonal vector measure if for any family $(P_\alpha)_{\alpha\in I}$ of mutually orthogonal projections from $\mathcal{A}$ the following two conditions are satisfied:

(i) the set $(m(P_\alpha))_{\alpha\in I}$ is orthogonal in K,

(ii) we have

$$m(\sum_{\alpha\in I} P_\alpha) = \sum_{\alpha\in I} m(P_\alpha),$$

where the series on the right-hand side is supposed to converge in the norm topology on K.

Let us exhibit a typical example of such measures (compare also with Corollary 1 and Corollary 2).

Example 1. Let $v \in H$. Then the mapping m: $L(\mathcal{A}) \to H$ defined by putting

$$m(P) = Pv \quad (P \in L(\mathcal{A}))$$

is an orthogonal vector measure.

A linear mapping $F : \mathcal{A} \to K$ is said to be an *orthogonal vector field on* $\mathcal{A}$ if F is normal (i.e. if F is continuous with respect to the ultraweak topology on $\mathcal{A}$ and weak topology on K) and if $\langle F(P), F(Q)\rangle = 0$ for any orthogonal elements $P, Q \in L(\mathcal{A})$. Let us define a *correlation function* C_F of F by putting $C_F(P,Q) = \langle F(P), F(Q)\rangle (P, Q \in L(\mathcal{A}))$ (see [5]). The basic result on correlation functions is the following theorem by S. Goldstein [4] (see also [5] for I_n-factors).

Theorem 1 ([4]). *Let F be an orthogonal vector field on $\mathcal{A}$. Then there are nonnegative trace class operators $A, B \in \mathcal{B}(H)$ such that*

$$\langle F(P), F(Q)\rangle = \mathrm{Tr}(APQ + BQP)$$

for any projections $P, Q \in \mathcal{A}$.

2. A characterization of orthogonal vector measures.

Let F be an orthogonal vector field on $\mathcal{A}$. Then $F \mid L(\mathcal{A})$ is an orthogonal vector measure on $L(\mathcal{A})$. We prove now the converse theorem which may be interpreted as a Gleason type theorem for orthogonal measures.

Theorem 2. *Let $m : L(\mathcal{A}) \to K$ be an orthogonal vector measure. Then there is an orthogonal vector field $F : \mathcal{A} \to K$ such that $F \mid L(\mathcal{A}) = m$.*

PROOF : Let us first define F on the set $S(K)$ of all self-adjoint operators from $\mathcal{A}$. Suppose that $A \in S(K)$. Let $L(A)$ be a projection lattice of the smallest abelian von Neumann subalgebra of $\mathcal{A}$ containing A. Then $L(A)$ is a Boolean algebra and we can extend m from $L(A)$ to a linear mapping F_A on $\mathrm{sp}\, L(A)$. Moreover, if $S = \sum_{i=1}^{n} \lambda_i Q_i$, where $\lambda_i \in C$ and $Q_1, ... Q_n \in L(A)$ are mutually orthogonal, then we can write

$$\begin{aligned} \| F_A(S) \|^2 &= \langle \sum_{j=1}^{n} \lambda_j m(Q_j), \sum_{j=1}^{n} \lambda_j m(Q_j)\rangle = \sum_{j=1}^{n} | \lambda_j |^2 \| m(Q_j) \|^2 \\ &\le max\{| \lambda_j |^2 \| m(\sum_{j=1}^{n} Q_j) \|^2 \mid 1 \le j \le n\} \le \| S \|^2 \| m(I) \|^2 . \end{aligned} \tag{1}$$

Thus, F_A is continuous on $\mathrm{sp}\, L(A)$ and we can put $F(A) = \lim_{n\to\infty} F_A(S_n)$ for an arbitrary sequence $(S_n) \subset \mathrm{sp}\, L(A)$ converging to A (the spectral theorem). Moreover, by the condition (1) we see that

$$\| F(A) \| \le \| A \| \, \| m(I) \| .$$

If $B \in \mathcal{A}$, then we can put $F(B) = F(ReB)+iF(ImB)$. By virtue of the preceeding estimation we see that

$$\| F(B) \| \leq 2 \| B \| \| m(I) \| .$$

To prove our theorem now, it is sufficient to show that F is linear. For this, let $u \in K$ be an arbitrary vector. Let us define $m_u : L(\mathcal{A}) \to C$ by putting

$$m_u(P) = \langle m(P), u \rangle \quad (P \in L(\mathcal{A})).$$

Then $\sup\{|m_u(P)| \mid P \in L(\mathcal{A})\} \leq \|m(I)\| \|u\|$. It follows that m_u is a bounded measure on $L(\mathcal{A})$. According to a result of M.S. Matveichuk [7],[8], there is a normal functional f_u on $\mathcal{A}$ extending m_u. Moreover, we can easily verify that for any $A \in S(K)$ we have $f_u(A) = \langle F(A), u\rangle$. Thus, $\langle F(A_1+A_2), u\rangle = \langle F(A_1)+F(A_2), u\rangle$ and F is a linear mapping extending m. ■

Let $m : L(\mathcal{A}) \to K$ be an orthogonal vector measure. Let us denote by $\mathcal{R}(m)$ the space $\overline{\mathrm{sp}}\{m(P) \mid P \in L(\mathcal{A})\}$. Further, call two orthogonal measures $m_1 : L(\mathcal{A}) \to H$ and $m_2 : L(\mathcal{A}) \to K$ unitarily equivalent if there is a unitary mapping $\mathcal{U}$ of $\mathcal{R}(m_1)$ onto $\mathcal{R}(m_2)$ such that $m_2 = \mathcal{U} \circ m_1$.

If $\dim H = \infty$, then H and the direct sum $C_2(H)\oplus C_2(H)$ are isomorphic Hilbert spaces. In this case we have the following lemma which enables us to construct H-valued orthogonal vector measures with a given correlation function (a contrapart of Theorem 1).

Lemma 1. *Let $A, B \in \mathcal{B}(H)$ be nonnegative trace class operators. Then a mapping $m_{A,B} : L(\mathcal{A}) \to C_2(H) \oplus C_2(H)$ defined by the formula*

$$m_{A,B}(P) = A^{1/2}P + PB^{1/2} \qquad P \in L(\mathcal{A})$$

is an orthogonal vector measure with the correlation function

$$\langle m_{A,B}(P), m_{A,B}(Q)\rangle = \mathrm{Tr}(APQ + BQP) \qquad P, Q \in L(\mathcal{A}).$$

PROOF : Let us first observe that the definition of $m_{A,B}$ is correct. For any orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H we have

$$\sum_{\alpha\in I} \| A^{1/2}e_\alpha \|^2 = \sum_{\alpha\in I}\langle Ae_\alpha, e_\alpha\rangle = \mathrm{Tr}\, A < \infty.$$

Thus, $A^{1/2}, B^{1/2} \in C_2(H)$ and we see that $\mathcal{R}(m_{A,B}) \subset C_2(H)\oplus C_2(H)$. Obviously, the mapping $m_{A,B}$ is finitely additive and we have to show that it is completely additive. For this, let $P = \sum_{\alpha\in I} P_\alpha$, where $(P_\alpha)_{\alpha\in I}$ is a family of mutually orthogonal projections from $\mathcal{A}$. Let $(Q_\beta)_{\beta\in J}$ be a net of finite partial sums of series $\sum_{\alpha\in I} P_\alpha$. Then $Q_\beta \nearrow P$. Making use of the properties of the trace, we have

$$\| m_{A,B}(P) - m_{A,B}(Q_\beta) \|^2 = \| A^{1/2}(P-Q_\beta) \|_2^2 + \| (P-Q_\beta)B^{1/2} \|_2^2$$
$$= \mathrm{Tr}\, A^{1/2}(P-Q_\beta)A^{1/2} + \mathrm{Tr}(P-Q_\beta)B^{1/2}B^{1/2}(P-Q_\beta) = \mathrm{Tr}(A+B)(P-Q_\beta) \to 0 .$$

It remains to prove the equation (2) in Lemma 1. Take $P, Q \in L(\mathcal{A})$. Then

$$\langle m_{A,B}(P), m_{A,B}(Q)\rangle = \langle A^{1/2}P, A^{1/2}Q\rangle + \langle PB^{1/2}, QB^{1/2}\rangle$$
$$= \mathrm{Tr}\, A^{1/2}PQA^{1/2} + \mathrm{T}r PB^{1/2}B^{1/2}Q = \mathrm{Tr}(APQ + BQP).$$

The proof is complete. ■

Theorem 3. *Let $m : L(\mathcal{A}) \to H$ be an orthogonal vector measure. Then m is unitarily equivalent with a measure $m_{A,B}$ defined by the formula*

$$m_{A,B}(P) = A^{1/2}P \oplus PB^{1/2} \qquad P \in L(\mathcal{A}),$$

where $A, B \in \mathcal{B}(H)$ are nonnegative trace class operators.

PROOF : By Theorem 1 there are nonnegative trace class operators $A, B \in \mathcal{B}(H)$ such that

$$\langle m(P), m(Q)\rangle = \langle m_{A,B}(P), m_{A,B}(Q)\rangle \quad P, Q \in L(\mathcal{A}).$$

Define now a mapping $\mathcal{U}$ on the set $D = \{m_{A,B}(P) \mid P \in L(\mathcal{A})\}$ such that

$$\mathcal{U}m_{A,B}(P) = m(P) \qquad \text{for any} \quad P \in L(\mathcal{A}).$$

We have to show that this definition is unambigous and that the mapping $\mathcal{U}$ can be linearly extended over the space $\mathcal{R}(m_{A,B})$. For this, let $\sum_{i=1}^{n} \alpha_i m_{A,B}(P_i) = m_{A,B}(Q)$, where $P_i, Q \in L(\mathcal{A})$ and α_i are complex numbers (i=1,...,n). Then

$$\begin{aligned} &\| \sum_{i=1}^{n} \alpha_i m(P_i) - m(Q) \|^2 \\ =&\| \sum_{i=1}^{n} m(P_i) \|^2 + \| m(Q) \|^2 - 2Re\langle \sum_{i=1}^{n} \alpha_i m(P_i), m(Q)\rangle \\ =&\| \sum_{i=1}^{n} \alpha_i m_{A,B}(P_i) - m_{A,B}(Q) \|^2 = 0\,. \end{aligned}$$

Thus $\mathcal{U}$ may be extended to a unitary mapping from $\mathcal{R}(m_{A,B})$ into H and the proof is complete. ∎

As a consequence of the analytic form of orthogonal measures given in the foregoing theorem we derive a geometrical expression of such measures.

Corollary 1. *Let $m : L(\mathcal{A}) \to H$ be an orthogonal vector measure. Then there is such a sequence $(v_k)_{k\in I} \subset H$ and a (real) linear isometry $\mathcal{U} : \mathcal{H} \to H$, where $\mathcal{H} = \overline{\mathrm{sp}}\{ \sum_{k\in I} \oplus Pv_k \mid P \in L(\mathcal{A}) \}$ (in $\sum_{k\in I} \oplus H$) such that*

$$m(P) = \mathcal{U}(\sum_{k\in I} \oplus Pv_k) \quad \textit{for any } P \in L(\mathcal{A}).$$

PROOF : Theorem 3 implies that m is unitarily equivalent with some measure $m_{A,B}$. Put $m_{A,B} = m_A + m_B$, where $m_A(P) = A^{1/2}P$ and $m_B(P) = PB^{1/2}$ ($P \in L(\mathcal{A})$). Let $(u_k)_{k\in I_1}$ (resp. $(u_k)_{k\in I_2}$) be an orthonormal set of eigenvectors corresponding to nonzero eigenvalues $(\lambda_k)_{k\in I_1}$ of A (resp. corresponding to nonzero eigenvalues $(\lambda_k)_{k\in I_2}$ of B) (we put $I_1, I_2 \subset N$ and $I_1 \bigcap I_2 = \emptyset$). Put $H_i =$

$\sum_{k\in I_i}\oplus\mathcal{H}_k$, where $\mathcal{H}_k=\overline{\mathrm{sp}}\{Pu_k \mid P\in L(\mathcal{A})\}$ (i=1,2). Let us define an H_i−valued orthogonal measure m_i (i=1,2) on $L(\mathcal{A})$ by putting

$$m_i(P)=\sum_{k\in I_i}\oplus\lambda_k^{1/2}Pu_k \quad (P\in L(\mathcal{A}))$$

($m_i(P)=0$ if $I_i=\emptyset$). Then

$$\langle m_1(P),m_1(Q)\rangle=\sum_{k\in I_1}\langle\lambda_k^{1/2}Pu_k,\lambda_k^{1/2}Qu_k\rangle=$$

$$=\sum_{k\in I_1}\lambda_k\langle QPu_k,u_k\rangle=\mathrm{Tr}\,AQP=\langle m_A(Q),m_A(P)\rangle \quad \text{for any } P,Q\in L(\mathcal{A}).$$

Following the technique of the proof of Theorem 3 we can find an antiunitary mapping $V_1:\mathcal{R}(m_1)\to\mathcal{R}(m_A)$ such that

$$m_A(P)=V_1m_1(P) \quad \text{for any } P\in L(\mathcal{A}).$$

Analogically, we can show that

$$\langle m_2(P),m_2(Q)\rangle=\langle m_B(P),m_B(Q)\rangle \quad \text{for any } P,Q\in L(\mathcal{A}),$$

and so there is a unitary mapping $V_2:\mathcal{R}(m_2)\to\mathcal{R}(m_A)$ satisfying

$$m_B(P)=V_2m_2(P) \quad \text{for any } P\in L(\mathcal{A}).$$

Let $\mathcal{V}$ be a unitary mapping such that $m=\mathcal{V}\circ m_{A,B}$. It remains to put $I=I_1\cup I_2$, $v_k=\lambda_k^{1/2}u_k$ $(k\in I)$, $\mathcal{U}=\mathcal{V}(V_1\oplus V_2)$ and the proof is complete. ∎

Thus, up to an isometry, every H−valued orthogonal measure can be represented as an orthogonal sum of measures stated in Example 1.

In the conclusion of this note let us shortly investigate the case of the type I_n-factors $(3\le n<\infty)$. Contrary to the infinite-dimensional case $(n=\infty)$, we obtain an essential restriction on operators A,B concerning a correlation function. Indeed, if $\mathcal{A}$ is an I_n-factor, then (using notation as in the proof of Corollary 1) one can verify that $\overline{\mathrm{sp}}\{Pv_k \mid P\in L(\mathcal{A})\}=H$ if $v_k\neq 0$ and so $\dim\mathcal{R}(m_1+m_2)>\dim H$ if card $I_1\cup I_2\ge 2$. This fact allows us to present a transparent description of orthogonal measures . (Let us recall that a symmetry on H is an arbitrary unitary or antiunitary mapping on H).

Corollary 2. *Let H be a finite-dimensional Hilbert space with $3\le\dim H$ and let $L(H)$ be the lattice of all orthogonal projections acting on H. Suppose that $m:L(H)\to H$ is an orthogonal vector measure. Then there is a vector $v\in H$ and a symmetry $\mathcal{U}:H\to H$ such that*

$$m(P)=\mathcal{U}Pv \quad \textit{for any } P\in L(H).$$

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(*Received* May 16, 1990)