Yurij Aleksandrovich Abramovich; Zbigniew Lipecki

On lattices and algebras of simple functions

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 627--635

Persistent URL: http://dml.cz/dmlcz/106898

On lattices and algebras of simple functions

Y. A. Abramovich, Z. Lipecki

Abstract. We prove that every (linear) sublattice of the lattice-ordered algebra $S(2^\Omega)$ of all real-valued simple (=step) functions on a set Ω is uniformly closed. It follows that the same is true for every subalgebra of $S(2^\Omega)$. In the proof we use a special case of the latter result due to P. Dierolf, S. Dierolf and L. Drewnowski. We also give a description of sublattices, subalgebras and ideals in lattice-ordered algebras of simple functions.

Keywords: Simple function, ring of sets, linear lattice, algebra, ideal, prime, maximal, homomorphism, hyper-Archimedean, uniformly closed

Classification: 46A40, 06F20, 28A20

1. Notation.

Let Ω be an arbitrary non-empty set. Given a family $\mathfrak{A}$ of subsets of Ω, we denote by $S(\mathfrak{A})$ the real linear space spanned by the characteristic functions $1_A, A \in \mathfrak{A}$. If $\mathfrak{A}$ is a ring of sets, then $S(\mathfrak{A})$ is both a sublattice[1] and a subalgebra of the lattice-ordered algebra $\mathbf{R}^\Omega$. For a subset Z of $S(2^\Omega)$ we denote by $\overline{Z}$ its uniform closure in $S(2^\Omega)$. We also define

$$\mathfrak{T}_Z = \{A \in 2^\Omega : 1_A \in Z\}.$$

(In [**3**, p. 982] this family of sets is called the trace of Z. It also appears in [**5**, Definition 2.3.7], where a different symbol is used.) Clearly, $S(\mathfrak{T}_Z) \subset \operatorname{lin} Z$. Sufficient conditions for the equality to hold are given in Lemma 1 below.

For $x \in S(2^\Omega)$ we define

$$\operatorname{supp} x = \{\omega \in \Omega : x(\omega) \neq 0\}.$$

In the sequel $\mathfrak{R}$ stands for an arbitrary ring of subsets of Ω.

Given a linear lattice X, we denote by $H(X)$ the set of all linear-lattice homomorphisms from X into $\mathbf{R}$. The notation $H(S(\mathfrak{R}))$ is abbreviated to $H(\mathfrak{R})$.

2. Uniform closedness of sublattices and subalgebras.

The following lemma is at least partially known; in particular, see [**5**, pp. 167–168] for another proof of part (b).

We are indebted to Prof. A.I. Veksler for suggesting the relevance of the class of hyper-Archimedean linear lattices to the material of the paper.

[1] By a sublattice of a linear lattice we always mean a linear sublattice. A similar convention applies to subspaces of linear spaces.

Lemma 1.

(a) *If Z is a sublattice or a subalgebra of $S(2^\Omega)$, then $\mathfrak{T}_Z$ is a ring of sets.*
(b) *If Z is a sublattice of $S(2^\Omega)$ such that $z \wedge 1_\Omega \in Z$ whenever $z \in Z$, then $Z = S(\mathfrak{T}_Z)$.*
(c) *If Z is a subalgebra of $S(2^\Omega)$, then $Z = S(\mathfrak{T}_Z)$.*

PROOF : For the lattice case of (a) see [**3**, Lemma 1] or [**5**, Proposition 2.3.8(a)]. The algebra case follows from the well-known formulas

$$1_{A_1 \cup A_2} = 1_{A_1} + 1_{A_2} - 1_{A_1} \cdot 1_{A_2} \text{ and } 1_{A_1 \setminus A_2} = 1_{A_1} - 1_{A_1} \cdot 1_{A_2}.$$

We shall establish (b) and (c) simultaneously. Fix $z_0 \in Z$ with $z_0 \neq 0$. The assumption of (b) implies that $1_{\mathrm{supp}\, z_0} \in Z$. Denote by Z_0 the sublattice [subalgebra] of $S(2^\Omega)$ generated by z_0 and $1_{\mathrm{supp}\, z_0}$ [z_0 alone]. Let $a_1, \ldots, a_n$ be all the non-zero values of z_0 (without repetitions). Now Z_0 can be identified with a sublattice [subalgebra] of $\mathbf{R}^n$. Therefore, applying a suitable version of the Stone–Weierstrass theorem (see [**4**, §4, théorème 2] and [**9**, Theorem 4E]), we get

$$Z_0 = \mathrm{lin}\{1_{z_0^{-1}(a_i)} : \ i = 1, \ldots, n\}.$$

Hence $z_0^{-1}(a_i) \in \mathfrak{T}_Z$ and $z_0 \in S(\mathfrak{T}_Z)$. ∎

Remark 1. We have used above some versions of the Stone–Weierstrass theorem with the underlying space being finite. In that case the standard proof of the lattice version can be somewhat simplified (cf. [**4**, n°1]) and a straightforward proof of the algebra version based on Vandermonde determinants is available. The latter proof also yields the complex counterpart of (c), since it does not require the subalgebra in question to be self-adjoint (cf. [**11**, Theorem 3]).

We shall now establish our main result. It generalizes simultaneously [**6**, Lemma 2] (see also [**8**, Lemma 2]) and [**1**, Example 1]. It is worth-while to compare it with [**1**, Theorem 6], asserting the existence of a variety of dense sublattices in every infinite-dimensional complete metrizable topological linear lattice. We also note in this connection that every infinite-dimensional metrizable topological linear space contains proper dense subspaces (see e.g. [**7**, Theorem 1]).

Theorem 1. *Every sublattice of $S(2^\Omega)$ is uniformly closed.*

PROOF : Let Z be a sublattice of $S(2^\Omega)$. We shall prove that $Z = \overline{Z}$ in four steps.

Step 1. The assertion holds if $1_\Omega \in Z$. Indeed, in view of Lemma 1(a) and (b), $\mathfrak{T}_Z$ is then an algebra of subsets of Ω and $Z = S(\mathfrak{T}_Z)$. Therefore, Z is uniformly closed by [**6**, Lemma 2[2]] (see also [**8**, Lemma 2]).

Step 2. The assertion holds if there is $z_0 \in Z$ with $z_0(\omega) > 0$ for all $\omega \in \Omega$. Indeed, by Step 1,

$$\{\frac{z}{z_0} : \ z \in Z\}$$

is uniformly closed in $S(2^\Omega)$, whence so is Z, because the map $x \to z_0 \cdot x$ is a homeomorphism of $S(2^\Omega)$ onto itself.

[2]Note that the condition $\bigcup_{i=1}^m A_i = \bigcup_{k=1}^n B_k = I$ is missing in that proof.

Step 3. The assertion holds if there is $z_0 \in Z$ with

$$\operatorname{supp} z \subset \operatorname{supp} z_0 \quad \text{for all} \quad z \in Z.$$

This follows from Step 2, since

$$\{x \in S(2^\Omega) : \operatorname{supp} x \subset \operatorname{supp} z_0\}$$

is uniformly closed in $S(2^\Omega)$.

Step 4. Let Z be arbitrary and suppose, to get a contradiction, that $\overline{Z} \neq Z$. Take $x_0 \in \overline{Z} \setminus Z$ and choose $z_0 \in Z_+$ with

$$\operatorname{supp} x_0 \subset \operatorname{supp} z_0.$$

Define

$$Z_0 = \{z \in Z : \operatorname{supp} z \subset \operatorname{supp} z_0\}.$$

Clearly, Z_0 is a sublattice of $S(2^\Omega)$ and $x_0 \notin Z_0$. We claim that $x_0 \in \overline{Z}_0$, which is in contradiction with Step 3. Let (z_n) be a sequence in Z tending to x_0 uniformly. Moreover, take $t \in \mathbf{R}$ with $|x_0| \leqslant tz_0$. Then the sequence

$$(z_n \wedge tz_0) \vee (-tz_0)$$

is in Z_0 and tends uniformly to x_0. ∎

Corollary 1. *Every subalgebra of $S(2^\Omega)$ is uniformly closed.*

This is a direct consequence of Theorem 1 and Lemma 1 (c).

Corollary 2. *Every sublattice [subalgebra] of $S(\mathfrak{R})$ is the intersection of a family of sublattices [subalgebras] of $S(\mathfrak{R})$ of codimension 1.*

PROOF : In view of the Stone representation theorem for Boolean algebras, we may and do assume that Ω is a compact space and $\mathfrak{R}$ is contained in the algebra of open-closed subsets of Ω. Let Z be a sublattice of $S(\mathfrak{R})$ and denote by $\tilde{Z}$ its uniform closure in $C(\Omega)$. In view of Theorem 1,

$$Z = \tilde{Z} \cap S(\mathfrak{R}).$$

Now, it is a consequence of a classical result due to S. Kakutani, and M. Krein and S. Krein ([**4**, §4, exerc. 4]) that every uniformly closed sublattice of $C(\Omega)$ is the intersection of a family of (uniformly closed) sublattices of $C(\Omega)$ of codimension 1. The lattice case of the assertion follows. Since every uniformly closed subalgebra of $C(\Omega)$ is a sublattice of $C(\Omega)$ ([**9**, Lemma 4D]), the algebra case also follows. ∎

Remark 2. The complex counterpart of Corollary 1 is also true. More precisely, every subalgebra Z of $S_{\mathbf{C}}(2^{\Omega})$, the algebra of all complex-valued simple functions on Ω, is uniformly closed and self-adjoint. Indeed, Lemma 2 of [**6**] holds, with the same proof, in the case of complex scalars. Therefore, in view of Remark 1, the assertion in question holds if $1_{\Omega} \in Z$. The general case follows, since Z is uniformly closed in $\mathrm{lin}(Z \cup \{1_{\Omega}\})$, which is a subalgebra of $S_{\mathbf{C}}(2^{\Omega})$. As a consequence, the algebra part of Corollary 2 is also valid in the complex situation.

We note that Theorem 1 and Corollary 1 fail if the topology of uniform convergence is replaced by that of pointwise convergence. Indeed, the subalgebra

$$\{x \in \mathbf{R}^{\Omega} : \operatorname{supp} x \text{ is finite }\}$$

is pointwise dense in $S(2^{\Omega})$.

3. Ideals.

We shall give a description of lattice and algebra ideals in $S(\mathfrak{R})$ and derive from it a version of Corollary 2 for ideals. We shall also characterize ideals of codimension 1 in $S(\mathfrak{R})$.

Proposition 1 (cf. [**13**, Proposition 5.1]). *For $I \subset S(\mathfrak{R})$ the following three conditions are equivalent:*

(i) *I is a lattice ideal in $S(\mathfrak{R})$.*
(ii) *I is an algebra ideal in $S(\mathfrak{R})$.*
(iii) *$\mathfrak{T}_I$ is an ideal in $\mathfrak{R}$ and $S(\mathfrak{T}_I) = I$.*

PROOF : (i)⇒(ii) It is enough to observe that, given $x, z \in S(2^{\Omega})$, we have $|xz| \leqslant t|z|$ for some $t \in \mathbf{R}$.

(ii)⇒(iii) This follows from Lemma 1(a) and (c) and the formula $1_{A\cap B} = 1_A 1_B$.

(iii)⇒(i) It is enough to show that, given $x \in S(\mathfrak{R})$ and $z \in I$ with $|x| \leqslant |z|$, we have $x \in I$. We may confine ourselves to the case where $x = 1_A$, $A \in \mathfrak{R}$. Then $A \subset \operatorname{supp} z$. It follows from (iii) that $A \in \mathfrak{T}_I$, and we are done. ■

The following result is essentially known (see [**2**, théorème 14.1.2]). It is also a consequence of [**10**, Theorem 33.5] and Proposition 2 below.

Corollary 3. *Every ideal in $S(\mathfrak{R})$ is the intersection of a family of ideals in $S(\mathfrak{R})$ of codimension 1.*

PROOF : Let I be an ideal in $S(\mathfrak{R})$. In view of Proposition 1, $\mathfrak{T}_I$ is an ideal in $\mathfrak{R}$. Therefore,

$$\mathfrak{T}_I = \bigcap_{\alpha\in M} \mathfrak{J}_{\alpha},$$

where $\{\mathfrak{J}_{\alpha} : \alpha \in M\}$ is the family of all maximal ideals in $\mathfrak{R}$ containing $\mathfrak{T}_I$ ([**10**, Theorem 4.4]). It follows from Proposition 1 that $S(\mathfrak{J}_{\alpha})$ is a maximal ideal in $S(\mathfrak{R})$ and

$$I = \bigcap_{\alpha\in M} S(\mathfrak{J}_{\alpha}).$$

An appeal to [**10**, Theorem 27.3(i)] completes the proof. ■

Recall that an ideal I in a linear lattice X is said to be prime if, for all $x_1, x_2 \in X$ with $x_1 \wedge x_2 \in I$, we have $x_1 \in I$ or $x_2 \in I$ ([**10**, Definition 33.1]).

Proposition 2. *For $I \subsetneq S(\mathfrak{R})$ the following four conditions are equivalent:*

(i) *I is a (lattice) ideal in $S(\mathfrak{R})$ of codimension 1.*
(ii) *I is a prime lattice ideal in $S(\mathfrak{R})$.*
(iii) *I is a prime algebra ideal in $S(\mathfrak{R})$.*
(iv) *$\mathfrak{T}_I$ is a prime ideal in $\mathfrak{R}$.*

PROOF : The implication (i)⇒(ii) holds in every linear lattice (see [**10**, Theorem 33.3(i)]).

We shall show that (ii) implies (iii). In view of Proposition 1, we only have to check that if $x, y \in S(\mathfrak{R})$ are non-zero and $xy \in I$, then $x \in I$ or $y \in I$. Represent x and y in the usual way:

$$x = \sum_{i=1}^{m} a_i 1_{A_i} \quad \text{and} \quad y = \sum_{j=1}^{n} b_j 1_{B_j} ,$$

where $a_1, \ldots, a_m \in \mathbf{R}$ are non-zero and $A_1, \ldots, A_m \in \mathfrak{R}$ are pairwise disjoint, and the same holds for $b_1, \ldots, b_n$ and $B_1, \ldots, B_n$. If $x \notin I$, then $1_{A_i} \notin I$ for some i. Since $1_{A_i} \wedge 1_{B_j} \in I$, we have $1_{B_j} \in I$ for all j. Consequently, $y \in I$.

The implication (iii)⇒(iv) is obvious.

To show that (iv) implies (i), observe that $S(\mathfrak{T}_I)$ is an ideal in $S(\mathfrak{R})$. Since $\mathfrak{T}_I$ is a maximal ideal in $\mathfrak{R}$ ([**10**, Theorem 4.3(i)]), $S(\mathfrak{T}_I)$ is a maximal ideal in $S(\mathfrak{R})$ by Proposition 1. The assertion now follows from [**10**, Theorem 27.3(i)], as $S(\mathfrak{T}_I) \subset I$. ■

4. Sublattices and subalgebras of codimension 1.

We shall need the following general lemma.

Lemma 2. *Let Z be a sublattice of a linear lattice X of codimension 1. If x_1, x_2 and x_3 are pairwise disjoint elements of X, then at least one of them is in Z.*

PROOF : We assume that the x_i's are non-negative. (The general case follows by considering $(x_i)_+, (x_i)_-$ for $i = 1, 2, 3$.) Suppose $x_1, x_2 \notin Z$. Then there exist $\lambda_1, \lambda_2 \in \mathbf{R}$ and $z_1, z_2 \in Z$ such that

$$x_3 = \lambda_i x_i + z_i, \quad i = 1, 2.$$

We have

$$(z_1)_+ \wedge (z_2)_+ = (x_3 - \lambda_1 x_1)_+ \wedge (x_3 - \lambda_2 x_2)_+ = x_3,$$

as x_1, x_2 and x_3 are pairwise disjoint. It follows that $x_3 \in Z$. ■

The trivial example: $X = \mathbf{R}^2$ with the canonical ordering, $Z = \{(\lambda, \lambda) : \lambda \in \mathbf{R}\}$ and $x_1 = (1, 0)$ and $x_2 = (0, 1)$ shows that the number "3" above is best possible.

Lemma 3. *Let Z be a sublattice of $S(\mathfrak{R})$ of codimension 1. Then there exist ideals I_1 and I_2 in $S(\mathfrak{R})$ of codimension 1 such that $I_1 \cap I_2 \subset Z$.*

PROOF : We may and do assume that Z is not an ideal in $S(\mathfrak{R})$. We define

$$I = \{z \in S(\mathfrak{R}) : \forall x \in S(\mathfrak{R})[|x| \leqslant |z| \Rightarrow x \in Z]\}.$$

It follows from [**10**, Corollary 15.6(i)] that I is an ideal in $S(\mathfrak{R})$. Moreover, $I \subsetneq Z$. Lemma 2 shows that if x_1, x_2 and x_3 are pairwise disjoint elements of X, then at least one of them is in I. We claim that there exist ideals I_1 and I_2 in $S(\mathfrak{R})$ of codimension 1 such that $I_1 \cap I_2 = I$. This can be established by two methods.

1. Proposition 2 shows that the ideal $\mathfrak{T}_I$ is not prime in $S(\mathfrak{R})$. Therefore, there are disjoint A_1 and A_2 in $\mathfrak{R} \backslash \mathfrak{T}_I$. Denote by $\mathfrak{J}_i$ the ideal in $\mathfrak{R}$ generated by $\mathfrak{T}_I \cup \{A_i\}$. Clearly, $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \mathfrak{T}_I$. As easily seen, $\mathfrak{J}_i$ is a prime ideal. Define $I_i = S(\mathfrak{J}_i), i = 1, 2$. It follows from Propositions 1 and 2 that I_1 and I_2 are as desired.

2. Since $S(\mathfrak{R})$ is hyper-Archimedean, i.e. every quotient of $S(\mathfrak{R})$ by an ideal is Archimedean (see [**10**, Theorem 61.1]), we conclude that $\operatorname{codim} I = 2$ ([**10**, Theorem 26.10] and [**12**, Lemma II.3.8]). Our claim now follows from [**1**, Lemma 3(b)]. ∎

Part (a) of the following result is not new. In fact, it holds in every linear lattice (see [**12**, Proposition II.2.6 and its Corollary]).

Theorem 2. *Let Z be a sublattice of $S(\mathfrak{R})$ of codimension 1.*

(a) *Z is an ideal in $S(\mathfrak{R})$ if and only if there exists $T \in H(\mathfrak{R})$ with $Z = T^{-1}(0)$.*
(b) *Z is not an ideal in $S(\mathfrak{R})$ if and only if there exist linearly independent $T_1, T_2 \in H(\mathfrak{R})$ such that $Z = (T_1 - T_2)^{-1}(0)$.*

PROOF : To show the "if" part of (b), take $x_1, x_2 \in S(\mathfrak{R})$ with $T_i(x_j) = 0$ or 1 according as $i \neq j$ or $i = j$. We may assume that $x_i \geqslant 0$. Then

$$x_1 + x_2 \in (T_1 - T_2)^{-1}(0) \quad \text{and} \quad x_i \leqslant x_1 + x_2.$$

Therefore, $(T_1 - T_2)^{-1}(0)$ is not an ideal in $S(\mathfrak{R})$.

To show the "only if" part of (b), take I_1 and I_2 according to Lemma 3. Clearly, $Z \not\subset I_1 \cup I_2$. It follows that there exists

$$x_0 \in Z_+ \setminus (I_1 \cup I_2).$$

Take $T_i \in H(\mathfrak{R})$ with $T_i^{-1}(0) = I_i$ and $T_i(x_0) = 1$ (see (a)). Then

$$(T_1 - T_2)^{-1}(0) \supset (I_1 \cap I_2) \cup \{x_0\},$$

which yields the assertion. ∎

Remark 3. Lemma 3 and Theorem 2 hold for arbitrary hyper-Archimedean linear lattices. This is so, since the second variant of the proof of Lemma 3 works in that generality.

Remark 4. In the case where $\mathfrak{R}$ is an algebra of sets, Theorem 2 could also be proved in a similar way as Corollary 2, that is, by an application of Theorem 1, the Stone representation theorem and the Kakutani–Kreins theorem. The proof given above is, however, more direct and does not involve the axiom of choice.

Corollary 4. *Let Z be a subalgebra of $S(\mathfrak{R})$ of codimension 1. Then Z is not an ideal in $S(\mathfrak{R})$ if and only if there exist (distinct) algebra homomorphisms $U_1, U_2 : S(\mathfrak{R}) \to \mathbf{R}$ such that $Z = (U_1 - U_2)^{-1}(0)$.*

PROOF : Lemma 1(c) and Proposition 2 enable us to apply Theorem 2(b). It yields the "if" part, since every algebra homomorphism from $S(\mathfrak{R})$ into $\mathbf{R}$ is a linear-lattice homomorphism. This last assertion is a special case of the following simple and well-known result: A non-zero linear functional $T : S(\mathfrak{R}) \to \mathbf{R}$ is a lattice [algebra] homomorphism if and only if

$$\{T(1_A) : A \in \mathfrak{R}\} = \{0, \lambda\},$$

where $\lambda > 0$ [$\lambda = 1$].

Suppose now Z is not an ideal in $S(\mathfrak{R})$, and take T_1 and T_2 as in Theorem 2(b). The set functions

$$\mathfrak{R} \ni A \to T_i(1_A) \in \mathbf{R}, \quad i = 1, 2,$$

are additive and two-valued. We claim that their ranges coincide. Otherwise,

$$\mathfrak{T}_Z = \{A \in \mathfrak{R} : T_1(1_A) = T_2(1_A) = 0\},$$

and so $\mathfrak{T}_Z$ would be an ideal in $\mathfrak{R}$. Therefore, $S(\mathfrak{T}_Z) \subsetneq Z$, a contradiction (see Lemma 1(c)). Let λ denote the non-zero element of the common range of these set functions, and put $U_i = \lambda^{-1} T_i$ for $i = 1, 2$. ∎

Remark 5. Corollary 4 carries over to the complex case verbatum. Indeed, denote by $S_{\mathbf{C}}(\mathfrak{R})$ the algebra of complex-valued $\mathfrak{R}$-simple functions. For every algebra homomorphism $U : S_{\mathbf{C}}(\mathfrak{R}) \to \mathbf{C}$ we have

$$U(S(\mathfrak{R})) \subset \mathbf{R}.$$

Let Z be a subalgebra of $S_{\mathbf{C}}(\mathfrak{R})$. Clearly,

$$Z_{\mathbf{R}} = \{z \in Z : z(\Omega) \subset \mathbf{R}\}$$

is a subalgebra of $S(\mathfrak{R})$. It follows from Remark 2 that $Z = \{z_1 + iz_2 : z_1, z_2 \in Z_{\mathbf{R}}\}$ and the codimension of Z in $S_{\mathbf{C}}(\mathfrak{R})$ coincides with that of $Z_{\mathbf{R}}$ in $S(\mathfrak{R})$. Moreover, Z is an ideal in $S_{\mathbf{C}}(\mathfrak{R})$ if and only if $Z_{\mathbf{R}}$ is an ideal in $S(\mathfrak{R})$.

In closing, we shall be concerned with the uniqueness of the representation given in Theorem 2(b). Throughout the rest of the paper X stands for an arbitrary linear lattice.

Lemma 4. *Let $I_1, \dots, I_n$ and I be ideals in X of codimension 1. If*

$$I_1 \cap \cdots \cap I_n \subset I,$$

then $I = I_i$ for some i.

PROOF : Assume $I_i \neq I_j$ for $i \neq j$ and define $Y = X/I_1 \cap \cdots \cap I_n$. Denote by Q the canonical homomorphism of X onto Y. Clearly, $\dim Y \leqslant n$ and $Q(I_i) \neq Q(I_j)$ for $i \neq j$. Moreover, $Q(I_i), i = 1, \dots, n$, and $Q(I)$ are ideals in Y of codimension 1 ([**12**, Proposition II.3.1]). It follows from [**12**, Theorem II.3.9] that $Q(I) = Q(I_i)$ for some i, which yields the assertion. ∎

From Lemma 4 we immediately obtain

Proposition 3. *If $T_1, \dots, T_n, T \in H(X)$ and $T \in \operatorname{lin}\{T_1, \dots, T_n\}$, then $T = \lambda T_i$ for some $\lambda \geqslant 0$ and i.*

Corollary 5. *Let $T_1, T_2, S_1, S_2 \in H(X)$ and suppose*

$$(T_1 - T_2)^{-1}(0) = (S_1 - S_2)^{-1}(0)$$

is not an ideal in X. Then $\{S_1, S_2\} = \{\lambda T_1, \lambda T_2\}$ for some $\lambda > 0$.

PROOF : By assumption, $S_1 - S_2 = \lambda(T_1 - T_2)$ and the sets $\{T_1, T_2\}$ and $\{S_1, S_2\}$ are linearly independent. Therefore, by Proposition 3, we get $S_1 = \alpha T_1$ or $S_1 = \alpha T_2$ for some $\alpha > 0$. In the first case, we have

$$(\lambda - \alpha)T_1 - \lambda T_2 + S_2 = 0.$$

Another application of Proposition 3 yields $S_2 = \beta T_2$ for some $\beta > 0$. It follows that

$$(\lambda - \alpha)T_1 = (\lambda - \beta)T_2\,,$$

whence $\lambda = \alpha = \beta$. The argument in the second case is similar. ∎

REFERENCES

[1] Abramovich Y.A., Lipecki Z., *On ideals and sublattices in linear lattices and F-lattices*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 79–87.

[2] Bigard A., Keimel K., Wolfenstein S., *Groupes et anneaux réticulés*, Lecture Notes in Math. **608**, Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[3] Bogdanowicz W.M., *Measurability and linear lattices of real functions closed under convergence everywhere*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 981–986.

[4] Bourbaki N., *Topologie générale*, Ch. **X**, Hermann, Paris, 1961.

[5] Constatinescu C., Weber K., Sontag A., *Integration Theory, vol. 1: Measure and Integral*, John Wiley & Sons, New York, 1985.

[6] Dierolf P., Dierolf S., Drewnowski L., *Remarks and examples concerning unordered Baire-like and ultra-barelled spaces*, Colloq. Math. **39** (1978), 109–116.

[7] Lipecki Z., *On some dense subspaces of topological linear spaces*, Studia Math. **77** (1984), 413–421.

[8] Lipecki Z., *On common extensions of two quasi-measures*, Czechoslovak Math. J. **36** (1986), 489–494.
[9] Loomis L.H., *An Introduction to Abstract Harmonic Analysis*, D. van Nostrand Company, New York, 1953.
[10] Luxemburg W.A.J., Zaanen A.C., *Riesz Spaces I*, North–Holland, Amsterdam–London, 1971.
[11] Rudin W., *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42.
[12] Schaefer H.H., *Banach Lattices and Positive Operators*, Springer–Verlag, Berlin–Heidelberg–New York, 1974.
[13] Weber H., *Minimal ring topologies*, preprint.

Department of Mathematical Sciences, IUPUI, Indianapolis, Indiana 46205, USA
Institute of Mathematics, Polish Academy of Sciences, Wrocław Branch, Kopernika 18, 51–617 Wrocław, Poland

(*Received* May 21, 1990)