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Abstract. In this note the complete characterization is given for continuity of the superposition operator acting from the space of all sequences or all functions Cesáro strongly summable to zero into the space l_1 or $L_1([0,\infty))$, respectively. Properties as well as criteria for uniform continuity of such kind of operator are essentially different from analogical ones for superposition operator acting from l_1 to l_1 or from $L_1([1,\infty))$ to $L_1([1,\infty))$.

Keywords: Space of all sequences Cesáro strongly summable to zero, space of all functions Cesáro strongly summable to zero, Lebesgue sequence space, Lebesgue function space, superposition operator

Classification: 46E30, 47B38

1. Introduction.

Let $\mathbf{R} = (-\infty,\infty)$ be the set of all real numbers, $\mathbf{N}$ the set of all natural numbers and S the set of all real sequences. We shall denote the n-th term of a sequence $x \in S$ by x_n and write $x = \{x_n\}$. By l_1 we denote the space of all $x \in S$ such that $\sum_{k=1}^{\infty}|x_k| < \infty$ equipped with the norm $\|\cdot\|_l$ defined as

$$\|x\|_l = \sum_{k=1}^{\infty}|x_k|$$

for every $x \in l_1$. Further, let w_0 be the space of all sequences which are Cesáro strongly summable to zero, i.e.

$$w_0 = \{x \in S : \frac{1}{n}\sum_{k=1}^{n}|x_k| \to 0 \text{ as } n \to \infty\}.$$

It is well known (see [10], [9] and [4]) that w_0 is a Banach space with the norm

$$\|x\|_w = \sup_r\{2^{-r}\sum\nolimits_r |x_k|\},$$

where $\sum_r$ denotes a sum over the range $2^r \leqslant k < 2^{r+1}$ and $r \in \mathbf{N}_0 = \{0,1,2,\dots\}$. For convenience, denote $I_r = \{k \in \mathbf{N} : 2^r \leqslant k < 2^{r+1}\}$.

Define the superposition operator F from S into S as follows:

$$Fx = \{f(k,x_k)\} \text{ for every } x \in S,$$

where the function $f : \mathbf{N} \times \mathbf{R} \to \mathbf{R}$. Moreover, sometimes we shall assume additionally some of the following conditions:

(1) $f(k,0) = 0$ for every $k \in \mathbf{N}$;

(2) $f(k,\cdot)$ is continuous for every $k \in \mathbf{N}$;

($\bar{2}$) for every $k \in \mathbf{N}$ the function $f(k,\cdot)$ is bounded on every bounded subset of real numbers.

The complete characterization of F acting from l_p into $l_q (p, q \geqslant 1)$ was given by F. Dedagich and P.P. Zabrejko [6]. Operator F defined on sequence Orlicz space was considered by J. Robert [14] and I.V. Shragin [15]. This note is a continuation of research made by Chew Tuan Seng (see [4], [5]) and R. Płuciennik [11]. For convenience of reading we shall present the following theorems:

Theorem 1. *Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ satisfy* (1) *and* (2). *The superposition operator F acts from w_0 to l_1 iff the following condition is satisfied*

(3) *there exist $a = \{a_k\} \in l_1$ and $c = \{c_k\} \in l_1$ with $a_k \geqslant 0$, $c_k \geqslant 0$ and $\eta > 0$ such that for $r \in \mathbb{N}_0, k \in I_r$, we have*
$$|f(k,u)| \leqslant a_k + c_r 2^{-r}|u|,$$
whenever $|u| \leqslant 2^r \eta$.

Remark 1. The above theorem was proved in this form by Chew Tuan Seng in [4]. It remains true without assumption (1). Moreover, using in the proof of that theorem (cf. [4]) the idea of the proof of Theorem 3 from [11], we can place assumption (2) for weaker one $(\bar{2})$

We say that the superposition operator F from Banach function space $(X, \|\cdot\|_X)$ into Banach function space $(Y, \|\cdot\|_Y)$ is locally bounded at the point $z \in X$ iff there exist constants $\alpha > 0$ and $\beta > 0$ such that for every $x \in B_\alpha(z) = \{x \in X : \|x - z\|_X \leqslant \alpha\}$ we have $\|Fx - Fz\|_Y \leqslant \beta$. The superposition operator is called bounded iff $\sup\{\|Fx\|_Y : x \in B_\varrho(0)\} < \infty$ for every $\varrho > 0$.

Theorem 2. *Suppose that the function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ satisfies* (3). *Then the operator F is locally bounded at every point $z \in w_0$ iff for every $k \in \mathbb{N}$ the function $f(k, \cdot)$ is bounded on every bounded set of real numbers, i.e. f satisfies* $(\bar{2})$.

Theorem 2 was proved by R. Płuciennik [11] in the case of f satisfying (1). Obviously, using a well-known technical trick, we can omit assumption (1).

Theorem 3. *The superposition operator F is bounded operator from w_0 into l_1 iff for every $\varrho > 0$ there are sequences $a(\varrho) = \{a_k(\varrho)\} \in l_1$ and $c(\varrho) = \{c_k(\varrho)\} \in l_1$ such that for $r \in \mathbb{N}_0$ and $k \in I_r$, the ineaquality*

$$|f(k,u)| \leqslant a_k(\varrho) + c_r(\varrho) 2^{-r}|u| \tag{4}$$

holds, whenever $|u| \leqslant 2^r \varrho$. Furthermore,

$$\mu_f(\varrho) \leqslant \nu_f(\varrho) \leqslant \|FO\|_l + 2\mu_f(\varrho)$$

for every $\varrho > 0$, where

$$\mu_f(\varrho) = \sup\{\|Fx\|_l : x \in B_\varrho(0)\}$$

and

$$\nu_f(\varrho) = \inf\{\|a(\varrho)\|_l + \|c(\varrho)\|_l : |f(k,u)| \leqslant a_k(\varrho) + c_r(\varrho)2^{-r}|u|, |u| \leqslant 2^r \varrho\}.$$

For the proof we refer to [**11**].

Results as well as proofs concerning boundedness, continuity and uniform continuity of the superposition operator in function spaces differ essentially from analogical ones in the sequence case. It is reason, why it is worth to consider the function case W_0 separately. To this end let M be the space of all Lebesgue-measurable real functions defined on $[1,\infty)$ (more precisely, equivalence classes of such functions with respect to equality almost everywhere). Define the space

$$W_0 = \{x \in M : \lim_{T\to\infty} \frac{1}{T} \int_1^T |x(t)|\,dt = 0\}$$

equipped with the norm

$$\|x\|_W = \sup_{r\in\mathbb{N}_0} \{2^{-r} \int_{A(r)} |x(t)|\,dt\},$$

where $A(r)$ denotes the interval $[2^r, 2r+1)$. $L_1([1,\infty))$ denotes the space of all integrable real functions defined on $[1,\infty)$ and $\|\cdot\|_L$ denotes the natural norm in $L_1([1,\infty))$. For the function case we shall assume that $f:[1,\infty)\times\mathbf{R}\to\mathbf{R}$ is the Carathéodory function, i.e. the function $f(t,\cdot)$ is continuous for almost all (a.a.) $t\in[1,\infty)$ and $f(\cdot,u)$ is measurable for every $u\in\mathbf{R}$. The superposition operator F from M into M is defined by the formula

$$[Fx](t) = f(t,x(t)) \text{ for every } x \in M.$$

The following theorem, proved by Chew Tuan Seng (see [**4**]), is fundamental:

Theorem 4. *The superposition operator F maps the space W_0 into the space $L_1([1,\infty))$ iff there exist $\alpha>0$ and $c=\{c_r\}\in l_1$ such that for each $r\in\mathbb{N}_0$ there exists $a_r(\cdot)\in L_1(A(r))$ with $\int_{A(r)} a_r(t)\,dt \leqslant c_r$ such that the inequality*

$$|f(t,u)| \leqslant a_r(t) + \alpha 2^{-r} c_r |u| \tag{5}$$

holds for a.a. $t\in A(r)$ and $u\in\mathbf{R}$.

Define the function $a(\cdot):[1,\infty)\to[0,\infty)$by the formula $a(t)=a_r(t)$ for $t\in A(r)$.

2. Results in sequence case.

Theorem 5. *Suppose that the superposition operator F acts from w_0 into l_1. Then the operator F is continuous at every point $z\in w_0$ iff the function $f(k,\cdot)$ is continuous for every $k\in\mathbb{N}$, i.e. f satisfies* (2).

Proof : For the proof of sufficiency suppose (2). By Theorem 1 there exist a number $\eta>0$ and sequences $\{a_k\}\in l_1, \{c_k\}\in l_1$ of non-negative terms such that for $r\in\mathbb{N}_0$ and $k\in I_r$ we have

$$|f(k,u)| \leqslant a_k + c_r 2^{-r}|u|,$$

provided $|u| \leqslant 2^r\eta$. Fix $z = \{z_k\} \in w_0$. Let $\bar{r}$ be the smallest natural number such that for $\bar{k} = 2^{\bar{r}}$

$$\|z\chi_{\{\bar{k},\bar{k}+1,\dots\}}\|_w \leqslant \frac{\eta}{2},$$

where $\chi_{\{\bar{k},\bar{k}+1,\dots\}}$ denotes the characteristic function of the set $\{\bar{k}, \bar{k}+1, \dots\}$. Then for every $x \in B_{\eta/2}(z)$ we have

$$\|x\chi_{\{\bar{k},\bar{k}+1,\dots\}}\|_w = \sup_{r\geqslant\bar{r}}\{2^{-r}\sum\nolimits_r |x_k|\} \leqslant$$
$$\leqslant \sup_{r\geqslant\bar{r}}\{2^{-r}\sum\nolimits_r |x_k - z_k|\} + \sup_{r\geqslant\bar{r}}\{2^{-r}\sum\nolimits_r |z_k|\} \leqslant \eta.$$

Hence $|z_k| \leqslant \frac{\eta}{2}2^r$ and $|x_k| \leqslant \eta 2^r$ for every $k \in I_r$, whenever $r \geqslant \bar{r}$. For fixed $\varepsilon > 0$ we define

$$r_\varepsilon = \min\{s \geqslant \bar{r} : \sum_{k=2^s}^{\infty} a_k < \frac{\varepsilon}{6} \quad \text{and} \quad \sum_{r=s}^{\infty} c_r < \frac{\varepsilon}{6\eta}\}.$$

Since $f(k,\cdot)$ is continuous for every $k \in \mathbf{N}$, so there is a $\delta \in (0,\eta)$ such that

$$\sum_{k=1}^{2^{r_\varepsilon}-1} |f(k,x_k) - f(k,z_k)| < \frac{\varepsilon}{3},$$

provided $\|x - z\|_w < \delta$, i.e. $|x_k - z_k| < \delta 2^{r_\varepsilon}$ for $k = 1, 2, \dots, 2^{r_\varepsilon} - 1$. Therefore, using inequality (3) for $r \geqslant r_\varepsilon$, we have

$$\|Fx - Fz\|_l = \sum_{k=1}^{\infty} |f(k,x_k) - f(k,z_k)| \leqslant$$
$$\leqslant \sum_{k=1}^{2^{r_\varepsilon}-1} |f(k,x_k) - f(k,z_k)| + \sum_{k=2^{r_\varepsilon}}^{\infty} |f(k,x_k)| + \sum_{k=2^{r_\varepsilon}}^{\infty} |f(k,z_k)| \leqslant$$
$$\leqslant \frac{\varepsilon}{3} + 2\sum_{k=2^{r_\varepsilon}}^{\infty} a_k + \sum_{r=r_\varepsilon}^{\infty} c_r\{2^{-r}[\sum\nolimits_r(|x_k| + |z_k|)]\} < \varepsilon.$$

This completes the proof of sufficiency.

Suppose conversely that the superposition operator is continuous on w_0. Let $i_k : \mathbf{R} \to w_0$ be the embedding defined for every $u \in \mathbf{R}$ by the formula $i_k(u) = \{u_j\} = \{u\delta_{jk}\}$, where δ^{jk} is Kronecker's symbol and let $p_k : l_1 \to \mathbf{R}$ be the projection defined for $v = \{v_j\} \in l_1$ by the formula $p_k(v) = v_k$. Then for every $k \in \mathbf{N}$ the function $f(k,\cdot)$ factors as follows

$$\begin{tikzcd} & w_0 \arrow[r, "F"] & l_1 \arrow[dr, "p_k"] & \\ \mathbf{R} \arrow[ur, "i_k"] \arrow[rrr, "f(k,\cdot)"] & & & \mathbf{R} \end{tikzcd}$$

i.e. $f(k,\cdot) = p_k \circ F \circ i_k$. Obviously, functions p_k and i_k are continuous for every $k \in \mathbb{N}$. Consequently, the function $f(k,\cdot)$ is continuous for each $k \in \mathbb{N}$ as a composition of continuous functions. Thus the proof of the theorem is complete. ∎

If the function $f(k,\cdot)$ is continuous on $\mathbb{R}$ for every k, then the superposition operator F generated by f is continuous (Theorem 5) and locally bounded (Theorem 2). The following example shows that F is not necessary uniformly continuous on bounded sets.

Example 1. Consider the operator F generated by the function

$$f(k,u) = 2^{-r}|u2^{-r}|^r \text{ for every } r \in \mathbb{N}_0, k \in I_r.$$

Obviously, $f(k,\cdot)$ is continuous for every $k \in \mathbb{N}$. Moreover, f satisfies inequality (3) with $a = \{a_k\} = \{0\}, c = \{c_r\} = \{2^{-r}\}$ and $\eta \leqslant 1$. Hence f is continuous and locally bounded operator from w_0 into l_1. We shall show that F is not uniformly continuous on bounded sets. To this end, consider two sequences

$$x^{(r)} = \{x_k^{(r)}\} = \{(2k+1)\chi_{\{2^r\}}\},$$
$$z^{(r)} = z_k^{(r)} = \{(2k-1)\chi_{\{2^r\}}\}, r = 0,1\ldots.$$

Obviously, $x^{(r)}$ and $z^{(r)}$ belong to $B_3(0)$ for every $r \in \mathbb{N}_0$. Further,

$$\|x^{(r)} - z^{(r)}\|_w = \frac{2}{2^r} \to 0 \text{ as } r \to \infty.$$

On the other hand, denoting $[r/2] = \max\{j \in \mathbb{N}_0 : j < \frac{r}{2}\}$, we have

$$\|Fx^{(r)} - Fz^{(r)}\|_l = \frac{1}{2^r}\left[\left(\frac{2^{r+1}+1}{2^r}\right)^r - \left(\frac{2^{r+1}-1}{2^r}\right)^r\right] =$$
$$= 2^{-r^2-r+1}\left[\sum_{i=0}^{[r/2]}\left(\binom{r}{2i+1}\right)2^{(r+1)(r-1)}\right] \geqslant 2$$

for $r \geqslant 1$. Hence F cannot be uniformly continuous on the ball $B_3(0)$.

Theorem 6. *The following three statements are equivalent:*

a) *F is a uniformly continuous (on bounded sets) operator from w_0 into l_1;*

b) *For every positive constants ϱ and δ one can find sequences of non-negative real numbers $a(\varrho,\delta) = \{a_k(\varrho,\delta)\}, b(\varrho,\delta) = \{b_k(\varrho,\delta)\}, c(\varrho,\delta) = \{c_k(\varrho,\delta)\}$ such that $a(\varrho,\delta) \in l_1, b(\varrho,\delta) \in l_1, c(\varrho,\delta) \in l_1, \|a(\varrho,\delta)\|_l + \|b(\varrho,\delta)\|_l \to 0$ as $\delta \to 0$ and for $r = 0,1,\ldots, k \in I_r$, the inequality*

$$(6) \qquad |f(k,u) - f(k,v)| \leqslant a_k(\varrho,\delta) + b_r(\varrho,\delta)2^{-r}(|u|+|v|) + c_r(\varrho,\delta)2^{-r}|u-v| \cdot$$

holds, whenever $|u| \leqslant 2^r\varrho, |v| \leqslant 2^r\varrho$ and $|u-v| \leqslant 2^r\delta$;

c) *f satisfies* (2) *and F is a bounded operator from w_0 into l_1.*

PROOF : *a*) $\Rightarrow$ *b*). Let $\omega_f(\varrho,\delta)$ be the modulus of continuity of the operator F, i.e.

$$\omega_f(\varrho,\delta)=\sup\{\|Fx-Fy\|_l : x,y\in B_\varrho(0) \text{ and } \|x-y\|_w\leqslant\delta\}.$$

Fix $\varrho>0$ and $\delta>0$. Define

$$\bar{c}_r(\varrho,\delta)=\sup\{\sum\nolimits_r|f(k,x_k)-f(k,y_k)| : \frac{1}{2^r}\sum\nolimits_r|x_k|\leqslant\varrho, \\ \frac{1}{2^r}\sum\nolimits_r|y_k|\leqslant\varrho \text{ and } \frac{1}{2^r}\sum\nolimits_r|x_k-y_k|\leqslant\delta\}.$$

By Theorem 5 the function $f(k,\cdot)$ is continuous for every k. Therefore, for each $r\in\mathbb{N}_0$ there are sequences of real numbers $\bar{x}_k$ and $\bar{y}_k, k\in I_r$ (depended on ϱ and δ) such that

$$\bar{c}_r(\varrho,\delta)=\sum\nolimits_r|f(k,\bar{x}_k)-f(k,\bar{y}_k)|, \\ \frac{1}{2^r}\sum\nolimits_r|\bar{x}_k|\leqslant\varrho, \frac{1}{2^r}\sum\nolimits_r|\bar{y}_k|\leqslant\varrho \text{ and } \frac{1}{2^r}\sum\nolimits_r|\bar{x}_k-\bar{y}_k|\leqslant\delta.$$

For any $r\in\mathbb{N}_0$ we define the sequences $s^{(r)}(\varrho,\delta)=\{s_k^{(r)}(\varrho,\delta)\}$ and $z^{(r)}(\varrho,\delta)=\{z_k^{(r)}(\varrho,\delta)\}$ by the following formulae

$$s_k^{(r)}(\varrho,\delta)=\begin{cases}\bar{x}_k & \text{for } k=1,2,\ldots,2^r\\ 0 & \text{for } k>2^r\end{cases} \qquad z_k^{(r)}(\varrho,\delta)=\begin{cases}\bar{y}_k & \text{for } k=1,2,\ldots,2^r\\ 0 & \text{for } k>2^r\end{cases}$$

Obviously, $s^{(r)}(\varrho,\delta)\in B_\varrho(0), z^{(r)}(\varrho,\delta)\in B_\varrho(0)$ and the difference $s^{(r)}(\varrho,\delta)-z^{(r)}(\varrho,\delta)\in B_\delta(0)$ for every $r\in\mathbb{N}_0$. Consequently, for every $n\in\mathbb{N}_0$ we obtain

$$\sum_{r=0}^{n}\bar{c}_r(\varrho,\delta)=\sum_{r=0}^{n}[\sum\nolimits_r|f(k,\bar{x}_k)-f(k,\bar{y}_k)|]= \\ =\|Fs^{(n)}(\varrho,\delta)-Fz^{(n)}(\varrho,\delta)\|_l\leqslant\omega_f(\varrho,\delta).$$

Hence, we conclude that $\bar{c}(\varrho,\delta)=\{\bar{c}_r(\varrho,\delta)\}\in l_1$ and $\|\bar{c}(\varrho,\delta)\|_l\leqslant\omega_f(\varrho,\delta)$. Consider

$$g_{\varrho,\delta}(k,s)=\sup\{|f(k,s+t)-f(k,s)|-2\bar{c}_r(\varrho,\delta)2^{-r}\delta^{-1}|t| : |t|\leqslant 2^r\delta \\ \text{and } -2^r-s\leqslant t\leqslant 2^r\varrho-s\}$$

for $r\in\mathbb{N}_0, k\in I_r$ and $|s|\leqslant 2^{-r}\varrho$. Evidently, $g_{\varrho,\delta}(k,s)\geqslant 0$ for every $k\in\mathbb{N}$. By the definition of supremum, for every $\varepsilon>0$ there is a sequence $\bar{t}(\varrho,\delta,\varepsilon,s)=\{\bar{t}_k(\varrho,\delta,\varepsilon,s)\}$ (denoting shorter $\bar{t}=\{\bar{t}_k\}$) such that $|\bar{t}_k|\leqslant 2^r\delta, -2^r\varrho-s\leqslant\bar{t}_k\leqslant 2^r\varrho-s$ and

$$g_{\varrho,\delta}(k,s)\leqslant|f(k,s+\bar{t}_k)-f(k,s)|-2\bar{c}_r(\varrho,\delta)2^{-r}\delta^{-1}|\bar{t}_k|+\frac{\varepsilon}{2^k}$$

for $k \in I_r, r = 0,1,2,\dots$ and $|s| \leqslant 2^r \varrho$. Further, for every $r \in \mathbb{N}_0$ a finite sequence $\{m_i\}$ (depended on $\varrho, \delta, \varepsilon, s$ and r) with $m_1 = 2^r < m_2 < \cdots < m_l = 2^{r+1} - 1$ can be found such that

$$\sum\nolimits_r |\bar{t}_k| = \sum_{k=2^r}^{m_2-1} |\bar{t}_k| + \sum_{k=m_2}^{m_3-1} |\bar{t}_k| + \cdots + \sum_{k=m_{l-1}}^{2^{r+1}-1} |\bar{t}_k|$$

and

$$2^{r-1}\delta \leqslant \sum_{k=m_i}^{m_{i+1}-1} |\bar{t}_k| \leqslant 2^r \delta \quad \text{for } i = 1,2,\dots,l-1,$$

and

$$0 \leqslant \sum_{k=m_{l-1}}^{2^{r+1}-1} |\bar{t}_k| \leqslant 2^r \delta.$$

Hence

$$\sum\nolimits_r g_{\varrho,\delta}(k,s) \leqslant \sum\nolimits_r |f(k,s+\bar{t}_k) - f(k,s)| - 2\bar{c}_r(\varrho,\delta)2^{-r}\delta^{-1}\sum\nolimits_r |\bar{t}_k| + \sum\nolimits_r \frac{\varepsilon}{2^k} \leqslant$$
$$\leqslant l\bar{c}_r(\varrho,\delta) - 2\bar{c}_r(\varrho,\delta)2^{-r}\delta^{-1}2^{r-1}\delta(l-1) + \sum\nolimits_r \frac{\varepsilon}{2^k} = \bar{c}_r(\varrho,\delta) + \sum\nolimits_r \frac{\varepsilon.}{2^k}$$

It follows that

$$\|G_{\varrho,\delta}x\|_l = \sum_{k=1}^{\infty} g_{\varrho,\delta}(k,x_k) \leqslant \sum_{r=1}^{\infty} \bar{c}_r(\varrho,\delta) + \varepsilon$$

for every $x \in B_\varrho(0)$, where $G_{\varrho,\delta}$ is the superposition operator generated by $g_{\varrho,\delta}$. Therefore, $G_{\varrho,\delta}$ is a bounded operator from w_0 into l_1 and by the definition of the sequence $\bar{c}(\varrho,\delta)$, we have

$$\sup\{\|G_{\varrho,\delta}x\|_l : x \in B_\varrho(0)\} \leqslant \omega_f(\varrho,\delta).$$

Consequently, by Theorem 3, there are sequences of non-negative terms $a(\varrho,\delta) = \{a_k(\varrho,\delta)\} \in l_1$ and $b(\varrho,\delta) = \{b_k(\varrho,\delta)\} \in l_1$ such that for each $k \in I_r, r = 0,1,2,\dots,$ the inequality

$$g_{\varrho,\delta}(k,s) \leqslant a_k(\varrho,\delta) + b_r(\varrho,\delta)2^{-r}|s|$$

holds, provided $|s| \leqslant 2^r\varrho$. Thus, by the definition of $g_{\varrho,\delta}(k,s)$, we have

$$|f(k,s+t) - f(k,s)| \leqslant a_k(\varrho,\delta) + b_r(\varrho,\delta)2^{-r}|s| + c_r(\varrho,\delta)2^{-r}|t|$$

for each $k \in I_r, r \in \mathbb{N}_0, |t| \leqslant 2^r\delta, |s+t| \leqslant 2^r\varrho, |s| \leqslant 2^r\varrho$, where $c_r(\varrho,\delta) = \delta^{-1}\bar{c}_r(\varrho,\delta)$. Taking into account the symmetry of our considerations and putting $s+t = u$, $s = v$, we obtain desirable inequality (6). Moreover, analysing the proof of Theorem 3 (cf.[11]), it is easy to notice that sequences $a(\varrho,\delta)$ and $b(\varrho,\delta)$ can be found such that

$$\|a(\varrho,\delta)\|_l \leqslant \|G_{\varrho,\delta}0\|_l + \omega_f(\varrho,\delta) \leqslant 2\omega_f(\varrho,\delta) \quad \text{and } \|b(\varrho,\delta)\|_l \leqslant \varrho^{-1}\omega_f(\varrho,\delta).$$

Consequently,

$$\|a(\varrho,\delta)\|_l + \|b(\varrho,\delta)\|_l \to 0 \text{ as } \delta \to 0,$$

which proves the implication $a) \Rightarrow b)$.

$b) \Rightarrow c)$. The continuity of $f(k,\cdot)$ for every $k \in \mathbb{N}$ is obvious. For the proof of boundedness of F fix $\varrho > 0$. Then, using inequality (6) with $\delta = \varrho$, we have

$$\big||f(k,u)| - |f(k,0)|\big| \leqslant |f(k,u) - f(k,0)| \leqslant a_k(\varrho,\varrho) + [b_r(\varrho,\varrho) + c_r(\varrho,\varrho)]2^{-r}|t|.$$

Putting $\bar{a}_k(\varrho) = |f(k,0)| + a_k(\varrho,\varrho), \bar{c}_r(\varrho) = b_r(\varrho,\varrho) + c_r(\varrho,\varrho)$, we obtain inequality (4) and by Theorem 3, the operator F is bounded from w_0 into l_1.

$c) \Rightarrow a)$. Let ε and ϱ be fixed positive constants. Define

$$r(\varepsilon) = \min\{s \in \mathbb{N}_0 : \sum_{k=2^s}^{\infty} a_k(\varrho) < \frac{\varepsilon}{6} \quad \text{and} \quad \sum_{r=s}^{\infty} c_r(\varrho) < \frac{\varepsilon}{6\varrho}\},$$

where sequences $a(\varrho) = \{a_k(\varrho)\}$ and $c(\varrho) = \{c_r(\varrho)\}$ are from Theorem 3. Let $x, z \in B_\varrho(0)$. By the continuity of $f(k,\cdot)$ for every $k \in \mathbb{N}$, there exists a $\delta \in (0,\varrho)$ such that

$$\sum_{k=1}^{2^{r(\varepsilon)}-1} |f(k,x_k) - f(k,z_k)| < \frac{\varepsilon}{3},$$

whenever $\|x - z\|_w < \delta$. Therefore, using inequality (4) for $r \geqslant r(\varepsilon)$, we have

$$\|Fx - Fz\|_l = \sum_{k=1}^{2^{r(\varepsilon)}-1} |f(k,x_k) - f(k,z_k)| + \sum_{k=2^{r(\varepsilon)}}^{\infty} (|f(k,x_k)| + |f(k,z_k)|) <$$
$$< \frac{\varepsilon}{3} + 2\sum_{k=2^{r(\varepsilon)}}^{\infty} a_k(\varrho) + \sum_{r=r(\varepsilon)}^{\infty} c_r(\varrho)2^{-r}(\sum\nolimits_r |x_k| + \sum\nolimits_r |z_k|) < \varepsilon,$$

provided $\|x - z\|_w < \delta$. This completes the proof of Theorem 6. ∎

The above theorem is rather surprising. It shows that in the case of continuous $f(k,\cdot)$ for each $k \in \mathbb{N}$, the boundedness of the superposition operator F acting from w_0 to l_1 is equivalent to the uniform continuity (on bounded sets) of this operator. Such theorem usually is not true, when we replace another sequence space instead w_0, for instance l_1. The following example shows this fact:

Example 2. Let $f(k,u) = u \sin k\pi u$ for $u \in \mathbb{R}$ and $k \in \mathbb{N}$. Then the superposition operator F generated by f is continuous and bounded (on every bounded set) from l_1 to l_1. Consider the sequences

$$x^{(n)} = \{x_k^{(n)}\} = \{\frac{2k+1}{2k}\chi_{\{n\}}\} \text{ and } z^{(n)} = \{z_k^{(n)}\} = \{\frac{2k-1}{2k}\chi_{\{n\}}\}.$$

Obviously, $x^{(n)}$ and $z^{(n)}$ belong to $B_3(0)$ for every $n \in \mathbb{N}$ and

$$\|x^{(n)} - z^{(n)}\|_l = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Nevertheless, since

$$\|Fx^{(n)} - Fz^{(n)}\|_l = 2$$

for every $n \in \mathbb{N}$, so F cannot be uniformly continuous on the ball $B_3(0)$.

Let us define

$$\begin{aligned}\pi_f(\varrho,\delta) = \inf\{\|a(\varrho,\delta)\|_l + 2\varrho\|b(\varrho,\delta)\|_l + \delta\|c(\varrho,\delta)\|_l : |f(k,u) - f(k,v)| \leqslant \\ \leqslant a_k(\varrho,\delta) + b_r(\varrho,\delta)2^{-r}(|u|+|v|) + c_r(\varrho,\delta)2^{-r}|u-v|, |u| \leqslant 2^r\varrho, |v| \leqslant 2^r\varrho \\ \text{and } |u-v| \leqslant 2^r\delta\}\end{aligned}$$

for every $\varrho > 0$ and $\delta > 0$.

From the proof of the first implication of Theorem 6 follows immediately

Corollary 1. *The functions $\omega_f(\cdot,\cdot)$ and $\pi_f(\cdot,\cdot)$ are equivalent in the sense that*

$$\omega_f(\varrho,\delta) \leqslant \pi_f(\varrho,\delta) \leqslant 5\omega_f(\varrho,\delta) \tag{7}$$

for all positive real numbers ϱ and δ.

3. Results in function case.

Theorem 7. *Every superposition operator F acting from W_0 into $L_1([1,\infty))$ is continuous and bounded*

Proof : The boundedness of F follows immediately from (5). It is sufficient to prove the continuity of F. Without loss of generality it can be assumed that $F0 = 0$. First, we shall show the continuity at zero of operators $G_r (r \in \mathbb{N}_0)$ defined by the formula

$$[G_r x](t) = \begin{cases} f(t, x(t)) & \text{for } t \in [1, 2^r) \\ 0 & \text{otherwise,} \end{cases} \qquad r = 0,1,2,\ldots.$$

Obviously, G_r maps W_0 into $L_1([1,\infty))$ for each $r \in \mathbb{N}_0$. Assume the contrary. Then there exist $\bar{r} \in \mathbb{N}_0$ and a sequence of functions $x_n \in W_0 (n \in \mathbb{N})$ which is convergent in norm to 0, whereas

$$\|G_{\bar{r}} x_n\|_L > \eta \text{ for every } n \in \mathbb{N}, \tag{8}$$

where η is a positive number. Without loss of generality it can be assumed that

$$\sum_{n=1}^{\infty} \|x_n\|_W < \infty. \tag{9}$$

Hereinafter, we shall construct sequences of numbers $\{\varepsilon_k\}$,of functions $\{x_{n_k}\}$ and of sets $A_k \subset [1, 2^{\bar{r}})(k \in \mathbb{N})$ such that the following conditions are satisfied:

(a) $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$,
(b) $\mu(A_k) \leqslant \varepsilon_k$,
(c) $\|G_{\bar{r}} x_{n_k} \chi_{A_k}\|_L > \frac{2}{3}\eta$,
(d) if $\mu(E) < 2\varepsilon_{k+1}$ for every set $E \subset [1, 2^{\bar{r}})$, then $\|G_{\bar{r}} x_{n_k} \chi_E\|_L < \frac{1}{3}\eta$, where μ is the Lebesgue measure.

Assume that $\varepsilon_1 = (2^{\bar r} - 1), x_{n_1}(t) = x_1(t), A_1 = [1, 2^{\bar r})$. In virtue of absolute continuity of the norm of the function $G_{\bar r}x_1$ and of condition (8), it is easy to verify that there exists an ε_2 such that conditions (a), (b), (c) and (d) are satisfied. Suppose that ε_k, x_{n_k} and A_k are already defined. Since $G_{\bar r}x_{n_k} \in L_1([1,\infty))$, so one can find an ε_{k+1} such that condition (d) will be fulfilled. Obviously, ε_{k+1} satisfies condition (a). Further, the fact that $x_n \to 0$ in the norm implies that the sequence $x_n\chi_{[1,2^{\bar r})}$ is convergent to zero in measure. Therefore, by Lemma 17.5 from [8] $G_{\bar r}x_n$ is convergent to zero in measure. Thus $G_{\bar r}x_n (n = 1, 2, \dots)$ cannot have equi-absolutely continuous norms because it would be convergent in norm, i.e. continuous at zero in contradiction to assumption (8). Hence there exist a set $A_{k+1} \subset [1, 2^{\bar r})$ and a function $x_{n_{k+1}}$ such that $\mu(A_{k+1}) < \varepsilon_{k+1}$ and

$$\|G_{\bar r}x_{n_{k+1}}\chi_{A_{k+1}}\|_L > \frac{2}{3}\eta.$$

In virtue of principle of mathematical induction we conclude that conditions (a), (b), (c) and (d) are satisfied for $k = 1, 2, \dots$.

Now, let us define a function y by the following formula

$$y(t) = \begin{cases} x_{n_k}(t) & \text{for } t \in B_k \quad (k = 1, 2, \dots) \\ 0 & \text{for } t \notin \bigcup_{k=1}^{\infty} B_k, \end{cases} \tag{10}$$

where $B_k = A_k \setminus \bigcup_{i=k+1}^{\infty} A_i, (k = 1, 2, \dots)$. Obviously, for $i \neq j$ we have $B_i \cap B_j = \emptyset$. Since

$$\mu\left(\bigcup_{i=k+1}^{\infty} A_i\right) \leqslant \sum_{i=k+1}^{\infty} \varepsilon_i < 2\varepsilon_{k+1},$$

then, by (c) and (d) it follows that

$$\|G_{\bar r}y\chi_{B_k}\|_L \geqslant \|G_{\bar r}x_{n_k}\chi_{A_k}\|_L - \|G_{\bar r}x_{n_k}\chi_{\bigcup_{i=k+1}^{\infty} A_i}\|_L > \frac{1}{3}\eta, \tag{11}$$

for $k = 1, 2, \dots$. Moreover, by (9), we have

$$\|y\|_W = \max_{r \leqslant \bar r} \frac{1}{2^r} \sum_{k=1}^{\infty} \int_{B_k \cap A(r)} |x_{n_k}(t)|\, dt \leqslant$$
$$\leqslant \sum_{k=1}^{\infty} \max_{r \leqslant \bar r} \frac{1}{2^r} \int_{B_k \cap A(r)} |x_{n_k}(t)|\, dt \leqslant \sum_{n=1}^{\infty} \|x_n\|_W < \infty,$$

whence $y \in W_0$. Applying the assumption of the theorem we obtain that $G_{\bar r}y \in L_1([1,\infty))$. On the other hand, in virtue of (11), we have

$$\|G_{\bar r}y\|_L = \int_1^{\infty} |f(t, y(t))|\, dt = \sum_{k=1}^{\infty} \int_{B_k} |f(t, x_{n_k}(t))|\, dt = \sum_{k=1}^{\infty} \|G_{\bar r}y\chi_{B_k}\|_L = \infty,$$

and consequently $G_{\bar{r}}y \notin L_1([1,\infty))$. We have thus arrived to a contradiction. Therefore G_r is continuous at zero for every $r \in \mathbb{N}_0$.

Fix $\varepsilon > 0$. Now, let $\bar{r}$ be such a large natural number that

$$\sum_{r=\bar{r}}^{\infty} \int_{A(r)} a_r(t)\,dt < \frac{\varepsilon}{3} \quad \text{and} \sum_{r=\bar{r}}^{\infty} c_r < \frac{\varepsilon}{3\alpha},$$

where $a_r(\cdot)$ and c_r are from Theorem 4. By the continuity $G_{\bar{r}}$ at the point zero and by mentioned Theorem 4 there exists a $\delta \in (0,1]$ such that

$$\|Fx\|_L \leqslant \|Fx\chi_{[1,2^{\bar{r}})}\|_L + \sum_{r=\bar{r}}^{\infty} \int_{A(r)} a_r(t)\,dt + \|x\|_W \alpha \sum_{r=\bar{r}}^{\infty} c_r < \varepsilon,$$

provided $\|x\|_W < \delta$. Hence F is continuous at zero. For the proof of continuity of F at an arbitrary point $x_0 \in W_0$. It is enough to remark that the continuity of the operator F at the point x_0 is equivalent to the continuity of the operator

$$F_1 x = F(x_0 + x) - Fx_0$$

at zero in $L_1([1,\infty))$. This completes the proof. ■

The application of many principles of nonlinear analysis to the study of different types of equations requires upper estimations for the operators which are generated by given functions. Let for every $\varrho > 0$

$$\nu_f(\varrho) = \inf\{\|a\|_L + \|c\|_L\varrho : |f(t,u)| \leqslant a_r(t) + 2^{-r}c_r|u| \quad \text{for a.a. } t \in A(r)\},$$

where $a_r(\cdot)$ and $c_r (r \in \mathbb{N}_0)$ are as in Theorem 4. Moreover, we associate with the operator F a function μ_f which is defined by

$$\mu_f(\varrho) = \sup\{\|Fx\|_L : \|x\|_W \leqslant \varrho\} \quad (\varrho > 0)$$

and describes the growth of F on balls centered at the origin. The next theorem gives a two-sided estimation for the operator considered by us.

Theorem 8. *The functions μ_f and ν_f are equivalent in the sense that*

$$\mu_f(\varrho) \leqslant \nu_f(\varrho) \leqslant 2\mu_f(\varrho).$$

PROOF : Fix $\varrho > 0$. By definition of ν_f, we have immediately

$$\|Fx\|_L \leqslant \|a\|_L + \|c\|_l\|x\|_W$$

for every $x \in B_\varrho(0)$ and consequently $\mu_f(\varrho) \leqslant \nu_f(\varrho)$

Since the operator F is bounded on the ball $B_\varrho(0)$, so we can define

$$c_r(\varrho) = \sup\left\{\int_{A(r)} |f(t,x(t))|\,dt : x \in W_0 \quad \text{and} \int_{A(r)} |x(t)|\,dt \leqslant \varrho 2^r\right\}$$

for each $r \in \mathbf{N}_0$. Therefore, for every $\varepsilon > 0$ there exists a function $y_{\varrho,\varepsilon}(\cdot)$ such that

$$\int_{A(r)} |y_{\varrho,\varepsilon}(t)|\, dt \leqslant 2^r \quad \text{and} \quad c_r(\varrho) \leqslant \int_{A(r)} |f(t, y_{\varrho,\varepsilon}(t))|\, dt + \frac{\varepsilon}{2^r}$$

for each $r \in \mathbf{N}_0$. Further, for every $n \in \mathbf{N}$ the function

$$z_{\varrho,\varepsilon}^{(n)}(\cdot) = y_{\varrho,\varepsilon}(\cdot)\chi_{[1,2^n)}(\cdot)$$

belongs to $B_\varrho(0)$. Consequently, for every $n \in \mathbf{N}_0$

$$\sum_{r=0}^{n} c_r(\varrho) \leqslant \sum_{r=0}^{n} \left(\int_{A(r)} |f(t, y_{\varrho,\varepsilon}(t))|\, dt + \frac{\varepsilon}{2^r} \right) \leqslant \|F z_{\varrho,\varepsilon}^{(n)}\|_L + \varepsilon \leqslant \mu_f(\varrho) + \varepsilon.$$

Hence, by the arbitrariness of $n \in \mathbf{N}_0$ and $\varepsilon > 0$, we conclude that $c(\varrho) = \{c_r(\varrho)\} \in l_1$ and $\|c(\varrho)\|_l \leqslant \mu_f(\varrho)$.

Define $h_\varrho : [1, \infty) \times \mathbf{R} \to \mathbf{R}$ by

$$h_\varrho(t, u) = \max\{0, |f(t, u)| - c_r(\varrho) 2^{-r} \varrho^{-1} |u|\}$$

for $t \in A(r)$ and $r \in \mathbf{N}_0$. Fix a function $x \in W_0$ and let A_r^+ denote the set of all points $t \in A(r)$ for which $h_\varrho(t, x(t))$ is positive. Now, choose $m \in \mathbf{N}_0$ and $\gamma \in [0, 1)$ such that

$$\int_{A_r^+} |x(t)|\, dt = (m + \gamma)\varrho 2^r$$

and divide A_r^+ into subsets $A_r^1, A_r^2, \ldots, A_r^{m+1}$ such that

$$\int_{A_r^i} |x(t)|\, dt \leqslant \varrho 2^r \quad (i = 1, 2, \ldots, m + 1).$$

From the definition of $c_r(\varrho)$ it follows that

$$\int_{A_r^i} |f(t, x(t))|\, dt \leqslant c_r(\varrho) \quad (i = 1, 2, \ldots, m + 1)$$

and therefore, by the definition of h_ϱ, we have

$$\int_{A(r)} h_\varrho(t, x(t))\, dt \leqslant (m + 1)c_r(\varrho) - c_r(\varrho)(m + \gamma) \leqslant c_r(\varrho).$$

Lemma 17.2 from [8] ensures that there exists a sequence $y_k(\cdot), |y_k(t)| \leqslant k$ such that

$$h_\varrho(t, y_k(t)) = \sup_{|u| \leqslant k} h_\varrho(t, u).$$

We put

$$a_{r,\varrho}(t) = \begin{cases} \sup\limits_{|u|<\infty} h_\varrho(t,u) = \lim\limits_{k\to\infty} h_\varrho(t,y_k(t)) & \text{for } t \in A(r) \\ 0 & \text{otherwise}, \end{cases}$$

$r = 0, 1, \ldots$. Hence, by Fatou Theorem, we have

$$\int_{A(r)} a_{r,\varrho}(t)\,dt \leqslant \sup_k \int_{A(r)} h_\varrho(t,y_k(t))\,dt \leqslant c_r(\varrho), (r = 0, 1, \ldots),$$

i.e. $a_\varrho(\cdot) = \sum_{r=0}^{\infty} a_{r,\varrho}(\cdot) \in L_1([1,\infty))$. Thus, by the definition of h_ϱ, we conclude

$$|f(t,u)| \leqslant a_{r,\varrho}(t) + c_r(\varrho)2^{-r}\varrho^{-1}|u| \quad (r = 0, 1, \ldots),$$

for a.a. $t \in A(r)$ and for $u \in \mathbb{R}$. Consequently,

$$\nu_f(\varrho) \leqslant \|a_\varrho\|_L + \frac{1}{\varrho}\sum_{r=0}^{\infty} c_r(\varrho)\varrho \leqslant 2\sum_{r=0}^{\infty} c_r(\varrho) \leqslant 2\mu_f(\varrho).$$

It completes the proof. ∎

By Theorem 7 the operator F is always continuous. On the other hand, F is not necessary uniformly continuous on bounded sets. The following example shows this fact:

Example 3. Let $f(t,u) = \chi_{[1,2)}(t)u\sin u$. Choose a sequence of subsets $D_n \subset [1,2)$ such that $\mu(D_n) = (4\pi n)^{-1}$. Consider the functions

$$x_n(t) = (4n+1)\frac{\pi}{2}\chi_{D_n}(t), \quad n \in \mathbb{N},$$

$$y_n(t) = (4n-1)\frac{\pi}{2}\chi_{D_n}(t), \quad n \in \mathbb{N}.$$

Obviously, $\|x_n\|_W < 1$ and $\|y_n\|_W < 1$ for every $n \in \mathbb{N}$. Moreover, $\|x_n - y_n\|_W = \frac{1}{4n} \to 0$ as $n \to \infty$. The superposition operator F generated by f maps the space W_0 into the space $L_1([1,\infty))$, because inequality (5) is satisfied with $a_r(t) \equiv 0$ for every $r \in \mathbb{N}_0, \alpha = 1$ and $\{c_r\} = \{1, 0, 0, \ldots\}$. By Theorem 7 the operator F is continuous on the whole space W_0. Nevertheless, since

$$\|Fx_n - Fy_n\|_L = \|4\pi n\chi_n(\cdot)\|_L = 1,$$

so F cannot be uniformly continuous on the ball $B_1(0)$.

Theorem 9. *The superposition operator F is uniformly continuous (on bounded sets) from W_0 into $L_1([1,\infty))$ iff for every positive constants ϱ and δ one can find sequences of non-negative terms $b(\varrho,\delta) = \{b_r(\varrho,\delta)\} \in l_1, c(\varrho,\delta) = \{c_r(\varrho,\delta)\} \in l_1$ and a non-negative function $a_{\varrho,\delta}(\cdot) \in L_1([1,\infty))$, such that*

(a) $\|a_{\varrho,\delta}\chi_{A(r)}\|_L \leqslant \varrho b_r(\varrho,\delta)$ *for every* $r \in \mathbb{N}_0$,
(b) $\|b(\varrho,\delta)\|_l \to 0$ *as* $\delta \to 0$ *for every fixed* ϱ,
(c) *the inequality*
$|f(t,u) - f(t,v)| \leqslant a_{\varrho,\delta}(t) + b_r(\varrho,\delta)2^{-r}(|u|+|v|) + c_r(\varrho,\delta)2^{-r}|u-v|$
holds for a.a. $t \in A(r)$ *and* $u, v \in \mathbb{R}$.

Since the proof of necessity of Theorem 9 is analogous to the proof of implication $a) \Rightarrow b)$ from Theorem 6, so we shall omit it. The proof of sufficiency is obvious.

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