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Gelfand—Phillips property in the completion of the space of Pettis integrable functions[1]

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Abstract. We consider the normed space $\mathcal{P}(\mu, X)$ of Pettis integrable functions with values in a Banach space X and we prove that if X has the Gelfand—Phillips property, then even the completion of $\mathcal{P}(\mu, X)$ has the same property.

Keywords: Pettis integrable functions, precompactness, Gelfand—Phillips property

Classification: 46E40, 46B20

Introduction.

Let (S, Σ, μ) be a finite measure space and X a Banach space. We consider the normed space $\mathcal{P}(\mu, X)$ of all (μ)-Pettis integrable functions, with values in X, equipped with the norm

$$\|f\| = \sup\left\{ \int_S |x^* f(s)|\, d\mu : x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

We say that X has the Gelfand—Phillips property (see [1]) if any bounded subset M such that

$$\lim_n \sup_M |x_n^*(x)| = 0 \textit{ for any } w^*\textit{-null sequence } (x_n^*) \subset X^* \tag{1}$$

is relatively compact. A set verifying (1) will be called "limited".

Purpose of this note is to prove that if X has the Gelfand—Phillips property, then the completion $\widehat{\mathcal{P}(\mu, E)}$ of $\mathcal{P}(\mu, E)$ has the same property.

In order to give our result we need the following remark done in [1].

Proposition 1. *If $f : S \to X$ is Pettis integrable and X has the Gelfand—Phillips property, then the set $\left\{\int_A f(s)\, d\mu : A \in \Sigma\right\}$ is relatively compact.*

PROOF : Using the μ-continuity of the indefinite integral of f, together with the finiteness of μ, it is very easy to show that $\left\{\int_A f(s)\, d\mu : A \in \Sigma\right\}$ is limited in X. ■

Result.

Our proof of the main result of the paper relies on the following theorem about the (strong) precompactness in the space $\mathcal{P}_c(\mu, X)$, the subspace of $\mathcal{P}(\mu, X)$ consisting of those f having an indefinite integral with compact range

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Theorem 1. *Let H be a bounded subset of $\mathcal{P}_c(\mu, X)$. If the following assumptions*

(i) *the set $\{x^*f : x^* \in X^*, \|x^*\| \le 1, f \in H\}$ is relatively compact in $L^1(\mu)$*

(ii) *the set $\left\{\int_S g(s)f(s)\, d\mu : g \in L^\infty(\mu), \|g\| \le 1, f \in H\right\}$ is relatively compact in X*

are verified, then H is precompact in $\mathcal{P}_c(\mu, X)$.

PROOF : Choose $(f_n) \subset H$ and observe that under (i) and (ii), H is weakly precompact ([3]). Then we can assume, by passing to a subsequence if necessary, that f_n is weak Cauchy. Now, suppose that f_n has no Cauchy subsequences. There are $\eta > 0, (f_{n_h}), (f_{m_h})$ such that

$$\eta < \|f_{n_h} - f_{m_h}\| \quad \text{for all } h \in \mathbb{N}$$

For suitable sequences $(x^*_h) \subset X^*, \|x^*_h\| \le 1$, $(g_h) \subset L^\infty(\mu), \|g_h\| \le 1$, we have

$$\eta < \int_S g_h(s)(f_{n_h}(s) - f_{m_h}(s))x^*_h\, d\mu \quad \text{for all } h \in \mathbb{N}$$

Now, suppose that $(x^*_{h_\gamma})$ and (g_{h_γ}) are suitable subnets weak* converging, respectively, to $x^* \in X^*, g \in L^\infty(\mu)$. Rewriting the last inequality for $(x^*_{h_\gamma})$ and (g_{h_γ}), we have

$$\begin{aligned}
\eta &< \int_S x^*_{h_\gamma} g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = \int_S x^*_{h_\gamma} g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu - \\
&- \int_S x^* g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu + \int_S x^* g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu - \\
&- \int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu + \int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = \\
&= (x^*_{h_\gamma} - x^*) \int_S g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu + \\
&+ \int_S x^*(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))(g_{h_\gamma}(s) - g(s))\, d\mu + \\
&+ \int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu.
\end{aligned}$$

Now observe that the following limit relations are verified

(j) $\lim_\gamma (x^*_{h_\gamma} - x^*) \int_S g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = 0$, because $x^*_{h_\gamma} - x^* \xrightarrow{w^*} \vartheta$ and (ii) holds true

(jj) $\lim_\gamma \int_S x^*(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))(g_{h_\gamma}(s) - g(s))\, d\mu = 0$, because $g_{h_\gamma} - g \xrightarrow{w^*} \vartheta$ and (i) holds true

(jjj) $\lim_\gamma \int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = 0$, because (f_n) is a weak Cauchy sequence.

The reached contradiction gives our thesis. ∎

Remark 1. It is possible to show that even the converse of Theorem 1 is true.

Remark 2. In a sense, the above result is the best possible; indeed, if H is a subset of $\mathcal{P}(\mu, X)$ (it doesn't matter how the range of the indefinite integral is) for which the above Theorem is true, then H must be a subset of $\mathcal{P}_c(\mu, X)$.This follows very easily from (ii) be choosing $g = \chi_A, A \in \Sigma$.

Now, we are ready to give our main result

Theorem 2. *Assume that X has the Gelfand—Phillips property. Then $\widehat{\mathcal{P}(\mu, X)}$ has the same property.*

PROOF : First of all, note that $\widehat{\mathcal{P}(\mu, X)} = \widehat{\mathcal{P}_c(\mu, X)}$, by virtue of Proposition 1. And so we have just to prove that $\widehat{\mathcal{P}_c(\mu, X)}$ enjoys the Gelfand—Phillips property. Let H be a limited subset of $\widehat{\mathcal{P}_c(\mu, X)}$ and (z_n) be a sequence in H. By virtue of the density of $\mathcal{P}_c(\mu, X)$ we can choose a sequence $(f_n) \subset \mathcal{P}_c(\mu, X)$ that is limited and such that $\lim_n \|z_n - f_n\| = 0$. It will be enough to show that (f_n) is relatively compact. This will be done by proving that (f_n) verifies (i) and (ii) of Theorem 1; then the completeness of $\widehat{\mathcal{P}_c(\mu, X)}$ will do the remaining job. First of all, assume that the set $A = \{x^* f_n : x^* \in X^*, \|x^*\| \leq 1, n \in \mathbb{N}\}$ is not limited in $L^1(\mu)$. There are $(g_h) \subset L^\infty(\mu), \|g_h\| \leq 1$, $g_h \xrightarrow{w^*} \vartheta$, $(x_h^* f_{n_h}) \subset A$ for which $\inf_h |g_h x_h^* f_{n_h}| > 0$.

Now, observe that $g_h x_h^* \in [\widehat{\mathcal{P}_c(\mu, X)}]^*$ for any $h \in \mathbb{N}$ and furthermore $g_h x_h^* \xrightarrow{w^*} \vartheta$. This last assertion can be shown as it follows.

Take $f \in \mathcal{P}_c(\mu, X)$ and calculate $(g_h x_h^*)(f) = g_h(x_h^* f), h \in \mathbb{N}$. Since $f \in \mathcal{P}_c(\mu, X)$, a result due to Edgar ([2]) tells us that $(x_h^* f)$ is relatively compact in $L^1(\mu)$ and so

$$\lim_h g_h(x_h^* f) = 0$$

because $g_h \xrightarrow{w^*} \vartheta$. Since $\mathcal{P}_c(\mu, X)$ is dense in $\widehat{\mathcal{P}_c(\mu, X)}$ we can conclude that $g_h x_h^* \xrightarrow{w^*} \vartheta$, as we wanted. Being (f_n) limited in $\mathcal{P}_c(\mu, X)$ (and so in $\widehat{\mathcal{P}_c(\mu, X)}$) we get a contradiction. Hence $\{x^* f_n : x^* \in X^*, \|x^*\| \leq 1, n \in \mathbb{N}\}$ is limited in $L^1(\mu)$, a Banach space with the Gelfand—Phillips property. (i) of Theorem 1 is then true. Now we pass to (ii). Again, assume the set $\left\{\int_S g(s) f_n(s)\, d\mu : g \in L^\infty(\mu), \|g\| \leq 1, n \in \mathbb{N}\right\}$ is not limited in X. There are a weak* null sequence $(x_h^*) \subset X^*, \|x_h^*\| \leq 1$, and $(g_h f_{n_h})$ such that $\inf_h |x_h^*(g_h f_{n_h})| > 0$. But once more $(g_h x_h^*)$ is a weak* null sequence in $[\widehat{\mathcal{P}_c(\mu, X)}]^*$. Indeed, if $f \in \mathcal{P}_c(\mu, X)$ we have

$$\left|\int_S x_h^* g_h(s) f(s)\, d\mu\right| \leq \int_S |x_h^* g_h(s) f(s)|\, d\mu \leq \int_S |x_h^* f(s)|\, d\mu \text{ for all } h \in \mathbb{N}.$$

Now, observe that $x_h^* f \to 0$ almost uniformly. Putting $S_h^+ = \{s : x_h^* f(s) \geq 0\}$ and $S_h^- = \{s : x_h^* f(s) < 0\}, h \in \mathbb{N}$ we get, for any $h \in \mathbb{N}$,

$$\text{(2)} \qquad \int_S |x_h^* f(s)|\, d\mu = \int_{S_h^+} x_h^* f(s)\, d\mu - \int_{S_h^-} x_h^* f(s)\, d\mu \leq$$

$$\leq \left|\int_{S_h^+} x_h^* f(s)\, d\mu\right| + \left|\int_{S_h^-} x_h^* f(s)\, d\mu\right|$$

Now, given $\varepsilon > 0$ there is $A_\varepsilon \in \Sigma, \mu(A_\varepsilon^c) < \varepsilon$, such that $x_h^* f \to 0$ uniformly on A_ε. On the other hand, the indefinite integral of f is μ-continuous and so given $\gamma > 0$ there is $\delta > 0$ such that $\left\|\int_A f(s)\, d\mu\right\| < \gamma$ whenever $\mu(A) < \delta$. Take $\varepsilon = \delta$. By (2) we have

$$\begin{aligned}\int_S |x_h^* f(s)|\, d\mu \leq & \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s)\, d\mu\right| + \left|\int_{S_h^+ \setminus A_\delta} x_h^* f(s)\, d\mu\right| + \\ & + \left|\int_{S_h^- \cap A_\delta} x_h^* f(s)\, d\mu\right| + \left|\int_{S_h^- \setminus A_\delta} x_h^* f(s)\, d\mu\right| \leq \\ & \leq \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s)\, d\mu\right| + \left|\int_{S_h^- \cap A_\delta} x_h^* f(s)\, d\mu\right| + \\ & + \left\|\int_{S_h^+ \setminus A_\delta} f(s)\, d\mu\right\| + \left\|\int_{S_h^- \cap A_\delta} f(s)\, d\mu\right\| \leq \\ \leq \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s)\, d\mu\right| + & \left|\int_{S_h^- \cap A_\delta} x_h^* f(s)\, d\mu\right| + 2\gamma \leq 2 \int_{A_\delta} |x_h^* f(s)|\, d\mu + 2\gamma.\end{aligned}$$

Since $x_h^* f \to 0$ uniformly on A_δ, we are done, i.e. we have reached the sought-for contradiction (use the density of $\mathcal{P}_c(\mu, X)$ in $\mathcal{P}_c\widehat{(\mu, X)}$, too). Being X a Banach space with the Gelfand—Phillips property, even (ii) in Theorem 1 is verified. The proof is complete. ∎

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