Weak subalgebra lattices*

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Abstract. The main result contains a characterization of the lattice of all weak subalgebras of a partial algebra. This is a distributive algebraic lattice with additional properties concerning its join-irreducible elements. It is also shown that for two unary algebras of arbitrary unary types, their weak subalgebra lattices are isomorphic iff their undirected graphs (representing operations) are isomorphic.

Keywords: Weak subalgebra, algebraic lattice, join-irreducible

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Since the characterization of subalgebra lattices of (total) algebras in [**BiFr**] a large number of papers appeared, relating properties of algebras with those of their subalgebra lattices. Several results characterize subalgebra lattices of algebras belonging to a particular variety, many others indicate the possible type of an algebra, given its subalgebra lattice. In recent papers [**EvGa**], [**Sha1**], [**Sha2**] modularity and distributivity of subalgebra lattices are investigated and these conditions turn out to imply rather strong properties on the underlying algebras.

Very few results of this kind are known for partial algebras. Here at least three structures may be considered (see e.g. [**Bur**]): the lattice of subalgebras, the lattice of relative subalgebras and the lattice of weak subalgebras. The second structure is clearly of no interest, since any subset of an algebra is the carrier of exactly one relative subalgebra. As for the first, it inherits many properties from the total case (every total algebra being in particular a partial one), but not much is known otherwise. In this paper we intend to introduce the third structure with the conviction that — together with the subalgebra lattice — it may yield more information on an algebra than the latter lattice alone. We present a complete characterization of lattices of weak subalgebras (some results on their structure appear already in [**Poy**]) and we show what information on an algebra is contained in the weak subalgebra lattice in the unary case. In a subsequent paper [**Ba**] we characterize those monounary algebras, which are completely determined by their weak subalgebra lattices.

It could be expected that a weak subalgebra lattice represents fairly well the structure of the algebra, but unfortunately this is not so in the general case. We show some simple cases when there is a connection between the structure of a partial algebra and that of its weak subalgebra lattice.

We represent a partial algebra of type (F, n) (F is a set of operation symbols and n is the arity function) as a pair $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$, where A is the universe of

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$\mathfrak{A}$ and for each $f \in F, f^{\mathfrak{A}}$ is an $n(f)$-ary partial function on A. A partial algebra $\mathfrak{B} = (B, (f^{\mathfrak{B}})_{f\in F})$ is a *weak subalgebra* of $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$ iff $B \subseteq A$ and for all $f \in F, f^{\mathfrak{B}} \subseteq f^{\mathfrak{A}}$. We write $\mathfrak{B} \subseteq \mathfrak{A}$ to indicate that $\mathfrak{B}$ is a weak subalgebra of $\mathfrak{A}$. Let $S_w(\mathfrak{A})$ be the set of all weak subalgebras of $\mathfrak{A}$. Then $\mathfrak{S}_w(\mathfrak{A}) = (S_w(\mathfrak{A}), \subseteq)$ is a complete lattice with the following supremum and infimum operations, respectively:

$$\bigvee_{\mathfrak{A}}\{\mathfrak{B}_i : i \in I\} = (\cup B_i, (\cup f^{\mathfrak{B}_i})_{f\in F})$$

$$\bigwedge_{\mathfrak{A}}\{\mathfrak{B}_i : i \in I\} = (\cap B_i, (\cap f^{\mathfrak{B}_i})_{f\in F})$$

for any family $\mathfrak{B}_i = (B_i, (f^{\mathfrak{B}_i})_{f\in F}), i \in I$ of weak subalgebras of $\mathfrak{A}$.

Given a partial algebra $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$, there is an obvious injective and inclusion-preserving correspondence between weak subalgebras of $\mathfrak{A}$ and subsets of the disjoint union $A\dot{\cup}\dot{\bigcup}\{f^{\mathfrak{A}} : f \in F\}$. This leads to the following

Proposition. *For any partial algebra $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$, $\mathfrak{S}_w(\mathfrak{A})$ is a complete sublattice of a complete and atomic Boolean algebra.*

Now we can obtain other basic properties of $\mathfrak{S}_w(\mathfrak{A})$ from the following theorem ([**CrDi**], p. 83):

Theorem 1. *For a lattice $\mathfrak{L}$, the following conditions are equivalent:*

(1) *$\mathfrak{L}$ is distributive and both $\mathfrak{L}$ and its dual are compactly generated*
(2) *$\mathfrak{L}$ is a compactly generated distributive lattice, in which each element is a join of completely join-irreducible elements*
(3) *$\mathfrak{L} \cong \mathfrak{O}(P)$ for some partially ordered set P (where $\mathfrak{O}(P)$ is the set of all order ideals of P, with inclusions)*
(4) *$\mathfrak{L}$ is a complete sublattice of some complete and atomic Boolean algebra.*

(Recall that an element $a \in L$ is *completely join-irreducible* iff for any non-empty $K \subseteq L, a = \bigvee K$ implies $a \in K$; it is *join-irreducible*, if the condition holds for all two-element subsets K.)

Thus, in particular

Corollary. *For any partial algebra $\mathfrak{A}$, $\mathfrak{S}_w(\mathfrak{A})$ is a distributive algebraic lattice, in which every element is a join of completely join-irreducible elements (cf.* [**Poy**]).

An easy example shows that these conditions are not sufficient for weak subalgebra lattices. A four-element chain has all the properties of the Corollary, though it is not isomorphic to $\mathfrak{S}_w(\mathfrak{A})$ for any $\mathfrak{A}$. Indeed, observe that $\mathfrak{S}_w(\mathfrak{A})$ should contain a principal ideal isomorphic to the power set of A (since any subset of A is the universe of a discrete weak subalgebra of $\mathfrak{A}$ — with all the operations empty) as well as a principal dual ideal isomorphic to the power set of the disjoint union $\dot{\cup} f^{\mathfrak{A}}$ (since any subset of this union determines a weak subalgebra on the set A), both generated by the discrete weak subalgebra on A. No element of a four-element chain (nor of any chain with at least 4 elements, of course) generates both an ideal and a dual ideal, which are complete and atomic Boolean lattices.

The full characterization of weak subalgebra lattices is given by the following

Theorem 2. *An algebraic lattice $\mathfrak{L}$ is isomorphic to $\mathfrak{S}_w(\mathfrak{A})$ for some partial algebra $\mathfrak{A}$ iff $\mathfrak{L}$ is distributive and*

(i) *every element is a join of join-irreducible elements*
(ii) *every non-zero join-irreducible element contains only a finite (and non-empty) set of atoms*
(iii) *the set of all non-zero and non-atomic join-irreducible elements is an antichain with respect to the lattice ordering of $\mathfrak{L}$.*

PROOF : Let $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$ be a partial algebra. Then by Corollary, $\mathfrak{S}_w(\mathfrak{A})$ is algebraic, distributive and each of its elements is a join of completely join-irreducible elements.

Lemma 3. *A weak subalgebra $\mathfrak{B} = (B, (f^{\mathfrak{B}})_{f\in F}$ of $\mathfrak{A}$ is a join-irreducible element in $\mathfrak{S}_w(\mathfrak{A})$ iff one of the following holds:*

1) *$B = \emptyset$ and $f^{\mathfrak{B}} = \emptyset$ for all $f \in F$*
2) *$B = \{a\}$ for some $a \in A$ and $f^{\mathfrak{B}} = \emptyset$ for all $f \in F$*
3) *There are a $g \in F$ and $(a_0, \ldots, a_{n(g)-1}, a) \in g^{\mathfrak{A}}$ such that $B = \{a_0, \ldots, a_{n(g)-1}, a\}$, $g^{\mathfrak{B}} = \{(a_0, \ldots, a_{n(g)-1}, a)\}$ and $f^{\mathfrak{B}} = \emptyset$ for all $f \in F$ with $f \neq g$.*

Moreover, $\mathfrak{B}$ is completely join-irreducible iff it is join-irreducible.

PROOF : is obtained by a simple verification and is therefore omitted. ∎

Proof of the Theorem (continued). Observe that weak subalgebras of the second form are exactly the atoms of $\mathfrak{S}_w(\mathfrak{A})$, while those of the third form are all the non-zero non-atomic join-irreducibles. Thus condition (ii) follows easily from Lemma 3, as well as (i) (we can omit the word "completely" in the Corollary). Note also that for two non-zero non-atomic join-irreducible elements $\mathfrak{B}_1$ and $\mathfrak{B}_2$, $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ iff $\mathfrak{B}_1 = \mathfrak{B}_2$. This proves condition (iii) and thus completes the proof of the necessity of the conditions (i)–(iii).

Now let $\mathfrak{L} = (L, \leqslant)$ be a distributive algebraic lattice satisfying (i)–(iii). Let A be the set of all atoms of $\mathfrak{L}$ with an arbitrary but fixed total order $\preccurlyeq$ on A. Let F be the set of all non-zero non-atomic join-irreducibles in $\mathfrak{L}$ and for any $z \in L$ let $A_z = \{a \in A : a \leqslant z\}$.

We define a partial algebra $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$ so that for any $f \in F$, if $A_f = \{a_0, \ldots, a_{n-1}, a_n\}$ with $a_0 \preccurlyeq \cdots \preccurlyeq a_{n-1} \preccurlyeq a_n$, then $f^{\mathfrak{A}} = \{(a_0, \ldots, a_{n-1}, a_n)\}$.

We shall prove that $\mathfrak{L} \cong \mathfrak{S}_w(\mathfrak{A})$.

For any $x \in L$, let I_x be the set of all join-irreducible elements of $\mathfrak{L}$ contained in x. By assumption , $x = \bigvee_{\mathfrak{L}} I_x$ for every $x \in L$. Define a mapping $h : L \to S_w(\mathfrak{A})$ as follows: for $x \in L$,

$$h(x) = (\bigcup\{A_z : z \in I_x\}, (f^{(x)})_{f\in F})$$

where

$$f^{(x)} = \begin{cases} f^{\mathfrak{A}} \text{ for } f \in I_x \\ \emptyset \text{ for } f \notin I_x \end{cases}$$

for any $f \in F$.

Thus e.g. if $0_{\mathfrak{L}}$ is the infimum of $\mathfrak{L}$, then $h(0_{\mathfrak{L}}) = (\emptyset, (f^{(0_{\mathfrak{L}})}_{f \in F})$ with $f^{(0_{\mathfrak{L}})} = \emptyset$ for all $f \in F$; for $a \in A, h(a) = (\{a\}, (f^{(a)})_{f \in F})$ with $f^{(a)} = \emptyset$ for all $f \in F$ and for any $g \in F, h(g) = (A_g, (f^{(g)})_{f \in F})$ with $f^{(g)} = \emptyset$ for $f \neq g$ and $g^{(g)} = g^{\mathfrak{A}}$ (observe that by assumption $f \notin I_g$ for any $f \neq g, \quad f, g \in F$). Moreover, by the definitions of h and $\mathfrak{S}_w(\mathfrak{A}), h(x) = \bigvee_{\mathfrak{A}}\{h(p) : p \in I_x\}$.

The mapping h is the required isomorphism of $\mathfrak{L}$ onto $\mathfrak{S}_w(\mathfrak{A})$. Indeed

(a) h is $1-1$:

Let $x, y \in L$ and $x \neq y$. Then $I_x \neq I_y$. Assume $v \in I_x - I_y$.If $v \in A$ then $v \notin A_z$ for all $z \in I_y$ and consequently v is not in the carrier of $h(y)$; on the other hand, $v \in A_v \subseteq \bigcup\{A_z : z \in I_x\}$. Thus $h(x) \neq h(y)$. If $v \in F$, then $v^{(x)} = v^{\mathfrak{A}} \neq \emptyset$ in $h(x)$, while $v^{(y)} = \emptyset$ in $h(y)$; again $h(x) \neq h(y)$.

(b) h is onto:

Let $\mathfrak{B} = (B, (f^{\mathfrak{B}})_{f \in F})$ be a weak subalgebra of $\mathfrak{A}$ and $F_{\mathfrak{B}} = \{f \in F : f^{\mathfrak{B}} = f^{\mathfrak{A}}\}$. Let $B_0 = B \cup F_{\mathfrak{B}}$ and define $c = \bigvee_{\mathfrak{L}} B_0$. Observe that by definition of $\mathfrak{A}$,

(1) $$\text{if } p \in A \text{ and } p \leqslant f \text{ for some } f \in F_{\mathfrak{B}}, \text{ then } p \in B$$

By the definition of h,

$$h(c) = (\bigcup\{A_z : z \in I_c\}, (f^{(c)})_{f \in F})$$

with $f^{(c)} = f^{\mathfrak{A}}$ for $f \in I_c$ and $f^{(c)} = \emptyset$ for $f \notin I_c$.

To prove that $h(c) = \mathfrak{B}$, we show that

(2) $$B_0 = I_c$$

Clearly $B_0 \subseteq I_c$. Suppose $p \in I_c - B_0$. Then $p = p \wedge c = p \wedge \bigvee_{\mathfrak{L}} B_0$. Being algebraic and distributive, $\mathfrak{L}$ is also infinitely distributive, so

$$p \wedge \bigvee_{\mathfrak{L}} B_0 = \bigvee_{\mathfrak{L}} \{p \wedge a : a \in B_0\}.$$

If p is an atom, then: for $a \in B, p \wedge a = 0_{\mathfrak{L}}$ (a is an atom, too) and for $a \in F_{\mathfrak{B}}, p \wedge a = 0_{\mathfrak{L}}$ by (1). Hence $p = 0_{\mathfrak{L}}$ — contrary to our assumption.

If $p \in F$, then: for $a \in B, p \wedge a \neq 0_{\mathfrak{L}}$ iff $a \leqslant p$ and then clearly $p \wedge a = a$; for $a \in F_{\mathfrak{B}}, p \wedge a = 0_{\mathfrak{L}}$ by (iii). Thus $p = \bigvee_{\mathfrak{L}} A_p < p$ (since p is join-irreducible) — a contradiction again. This proves (2). Consequently

$$\bigcup\{A_z : z \in I_c\} = \bigcup\{A_z : z \in B_0\} = B$$

and for any $f \in F$,

$$f^{(c)} = f^{\mathfrak{A}} \text{ iff } f \in I_c \text{ iff } f \in B_0 \text{ iff } f \in F_{\mathfrak{B}} \text{ iff } f^{\mathfrak{B}} = f^{\mathfrak{A}}$$

and for any $f \in F$,

$$f^{(c)} = f^{\mathfrak{A}} \text{ iff } f \in I_c \text{ iff } f \in B_0 \text{ iff } f \in F_{\mathfrak{B}} \text{ iff } f^{\mathfrak{B}} = f^{\mathfrak{A}}$$

i.e. $f^{(c)} = f^{\mathfrak{B}}$ for all $f \in F$.

Thus $h(c) = \mathfrak{B}$.

(c) h is an order isomorphism:

Assume $x \leqslant y$ in $\mathfrak{L}$. Then $I_x \subseteq I_y$ and consequently

$$h(x) = \bigvee_{\mathfrak{A}}\{h(p) : p \in I_x\} \subseteq \bigvee_{\mathfrak{A}}\{h(p) : p \in I_y\} = h(y).$$

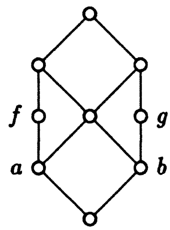
On the other hand, if $h(x) \subseteq h(y)$, then

$$\bigvee_{\mathfrak{A}}\{h(p) : p \in I_x\} \subseteq \bigvee_{\mathfrak{A}}\{h(p) : p \in I_y\}$$

which implies $A \cap I_x \subseteq A \cap I_y$ and $F \cap I_x \subseteq F \cap I_y$; hence $I_x \subseteq I_y$ and $x \leqslant y$.

By (a),(b),(c) h is an isomorphism of $\mathfrak{L}$ onto $\mathfrak{S}_w(\mathfrak{A})$. This completes the proof of the theorem. ■

Consider now the following finite lattice $\mathfrak{L}$:



It has two non-zero non-atomic join-irreducible elements and two atoms. It follows from the proof of Theorem 2 that it is isomorphic to the weak subalgebra lattice of an algebra $\mathfrak{A} = (\{a, b\}, f^{\mathfrak{A}}, g^{\mathfrak{A}})$ with f and g nullary, $f^{\mathfrak{A}} = a, g^{\mathfrak{A}} = b$. But it may be easily verified that it is also isomorphic to the weak subalgebra lattice of an algebra $\mathfrak{B} = (\{a, b\}, f^{\mathfrak{B}}, g^{\mathfrak{B}})$ with f n-ary and g m-ary for some arbitrary fixed natural numbers n and m and $f^{\mathfrak{B}} = \{(\underbrace{a, \ldots, a}_{n \text{ times}}, a)\}, g^{\mathfrak{B}} = \{(\underbrace{b, \ldots, b}_{m \text{ times}}, b)\}$. Moreover, the same is true of $\mathfrak{C} = (\{a, b\}, f^{\mathfrak{C}})$ with f n-ary and $f^{\mathfrak{C}} = \{(\underbrace{a, \ldots, a}_{n \text{ times}}, a), (\underbrace{b, \ldots, b}_{n \text{ times}}, b)\}$. This simple example shows that neither the type of a partial algebra nor the arity of its operations can be deduced from the structure of its weak subalgebra lattice: in this respect this lattice contains no more information on a partial algebra in the general case than the subalgebra lattice of a total algebra. The cardinality of the universe is implicit in the weak subalgebra lattice, though.

Some additional structural information may be obtained if we restrict the class of partial algebras considered to be unary algebras only (i.e. with all operations unary). Let $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f \in F})$ be a unary partial algebra. From Lemma 3 we obtain immediately the following

Lemma 4. *For $\mathfrak{A}$ unary, each non-zero join-irreducible element in $\mathfrak{S}_w(\mathfrak{A})$ contains either exactly one or exactly two atoms.*

Now define for $\mathfrak{A}$ a directed graph $\mathbf{G}(\mathfrak{A}) = (V_{\mathfrak{A}}, E_{\mathfrak{A}})$ as follows:
$V_{\mathfrak{A}} = A$, $\quad E_{\mathfrak{A}} = \bigcup\{E_{ab} : (a,b) \in A^2\}$, where $E_{ab} \subseteq \{a\} \times F \times \{b\}$ and $(a,f,b) \in E_{ab}$ iff $(a,b) \in f^{\mathfrak{A}}$. E_{ab} is the set of edges with initial vertex a and final vertex b.

Let $\mathbf{G}'(\mathfrak{A})$ be the corresponding undirected graph, obtained from $\mathbf{G}(\mathfrak{A})$ by omitting the orientations of edges. In particular, if e.g. $(a,b) \in f^{\mathfrak{A}}$ and $(b,a) \in f^{\mathfrak{A}}$, then we still have two different edges in $\mathbf{G}'(\mathfrak{A})$, for which we preserve the names (a,f,b) and (b,f,a).

Note that by definition of $\mathbf{G}'(\mathfrak{A})$, there is a bijective correspondence between the set of all atoms of $\mathfrak{S}_w(\mathfrak{A})$ and the set of all vertices of $\mathbf{G}'(\mathfrak{A})$, as well as between the set of all non-zero non-atomic join-irreducibles of $\mathfrak{S}_w(\mathfrak{A})$ and the set of all edges of $\mathbf{G}'(\mathfrak{A})$ (use Lemma 4 for the latter). Moreover, a vertex a is incident to an edge (x,g,y) in $\mathbf{G}'(\mathfrak{A})$ iff the atom $(\{a\},(f^{(a)})_{f\in F})$ of $\mathfrak{S}_w(\mathfrak{A})$ is contained in the join-irreducible element $(\{x,y\},(f^{(g)})_{f\in F})$ where $f^{(g)} = \emptyset$ for all $f \neq g$ and $g^{(g)} = \{(x,y)\}$.

This leads to the following

Theorem 5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unary partial algebras (which may have different types). Then $\mathfrak{S}_w(\mathfrak{A}) \cong \mathfrak{S}_w(\mathfrak{B})$ iff $\mathbf{G}'(\mathfrak{A}) \cong \mathbf{G}'(\mathfrak{B})$.*

PROOF : A lattice isomorphism of $\mathfrak{S}_w(\mathfrak{A})$ onto $\mathfrak{S}_w(\mathfrak{B})$ induces an order isomorphism of the set of all join-irreducibles in $\mathfrak{S}_w(\mathfrak{A})$ onto the set of all join-irreducibles in $\mathfrak{S}_w(\mathfrak{B})$. By the observation preceding the theorem, this in turn yields a graph isomorphism of $\mathbf{G}'(\mathfrak{A})$ onto $\mathbf{G}'(\mathfrak{B})$. Similarly, a graph isomorphism of $\mathbf{G}'(\mathfrak{A})$ onto $\mathbf{G}'(\mathfrak{B})$ induces an order isomorphism of the join-irreducibles of $\mathfrak{S}_w(\mathfrak{A})$ onto the join-irreducibles of $\mathfrak{S}_w(\mathfrak{B})$. Since each element of these two lattices is the join of all join-irreducibles it contains, the order isomorphism extends to a lattice isomorphism of $\mathfrak{S}_w(\mathfrak{A})$ onto $\mathfrak{S}_w(\mathfrak{B})$. ■

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