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Abstract. In this paper, a cardinal function, denoted by $sqL(X)$, which is less than both the spread and the density, is investigated in some details. We prove that, in several known inequalities involving the spread $s(X)$, the spread $s(X)$ can be replaced by $sqL(X)$. A related cardinal function, denoted by $qL(X)$, is also discussed.

Keywords: Cardinal function, cardinal inequality, spread, density, κ-quasi-dense.

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1. Introduction

It is well known that in the theory of cardinal function, there are some fundamental inequalities involving the spread $s(X) = \sup\{|D| : D \subseteq X, D$, is discrete$\}.\omega$, for example,

$$\text{“For } X \in \mathcal{T}_2, \psi(X) \leq 2^{s(X)}\text{”;}$$

$$\text{“For } X \text{ compact, } |RO(X)| \leq 2^{s(X)}\text{”}$$

and the Šapirovskii's theorem [**2**, Theorem 5.1]: “If $X \in \mathcal{T}_2$ with $s(X) \leq \kappa$, then there is a subset S of X with $|S| \leq 2^\kappa$ such that $X = \bigcup\{\overline{D} : D \in [S]^{\leq\kappa}\}$.”

In this paper, we will prove that, in the above inequalities, $s(X)$ can be replaced by another cardinal function, denoted by $sqL(X)$, which is less than both the spread and the density. Here we define a subset A of a space X with $|A| \leq 2^\kappa$, where κ is a cardinal, to be a strong κ-quasi-dense subset of X if for each family $\mathcal{U}$ of open subsets of X, there exist a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ and a $B \in [A]^{\leq\kappa}$ such that

$$(\cup\mathcal{V}) \cup \overline{B} \supseteq (\cup\mathcal{U}),$$

where $[A]^{\leq\kappa}$ denotes the set $\{B : B \subseteq A, |B| \leq \kappa\}$. If the above property holds only for open cover $\mathcal{U}$ of X, then we say that A is κ-quasi-dense. Now let us write

$$sqL(X) = \min\{\kappa : \text{there is a strong } \kappa\text{-quasi-dense subset of } X\},$$

$$qL(X) = \min\{\kappa : \text{there is a } \kappa\text{-quasi-dense subset of } X\}.$$

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Remark: Both the cardinal functions $sqL(X)$ and $qL(X)$ were introduce in [4], while the function $qL(X)$ has been discussed further in [5].

It is immediate that $qL(X) \leq sqL(X) \leq d(X)$, where $d(X)$ is the density of X. We now prove that if $X \in \mathcal{T}_2$, then $sqL(X) \leq s(X)$. In fact, let κ be such that $s(X) \leq \kappa$. By the theorem of Šapirovskii as quoted above, there is an $S \subseteq X$ with $|S| \leq 2^\kappa$ such that $X = \bigcup\{\overline{D} : D \in [S]^{\leq\kappa}\}$. Thus we need only to show that the subset S is strong κ-quasi-dense in X. Let $\mathcal{U}$ be a family of open subsets of X and let $Y = \cup\mathcal{U}$ be the subspace of X. Then $s(Y) \leq \kappa$. By another theorem of Šapirovskii ([**2**,Proposition 4.8]), there is a subset B of Y with $|B| \leq \kappa$ and a subcollection $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ such that $Y = \overline{B} \cup (\cup\mathcal{V})$. Therefore for each $b \in B$, there is a subset $A(b)$ of S with $|A(b)| \leq \kappa$ such that $b \in \overline{A(b)}$. Let $A = \bigcup\{A(b) : b \in B\}$. Then A is a subset of S with $|A| \leq \kappa$ such that $(\cup\mathcal{U}) \subseteq \overline{A} \cup (\cup\mathcal{V})$. Hence S is strong κ-quasi-dense and so $sqL(X) \leq s(X)$. Moreover both $sqL(X) \leq d(X)$ and $sqL(X) \leq s(X)$ can be strict.

For undefined notations and terminologies, we refer to [**3**]. We will use the Pol-Šapirovskii technique for the proofs of our main results.

2. Main theorems

First let us recall the following definition. Let X be a topological space. Then a family $\mathcal{U}$ of nonempty open subsets of X is said to be a pseudo-local base for a point $p \in X$, if $\{p\} = \bigcap\{U : U \in \mathcal{U}\}$. Then

$$\psi(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a pseudo-local base for } p\}.\omega_0,$$

$$\psi(X) = \sup\{\psi(p, X) : p \in X\}.$$

Theorem 1. *For $X \in \mathcal{T}_2, \psi(X) \leq 2^{sqL(X)}$.*

PROOF : Let $sqL(X) = \kappa$ and A with $|A| \leq 2^\kappa$ be a strong κ-quasi-dense subset in X. Let p be any point in X. If $q \in X$ with $q \neq p$, then there is an open subset V_q of q with $p \notin \overline{V_q}$ since $X \in \mathcal{T}_2$. Thus

$$\bigcup\{V_q : q \in X - p\} = X - \{p\}.$$

Since A is strong κ-quasi-dense, for such family $\{V_q\}$, we can find $\{q_\alpha\}_{\alpha<\kappa} \subseteq X-\{p\}$ and $B \in [A]^{\leq\kappa}$ with

$$\bigcup_{\alpha<\kappa} V_{q_\alpha} \cup \overline{B} \supseteq X - \{p\}.$$

Let $\mathcal{U}_1 = \{X - \overline{D} : D \subseteq B, p \notin \overline{D}\}$. Then $|\mathcal{U}_1| \leq 2^\kappa$. Let $\mathcal{U}_2 = \{X - \overline{V_{q_\alpha}} : \alpha < \kappa\}$ and let $\mathcal{U}_p = \mathcal{U}_1 \cup \mathcal{U}_2$. Then $\{p\} = \cap\mathcal{U}_p$. In fact, let

$$q \in X - \{p\} \subseteq \bigcup_{\alpha<\kappa} V_{q_\alpha} \cup \overline{B}.$$

If $q \in \bigcup_{\alpha<\kappa} V_{q_\alpha}$, then $q \in V_{q_{\alpha'}}$ for some $\alpha' < \kappa$, so that $q \notin X - \overline{V_{q_{\alpha'}}}$ and $q \notin \cap\mathcal{U}_2$. If $q \in \overline{B}$, then $q \in \overline{B} \cap V_q \subseteq \overline{B \cap V_q} \subset \overline{V_q}$, and by choosing $D = B \cap V_q$, we see that $q \in \overline{D}$. But $p \notin \overline{V_q}$, thus $p \notin \overline{D}$ and so $X - \overline{D} \in \mathcal{U}_1$. Hence $q \notin \cap\mathcal{U}_1$ and therefore $\{p\} = \cap\mathcal{U}_p$. As $|\mathcal{U}_p| \leq |\mathcal{U}_1| + |\mathcal{U}_2| \leq 2^\kappa$, we conclude that $\psi(p, X) \leq 2^\kappa$ and hence $\psi(X) \leq 2^\kappa = 2^{sqL(X)}$. This completes the proof. ■

Corollary. ([2, Proposition 4.11]). *For* $X \in \mathcal{T}_2, \psi(X) \leq 2^{s(X)}$.

Example. Let X_1 be the Niemytzki plane, X_2 be the space $\mathbf{R}$ with the topology $\tau = \{V - A : V$ is the usual open set in $\mathbf{R}$ and A is countable$\}$, and let $Y = X_1 \oplus X_2$. Then $sqL(X) = \omega$, but $d(Y) \geq d(X_2) > \omega$ and $s(Y) \geq s(X_1) \geq 2^\omega$.

Theorem 2. *If* $X \in \mathcal{T}_2$ *with* $sqL(X) \leq \kappa$, *then there is a subset* S *of* X *with* $|S| \leq 2^\kappa$ *such that* $X = \bigcup\{\overline{D} : D \in [S]^{\leq\kappa}\}$. *In particular,* $d(X) \leq 2^{sqL(X)}$.

PROOF : Let A be a strong κ-quasi-dense subset of X. Since $X \in \mathcal{T}_2$, it follows from Theorem 1 that $\psi(X) \leq 2^\kappa$.

For each $p \in X$, let $\mathcal{U}_p$ be a pseudo-local base for p with $|\mathcal{U}_p| \leq 2^\kappa$. By transfinite induction, construct a sequence $\{S_\alpha : 0 \leq \alpha < \kappa^+\}$ and a sequence $\{\mathcal{U}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(i) $|S_\alpha| \leq 2^\kappa, \qquad 0 \leq \alpha < \kappa^+$;

(ii) $\mathcal{U}_\alpha = \{V \in \mathcal{U}_p : p \in \bigcup_{\beta<\alpha} S_\beta\}, \qquad 0 < \alpha < \kappa^+$;

(iii) if $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}, B \in [A]^{\leq\kappa}$ and $\overline{B} \cup (\cup\mathcal{V}) \neq X$, then $S_\alpha - (\overline{B} \cup (\cup\mathcal{V})) \neq \emptyset$.

Now let

$$S = \left(\bigcup_{\alpha<\kappa^+} S_\alpha\right) \cup A.$$

Then S is the required subset. First, we note that $|S| \leq \kappa^+ . 2^\kappa + 2^\kappa = 2^\kappa$. Next, let $p \in X$. If $p \in S$, then nothing to prove. If $p \notin S$, then $p \notin \bigcup_{\alpha<\kappa^+} S_\alpha$. For each $q \in \bigcup_{\alpha<\kappa^+} S_\alpha$, $q \neq p$ so that we can choose a $V_q \in \mathcal{U}_q$ such that $p \notin V_q$, and hence $p \notin \bigcup\{V_q : q \in \bigcup_{\alpha<\kappa^+} S_\alpha\}$. On the other hand, since A is strong κ-quasi-dense, there is $B \in [A]^{\leq\kappa}$ and $M \in \left[\bigcup_{\alpha<\kappa^+} S_\alpha\right]^{\leq\kappa}$ such that

$$\left(\bigcup_{q\in M} V_q\right) \cup \overline{B} \supseteq \bigcup\{V_q : q \in \bigcup_{\alpha<\kappa^+} S_\alpha\} \supseteq \bigcup_{\alpha<\kappa^+} S_\alpha.$$

It remains to prove that $p \in \overline{B}$. If $p \notin \overline{B}$, then $\left(\bigcup_{q\in M} V_q\right) \cup \overline{B} \neq X$. Since $|M| \leq \kappa$, there is $\alpha' < \kappa^+$ such that $M \subseteq S_{\alpha'}$; that is $\{V_q : q \in M\} \in [\mathcal{U}_{\alpha'}]^{\leq\kappa}$. Hence, by (iii), $S_{\alpha'+1} - \left(\left(\bigcup_{q\in M} V_q\right) \cup \overline{B}\right) \neq \emptyset$, which contradicts the fact that

$$\left(\bigcup_{q\in M} V_q\right) \cup \overline{B} \supseteq \bigcup_{\alpha<\kappa^+} S_\alpha \supseteq S_{\alpha'+1}.$$

This completes the proof. ∎

Remark. Our results generalize the theorem of Šapirovskii [**2**, Theorem 5.1] as quoted above.

Now, recall another inequality [**2**, Theorem 5.3]: For $X \in \mathcal{T}_3$, $nw(X) \leq 2^{s(X)}$, where $nw(X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a net for } X\}$ is the net weight for X. Using Theorem 2, we can strengthen the above result in replacing the spread $s(X)$ by $sqL(X)$.

Theorem 3. *For* $X \in \mathcal{T}_3, nw(X) \leq 2^{sqL(X)}$.

PROOF : Let $sqL(X) = \kappa$. By Theorem 2, there is a subset S of X with $|S| \leq 2^\kappa$ and $X = \bigcup\{\overline{A} : A \subseteq S, |A| \leq \kappa\}$. Then the family $\mathcal{N} = \{\overline{N} : N \subseteq S, |N| \leq \kappa\}$ can be easily checked to be a net in X of cardinality $\leq 2^\kappa$ (cf. [**2**, Theorem 5.3]). Hence $nw(X) \leq 2^{sqL(X)}$. ∎

Remark. The result of Theorem 3 was also announced in [**4**, Theorem 2.12]; but in our proof, the result of Theorem 2, which was not mentioned in [4], is essential.

Remark. The following inequality follows immediately from Theorem 3: For $X \in \mathcal{T}_3, |X| \leq 2^{sqL(X)\psi(X)}$.

Recall that a space X is said to be of point-countable type if for each point $p \in X$, there is a compact set K such that $p \in K$ and K has countable character. Note that for $X \in \mathcal{T}_2$ of point-countable type, $\psi(p, X) = \chi(p, X)$ and $\psi(X) = \chi(X)$, where $\chi(p, X)$ is the character at p and $\chi(X)$ is the character of X.

Next, consider the following inequality involving the cardinality $|RO(X)|$ of regular open subsets of X [**2**, Corollary 7.7]: If $X \in \mathcal{T}_2$ is compact, then $|RO(X)| \leq 2^{s(X)}$. In fact, the above inequality holds if $X \in \mathcal{T}_2$ is of point-countable type (cf. [**2**, p.30]). We will prove that this inequality can be improved.

Theorem 4. *If* $X \in \mathcal{T}_2$ *is of point-countable type, then*

$$|RO(X)| \leq 2^{sqL(X)}.$$

PROOF : For any space X, we have $|RO(X)| \leq \pi w(X)^{c(X)}$, where $\pi w(X)$ is the

π-weight of X and $c(X)$ is the cellularity of (X). Clearly $c(X) \leq sqL(X)$. Hence

$$\begin{aligned}
|RO(X)| &\leq \pi w(X)^{sqL(X)} \\
&= (\pi\chi(X)d(X))^{sqL(X)} \\
&= \pi\chi(X)^{sqL(X)}d(X)^{sqL(X)} \\
&\leq \pi\chi(X)^{sqL(X)}\left(2^{sqL(X)}\right)^{sqL(X)}, \qquad \text{(using Theorem 2)} \\
&= \pi\chi(X)^{sqL(X)} \\
&\leq \chi(X)^{sqL(X)} \\
&= \psi(X)^{sqL(X)}, \qquad \text{(since } X \text{ is of point-countable type)} \\
&\leq \left(2^{sqL(X)}\right)^{sqL(X)}, \qquad \text{(using Theorem 1)} \\
&= 2^{sqL(X)}.
\end{aligned}$$

■

As an immediate consequence, we have:

Corollary. *If $X \in T_3$ is of point-countable type, then $w(X) \leq 2^{sqL(X)}$, where $w(X)$ is the weight of X.*

In the last part of the paper, we will establish some new inequalities involving the cardinal function $qL(X)$. First, let us recall the following definitions (cf. [**2**, p.54]). Let $\mathcal{U}$ be an open collection of X and $p \in X$. Then

$$\begin{aligned}
\text{ord}(p,\mathcal{U}) &= |\{U \in \mathcal{U} : p \in U\}|; \\
\text{ord}(\mathcal{U}) &= \sup\{\text{ord}(p,\mathcal{U}) : p \in X\}; \\
psw(X) &= \min\{\text{ord}(\mathcal{U}) : \text{ for any } p \in X, \bigcap\{U \in \mathcal{U} : p \in U\} = \{p\}\}.
\end{aligned}$$

Theorem 5. *For $X \in T_1, d(X) \leq psw(X)^{qL(X)}$.*

Proof : Let $psw(X) = \lambda, qL(X) = \kappa$, $\mathcal{U}$ an open cover of X such that for any $p \in X, \{p\} = \bigcap\{U \in \mathcal{U} : p \in U\}$ and $\text{ord}(\mathcal{U}) = \lambda$ and let A be κ-quasi-dense subset of X. We write $\mathcal{U}_p = \{U \in \mathcal{U} : p \in U\}$. Use transfinite induction to construct a sequence $\{B_\alpha : 0 \leq \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(i) $|B_\alpha| \leq \lambda^\kappa, \quad 0 \leq \alpha < \kappa^+$;
(ii) $\mathcal{U}_\alpha = \{V : V \in \mathcal{U}_p, p \in \bigcup_{\beta<\alpha} B_\beta\}, \quad 0 < \alpha < \kappa^+$;
(iii) If $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}, D \in [A]^{\leq\kappa}$ with $(\cup\mathcal{V}) \cup \overline{D} \neq X$, then $B_\alpha - ((\cup\mathcal{V}) \cup \overline{D}) \neq \emptyset$.

Let $S = \bigcup_{\alpha<\kappa^+} B_\alpha \cup A$, then $|S| \leq \kappa^+.\lambda^\kappa + 2^\kappa = \lambda^\kappa$. It remains to show that $\overline{S} = X$.

If $p \in X - \overline{S}$, then $p \notin \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}$, so each $q \in \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}$, we have $q \neq p$ and thus there is $V_q \in \mathcal{U}_q$ with $p \notin V_q$, and that $\{V_q : q \in \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}\} \supseteq \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}$. Hence there is a subset $M \subseteq \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}$ with $|M| \leq \kappa$ and $D \in [A]^{\leq\kappa}$ such that

$$\bigcup_{q\in M} V_q \cup \overline{D} \supseteq \overline{\bigcup_{\alpha<\kappa^+} B_\alpha}, \tag{1}$$

since A is κ-quasi-dense. Now for any $q \in M$, $V_q \cap (\bigcup_{\alpha<\kappa^+} B_\alpha) \neq \emptyset$, thus we can choose $b(q) \in V_q \cap (\bigcup_{\alpha<\kappa^+} B_\alpha)$ so that $V_q \in \mathcal{U}_{b(q)}$. Since $|\{b(q) : q \in M\}| \leq \kappa$, there is $\alpha' < \kappa^+$ with $\{b(q) : q \in M\} \subseteq B_{\alpha'}$; that is $\{V_q : q \in M\} \in [\mathcal{U}_{\alpha'}]^{\leq\kappa}$. Since $p \notin \overline{S}, p \notin \overline{A}$ and so $p \notin \overline{D}$; that is $\left(\bigcup_{q\in M} V_q\right) \cup \overline{D} \neq X$. Then use (iii) to conclude that $B_{\alpha'} - \left(\left(\bigcup_{q\in M} V_q\right) \cup \overline{D}\right) \neq \emptyset$, contradicting (1). This completes the proof. ∎

Remark. It follows from the theorem that, for $X \in \mathcal{T}_1$,

$$|X| \leq psw(X)^{qL(X)psw(X)} = 2^{qL(X)psw(X)}.$$

However, a better inequality has been proved in [6],: For $X \in \mathcal{T}_1$,

$$|X| \leq 2^{L^*(X)psw(X)},$$

and it is easy to show that $L^*(X) \leq qL(X)$.

Corollary. [4, Theorem 1.11]. *For $X \in \mathcal{T}_3$, $d(X) \leq psw(X)^{qL(X)}$.*

Lemma. *For any topological space X, $sqL(X) \leq \Psi(X)qL(X)$, where $\Psi(X) = \min\{\kappa :$ every closed subset in X is the intersection of $\leq \kappa$ open sets $\}$.*

Theorem 6. *For $X \in \mathcal{T}_3, K(X) \leq 2^{qL(X)\Psi(X)}$, where $K(X)$ denotes the number of all compact subsets of X.*

Proof : Let $qL(X)\Psi(X) = \kappa$. By the above lemma, we have $sqL(X) \leq \kappa$. Then using second remark of Theorem 3 to conclude that $|X| \leq 2^{sq(X)\psi(X)} = 2^\kappa$. Since $\psi(X) \leq \Psi(X) \leq \kappa$ and $X \in \mathcal{T}_3$, for each $p \in X$, we can choose a collection $\mathcal{V}_p$ of open neighborhoods of p, closed under finite intersections, such that $|\mathcal{V}_p| \leq \kappa$ and $\bigcap\{\overline{V} : V \in \mathcal{V}_p\} = \{p\}$. Let $\mathcal{V} = \bigcup_{p\in X} \mathcal{V}_p$, let $\mathcal{W}$ be all unions of $\leq \kappa$ elements of $\mathcal{V}$, and let $\mathcal{G} = \{W \cup (\overline{D} \cap (X - K)) : W \in \mathcal{W}, D \in [A]^{\leq\kappa}\}$, where A is a strong κ-quasi-dense subset in X. It remains to prove that the complement of every compact subset of X is the union of $\leq \kappa$ elements of $\mathcal{G}$. Let $K \subseteq X$ be compact. Since $\Psi(X) \leq \kappa$, we see that $X - K = \bigcup\{F_\alpha : 0 \leq \alpha < \kappa\}$ with each

F_α closed. Fix $\alpha < \kappa$. Then for each $p \in F_\alpha$, use compactness of K to obtain $V_p \in \mathcal{V}_p$ such that $K \cap V_p = \emptyset$. Since $\{V_p : p \in F_\alpha\} \supseteq F_\alpha$ and $sqL(X) \leq \kappa$, we can find $W_\alpha \in \mathcal{W}$ and $D_\alpha \in [A]^{\leq\kappa}$ such that $W_\alpha \cup \overline{D_\alpha} \supseteq \{V_p : p \in F_\alpha\} \supseteq F_\alpha$. Let $G_\alpha = W_\alpha \cup (\overline{D_\alpha} \cap (X - K))$. Then $G_\alpha \in \mathcal{G}, G_\alpha \cap K = \emptyset$ and $X - K = \bigcup_{\alpha<\kappa} G_\alpha$. This completes the proof. ■

Remark. This result gives a partial extension of the following [**2**, Theorem 9.5]: For $X \in \mathcal{T}_2, K(X) \leq 2^{e(X)\Psi(X)}$. In fact, it can be easily checked that $e(X)\Psi(X) = s(X)\Psi(X)$ and so $e(X)\Psi(X) \geq qL(X)\Psi(X)$.

Example. The following example shows that the inequality in the remark can be strict. Let X be the Niemytzki plane. Then $e(X)\Psi(X) = 2^\omega\omega = 2^\omega$. But $qL(X)\Psi(X) = d(X)\Psi(X) = \omega$ so that $e(X)\Psi(X) > qL(X)\Psi(X)$.

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