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Topologies in product which preserve Baire spaces

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Abstract. We give some conditions on a topology on the product of finitely many Baire spaces with countable pseudo-base which ensure that the product is a Baire space. In particular, R^n with the algebraic topology is a Baire space.

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Let X_i be topological spaces, $i \in \mathbb{N}$. In what follows, by $X^{(n)}$ we denote the product $\prod_{i=1}^{n} X_i$. We identify $X^{(n+1)}$ with $X^{(n)} \times X_{n+1}$. For $G \subset X^{(n+1)}, x \in X^{(n)}$ and $y \in X_{n+1}$ we put $G|_x = \{z \in X_{n+1} : (x,z) \in G\}$ and $G|^y = \{z \in X^{(n)} : (z,y) \in G\}$.

We regard a topology on the product of finitely many topological spaces which for each n satisfies the following conditions:

(1) For $n = 1$ it coincides with the topology on X_1.
(2) If $x \in X^{(n)}$ and $U \subset X^{(n+1)}$ is open then $U|_x$ is open in X_{n+1}.
(3) If $y \in X_{n+1}$ and $U \subset X^{(n+1)}$ is open then $U|^y$ is open in $X^{(n)}$.
(4) If $V \subset X^{(n)}$ is open and $W \subset X_{n+1}$ is open then $V \times W$ is open in $X^{(n+1)}$.

Of course, the Tychonoff topology on the product satisfies conditions (1)-(4), but there are other natural topologies which satisfy these conditions (cf. Examples 1 and 2).

Recall that a topological space is called a Baire space if it satisfies the Baire category theorem. A family of non-empty open sets is called a pseudo-base if every non-empty open set contains at least one member of this family.

The proof of the following theorem is essentially the same as the proof of the corresponding result for the Tychonoff topology (cf. [1]).

Theorem. *The product of finitely many Baire spaces, each of which (except the first) has a countable pseudo-base, with a topology satisfying* (1)-(4) *is a Baire space.*

PROOF : At first observe that if $G \subset X^{(n+1)}$ is open and dense, $W \subset X_{n+1}$ is open and non-empty, then $A = \{x \in X^{(n)} : G|_x \cap W \neq \emptyset\}$ is open and dense in $X^{(n)}$. In fact, let $U \subset X^{(n)}$ be open and non-empty. By (4) we obtain $(U \times W) \cap G \neq \emptyset$. For $(x,y) \in (U \times W) \cap G$ we have $y \in G|_x \cap W$, therefore $x \in U \cap A$. Hence A is dense. Now, for $x \in A$ we take $y \in W$ such that $(x,y) \in G$. By (3) the set $G|^y$ is open in $X^{(n)}$ and $x \in G|^y$. If $z \in G|^y$ then $y \in G|_z \cap W$, therefore $G|^y \subset A$. Hence A is open.

Now we prove by induction. For $n = 1$ the thesis follows from (1).

Let $\{G_m : m \in \mathbb{N}\}$ be a family of open and dense subsets in $X^{(n+1)}$, and $\{W_m : m \in \mathbb{N}\}$ be a pseudo-base in X_{n+1}. Let $A_{mk} = \{x \in X^{(n)} : G_m|_x \cap W_k \neq \emptyset\}$ and

$A = \bigcap\{A_{mk} : m, k \in \mathbb{N}\}$. Since A_{mk} are open and dense in $X^{(n)}$, by the induction hypothesis A is dense in $X^{(n)}$. Let $U \subset X^{(n+1)}$ be non-empty and open. Take a point $(\overline{x}, \overline{y}) \in U$. Take $x \in U|^{\overline{y}} \cap A$, by (3) the set $U|^{\overline{y}} \cap A$ is non-empty. It follows from (2) that the sets $G_m|_x$ are open in X_{n+1}. Since $x \in A$, we obtain $G_m|_x \cap W_k \neq \emptyset$ for each m and k. It implies that $G_m|_x$ are open and dense in X_{n+1}, because the family $\{W_k : k \in \mathbb{N}\}$ is a pseudo-base. Because of (2) and $(x, \overline{y}) \in U$, the set $U|_x$ is open and non-empty in X_{n+1}. Since X_{n+1} is a Baire space, there exists a point $y \in U|_x \cap \bigcap_{m \in \mathbb{N}} G_m|_x$, i.e., $(x, y) \in U \cap \bigcap_{m \in \mathbb{N}} G_m$. Hence $X^{(n+1)}$ is a Baire space. ∎

Example 1. Algebraic topology. Let X be a linear topological space. A set $U \subset X$ is called algebraically open (a-open) if for each $x \in U$ and $h \in X$ there exists an $\varepsilon > 0$ such that $x + th$ belongs to U, whenever $|t| < \varepsilon$. The family of all a-open sets is a topology on X, which we call the algebraic topology (a-topology). It is also called the core topology (cf. [2]). It is easy to check that the a-topology on R^n satisfies conditions (1)-(4). Hence R^n with the a-topology is a Baire space.

Example 2. Cross-topology. Let $\{X_t : t \in T\}$ be a family of topological spaces and $X = \prod_{t \in T} X_t$. We say that $U \subset X$ is cross-open (c-open) if for each $x \in U$ and $s \in T$ there exists an open $V_s \subset X_s$ such that $x \in V \subset U$, where $V = \prod_{t \in T} W_t$ and $W_t = \{x_t\}$ for $s \neq t$, and $W_s = V_s$. Of course the family of all c-open sets is a topology on X which we call the cross-topology (c-topology). The product of finitely many topological spaces with the c-topology satisfies conditions (1)-(4). Hence, the product of finitely many Baire spaces, each of which has a countable pseudo-base (except one), with the c-topology is a Baire space.

Remarks:

1. Note that in the case of R^n, the set U is c-open iff for each $x \in U$ there exists an $\varepsilon > 0$ such that $x + te_i \in U$, whenever $|t| < \varepsilon$, $\quad i = 1, 2, \ldots, n$, where $e_i = \{\delta_k^i\}$. Of course, if we replace e_i by a_i, where $a_1, \ldots, a_n$ are linearly independent, then R^n with this topology is also a Baire space.

2. In the case of R^n or $\mathrm{R}^{\mathbb{N}}$ the Tychonoff topology is essentially weaker than the a-topology and the a-topology is essentially weaker than the c-topology.

3. Combining topologies satisfying (1)-(4), one can obtain other such topologies. For example, consider the topology on R^3 consisting of all sets U satisfying the condition: for each $x \in U$ and $h = (h_1, h_2, 0) \in \mathrm{R}^3$ there exists $\varepsilon > 0$ such that $x + th \in U$ and $x + te_3 \in U$, whenever $|t| < \varepsilon$. Here R^2 is endowed with the a-topology.

4. Note, that the first part of the proof of our Theorem gives the Kuratowski-Ulam Theorem for the topology satisfying (1)-(4). Namely, it is the first step of the proof, that if $F \subset X^{(n+1)}$ is of first category and $X^{(n+1)}$ has a countable pseudo-base, then $F|_x$ are also of first category (in X_{n+1}), for all $x \in X^{(n)}$ except a set of first category. Using this fact, one can easily obtain (also by induction) another proof of the Theorem, by showing that every open $U \subset X^{(n+1)}$ is of second category.

5. Z. Kominek has proved that any infinite dimensional linear space, in particular $\mathrm{R}^{\mathbb{N}}$, with the a-topology is not a Baire space. In a similar way we can show that $\mathrm{R}^{\mathbb{N}}$ with the c-topology is not a Baire space too. Indeed, let H be an algebraic

base of the linear space $\mathrm{R}^{\mathbf{N}}$ such that $\{e_n : n \in \mathbf{N}\} \subset H$, $e_n = \{\delta_n^k\}_{k\in\mathbf{N}}$. Let λ_n be the coordinate function corresponding to e_n in this base. The functions λ_n satisfy conditions:
(i) $\{n : \lambda_n(x) \neq 0\}$ is finite for each $x \in \mathrm{R}^{\mathbf{N}}$,
(ii) $\lambda_n(tx + sy) = t\lambda_n(x) + s\lambda_n(y)$ for each $x, y \in \mathrm{R}^{\mathbf{N}}$ and $t, s \in \mathrm{R}$,
(iii) $\lambda_n(e_k) = \delta_k^n$.
Put $G_n = \{x \in \mathrm{R}^{\mathbf{N}} : \lambda_n(x) \neq 0\}$. Using (ii) and (iii) it is easy to check that G_n are c-open and c-dense. By (i) the intersection $\bigcap_{n\in\mathbf{N}} G_n$ is empty.

References

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