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Abstract. Recently Li & Yorke introduced the notion of chaos for continuous maps. The recent papers by Janková, Smítal and Kuchta, Smítal give a characterization of chaos for continuous maps of compact real interval into itself. In the present paper we give a characterization of chaos for continuous maps of the circle. The class of continuous maps of the circle extends the class of continuous maps of the interval, if we consider dynamical systems generated by the maps from these classes.

Keywords: Chaos, circle, iteration, topological dynamics

Classification: 54H20, 26A18

1. INTRODUCTION

Let M denote a compact real interval or a circle and $C(M,M)$ denote the set of continuous maps of M into itself. Let $f \in C(M,M); \delta \geq 0$ and $A \subset M$ be a set such that for any $x, y \in A; x \neq y$ and any periodic point p of f:

$$\limsup_{n\to\infty} \|f^n(x), f^n(y)\| > \delta \tag{1}$$

$$\liminf_{n\to\infty} \|f^n(x), f^n(y)\| = 0 \tag{2}$$

$$\limsup_{n\to\infty} \|f^n(x), f^n(p)\| > \delta \tag{3}$$

Here f^n is the n-th iterate of f and $\|x,y\|$ is the distance of points x, y. Then A is called a scrambled set of f, or (when $\delta > 0$) a δ-scrambled set. The function f is chaotic in the sense of Li & Yorke [7] if f has an uncountable scrambled set. But, indeed, there are some other definitions of chaos. This paper gives a characterization of chaos for the maps from $C(M,M)$. Before we state the results, we recall some terminology.

Let $f \in C(M,M)$. We say that an interval $J \subset M$ is an f-periodic interval of period $k \in N$ if $f^k(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for $i \neq j$; $i, j = 1 \dots k$. Interval J may be degenerated to a point and be called a periodic point. Let $Per(f)$ denote the set of all periodic points of f and $P(f)$ denote the set of all periods of periodic points of f. Two points $u, v \in M$ are f-separable if there are disjoint periodic closed intervals $J_u, J_v \subset M$ (may be degenerated to a point) with $u \in J_u$; $v \in J_v$. Otherwise u, v are f-nonseparable. The set of all limit points of a trajectory $\{f(x)\}_{k=1}^{\infty}$ is called the attractor of f and x, and is denoted by $\omega(f, x)$.

Now we give main results of this paper.

Theorem A. *Let $f \in C(M,M)$ and $Per(f) \neq \emptyset$. The following conditions are equivalent:*

(a) *f is chaotic in the sense of Li & Yorke;*
(b) *f has an infinite attractor containing two f-nonseparable points;*
(c) *for some $\delta > 0$, f has a nonempty perfect δ-scrambled set;*
(d) *f has a trajectory which is not approximated by cycles;*
(e) *f is topologically conjugated to a function, which has a scrambled set of positive Lebesgue measure;*
(f) *for some $\delta > 0$, f has a nonempty δ-scrambled set;*
(g) *f has a scrambled set containing two points.*

Theorem B. *Let $f \in C(M,M)$ and $Per(f) = \emptyset$. Then f is topologically semi-conjugated to a rotation of a circle and f has no two-point scrambled set.*

Remark 1.1. Theorem A was proved in papers by Janková, Smítal [**5**] and Kuchta, Smítal [**6**] for maps of compact real interval into itself.

2. Maps with periodic points

Let N, Z, R denote the sets of natural, integer, real numbers respectively and $S = R \setminus Z$ denote the circle. We shall denote by $\Pi : R \to S$ the natural projection.

Let $x_1, x_2 \in S$. We define: $\|x_1, x_2\| = \min\{|y_1 - y_2|;\ x_i = \Pi(y_i)\}$

Let $F \in C(R,R)$ and $f \in C(S,S)$. Then F is a lifting of f if $f \circ \Pi = \Pi \circ F$.

Let F be a lifting of f. Then there is $k_f \in Z$ such that for any $x \in R, z \in Z$: $F(x+z) = F(x) + zk_f$. So we can define the degree of $f : deg(f) = k_f$

Let $f \in C(S,S)$, $deg(f) = 1$ and F be a lifting of f. Then:

$$\rho(F,x) = \limsup_{n\to\infty} \frac{1}{n}(F^n(x) - x) \qquad \text{for } x \in R$$
$$\rho(F) = \{\rho(F,x); x \in R\}.$$

The class $C(S,S)$ can be decomposed into following disjoint sets (see [**1**], [**2**], [**8**]):

$W_0 = \{f \in C(S,S); Per(f) = \emptyset\}$
$W_1 = \{f \in C(S,S); P(f^n) = \{1\} \text{ for some } n \in N\}$
$W_2 = \{f \in C(S,S); P(f^n) = \{2^i; i = 0,1,2,\dots\} \text{ for some } n \in N\}$
$W_3 = \{f \in C(S,S); P(f^n) = N \text{ for some } n \in N\}$

Remark 2.1. Let $f \in C(S,S)$ and $n \in N$. It is easy to see that f has some property from the properties (a), …,(g) from Theorem A if and only if f^n has the same property.

Lemma 2.2. *Let $f \in C(S,S)$ and $1,3 \in P(f)$. The following conditions hold:*

(i) *for some $\delta > 0$, f has a nonempty perfect δ-scrambled set;*
(ii) *f has an infinite attractor containing two f-nonseparable points.*

Proof : Without any loss of generality we may assume that:
$0 < x_1 < x_2 < x_3 < 1$ and $f(\Pi(0)) = \Pi(0)$; $f(\Pi(x_1)) = \Pi(x_2)$; $f(\Pi(x_2)) = \Pi(x_3)$;

$f(\Pi(x_3)) = \Pi(x_1)$. ($\{\Pi(x_1), \Pi(x_2), \Pi(x_3)\}$ is three cycle and $\Pi(0) = \Pi(1)$ is fixed point.)

It is easy to see, that for any given function f there exists a code, consisting from four numbers (0 or 1), such that Table 1 gives us an information about properties of f.

Then according to Table 2 we obtain from this code some intervals I_0, I_1 and some $m \in N$, such that for all $n \in N$ and $s_i \in \{0;1\}$ there exists a closed interval $I_{s_1 \ldots s_{n+1}}$ such that:

$$I_{s_1 \ldots s_{n+1}} \subset I_{s_1 \ldots s_n} \qquad \text{and} \qquad f^m(\Pi(I_{s_1 \ldots s_{n+1}})) = \Pi(I_{s_2 \ldots s_{n+1}})$$

Let $I_s = \bigcap_{n=1}^{\infty} I_{s_1 \ldots s_n}$ for every $s = \{s_i\}_{n=1}^{\infty}$ $s_i \in \{0;1\}$.

Let $A = \{I_s;\ I_s \text{ consist of one point}\}$.

There exists such $B \subset A$ that $\Pi(B)$ is nonempty perfect δ-scrambled set ($\delta > 0$) for function f^m (see [5]). There exists $x \in S$ such that $\Pi(A) \subset \omega(f^m, x)$ (for more details see [3], Symbolic dynamic, pp.39-43). For any $y \in \Pi(B)$ and $J \subset S$ closed f^m-periodic interval of period $k \in N$ it holds: if $y \in J$ then $\omega(f^m, x) \subset \bigcup_{i=1}^{k} f^{mi}(J)$ (y is not periodic). Because for any two points of $\Pi(B)$ (2) holds for function f^m we obtain $\Pi(B) \subset J$ and this proves that any two points of $\Pi(B)$ are f^m-nonseparable. Now it suffices to consider Remark 2.1 and we are done. ∎

Table 1

Code:	Property: $*$ = There exists an interval	
0 . . .	$* A_0 \subset [0; x_1]$;	$f(\Pi(A_0)) = \Pi([0; x_2])$
1 . . .	$* A_1 \subset [0; x_1]$;	$f(\Pi(A_1)) = \Pi([x_2; 1])$
. 0 . .	$* B_0 \subset [x_1; x_2]$;	$f(\Pi(B_0)) = \Pi([x_2; x_3])$
. 1 . .	$* B_1 \subset [x_1; x_2]$;	$f(\Pi(B_1)) = \Pi([x_3; 1 + x_2])$
. . 0 .	$* C_0 \subset [x_2; x_3]$;	$f(\Pi(C_0)) = \Pi([x_3; 1 + x_1])$
. . 1 .	$* C_1 \subset [x_2; x_3]$;	$f(\Pi(C_1)) = \Pi([x_1; x_3])$
. . . 0	$* D_0 \subset [x_3; 1]$;	$f(\Pi(D_0)) = \Pi([x_1; 1])$
. . . 1	$* D_1 \subset [x_3; 1]$;	$f(\Pi(D_1)) = \Pi([0; x_1])$

Table 2

Code:	Result:		
0 1 . .	$m = 1$,	$I_0 = [0; x_1]$,	$I_1 = [x_1; x_2]$,
. 0 1 .	$m = 2$,	$I_0 = [x_1; x_2]$,	$I_1 = [x_2; x_3]$,
0 0 0 .	$m = 3$,	$I_0 = [0; x_1]$,	$I_1 = [x_1; x_2]$,
. . 0 0	$m = 2$,	$I_0 = [x_2; x_3]$,	$I_1 = [x_3; 1]$,
1 . 0 1	$m = 2$,	$I_0 = [x_2; x_3]$,	$I_1 = [x_3; 1]$,
1 1 1 .	$m = 3$,	$I_0 = [x_1; x_2]$,	$I_1 = [x_2; x_3]$,

Lemma 2.3. *Let $f \in C(S,S)$, $f \in W_1 \cup W_2$ and f have a fixed point. Then there exists F lifting of f and a closed bounded interval $I \subset R$ greater than one, such that $F \setminus I \in C(I,I)$.*

PROOF : We know that $|deg(f)| < 1$ (see [2]), so we can divide our problem to the three cases.

* Let $deg(f) = 0$. In this case F is periodic and we are easily done.
* Let $deg(f) = 1$. There is F lifting of f such that F has a fixed point. It follows that $0 \in \rho(F)$. Since $f \in W_1 \cup W_2$ the set $\rho(F)$ must have only one point (see [4]). Then $\rho(F) = \{0\}$. We claim there exists a point $a \in R$ such that:
$$\text{if} \qquad x \geq a \qquad \text{then} \qquad F(x) \geq a.$$
If it is not true, then there exists $\delta > 0$ such that for every $x \in R$ there is some $y \geq x + \delta$ such that $F(y) = x$. Now we can easily obtain that there is $z \in R$ such that $\rho(F,z) \leq -\delta$ which is impossible. In a similar way we can find $b \in R$ such that:
$$\text{if} \qquad x \leq b \qquad \text{then} \qquad F(x) \leq b.$$
Since the properties of a,b are not changed by addition of an integer we can now define $I = [a;b]$ where $b - a > 1$.
* Let $deg(f) = -1$. Let F be a lifting of f. We know that F has a fixed point, $deg(f^2) = 1$ and F^2 is the lifting of f^2. Let J be an interval for function F^2 which contains some fixed point of F and $F^2(J) \subset J$ as we found it in the previous case. Now we can define $I = J \cup F(J)$. ∎

Lemma 2.4. *Let $f \in C(S,S)$, F be a lifting of f and $I \subset R$ be the closed bounded interval greater than one such that $F \setminus I \in C(I,I)$. Then the following conditions hold:*

(i) *f has an infinite attractor containing two f-nonseparable points if and only if $F \setminus I$ has an infinite attractor containing two $F \setminus I$-nonseparable points;*
(ii) *if f has a nonempty δ-scrambled set then $F \setminus I$ has a nonempty δ-scrambled set;*
(iii) *if f has two-point scrambled set then $F \setminus I$ has two-point scrambled set;*
(iv) *if for some $\delta_F > 0$ function $F \setminus I$ has a nonempty perfect δ_F-scrambled set then there is $\delta_f > 0$ such that f has a nonempty perfect δ_f-scrambled set.*

PROOF : It is easy to see:

$$\omega(f, \Pi(x)) = \Pi(\omega(F,x)) \qquad \text{for all } x \in R \tag{4}$$
$$Per(f) = \Pi(Per(F \setminus I)) \tag{5}$$
$$P(f) = P(F \setminus I) \tag{6}$$
$$|deg(f)| < 1 \tag{7}$$

(i) Let us check the following possibilities:

Let $f \in W_0$. This is impossible (see (6)).

Let $f \in W_1$. Then $\omega(F,x)$ is finite for all $x \in R$ (see [11]) and also $\omega(f,y)$ is finite for all $y \in S$ (see (4)).

Let $f \in W_3$. In this case functions $f, F\setminus I$ have infinite attractors containing two nonseparable points (see Remark 2.1, Lemma 2.2 and [5]).

Thus the only interesting case is $f \in W_2$.

$$\text{Let} \quad x \in I \quad \text{and} \quad a, b \in \omega(F,x),\ a \neq b.$$

If a, b are $F \setminus I$-separable then either a, b are periodic points and (see (5)) $\Pi(a), \Pi(b)$ are also periodic and they are f-separable (if $\Pi(a) = \Pi(b)$ then they are not two f-nonseparable points), or there exist a closed periodic interval $J \subset I$ and $n \in N$ such that $a \in J$, $b \in F^n(J)$ and $J \cap F^n(J) = \emptyset$. But $\Pi(J)$ is also periodic and $\Pi(J) \cap f^n(\Pi(J)) = \emptyset$ and so we have that $\Pi(a), \Pi(b)$ are f-separable.

If a, b are F-nonseparable then there is an interval $J \subset I$ such that $a, b \in J$ and $F^i(J) \cap F^j(J) = \emptyset$ for all $i \neq j$ (see [10]). Then $\Pi(a), \Pi(b) \in \Pi(J)$ and $f^i(\Pi(J)) \cap f^j(J) = \emptyset$ for all $i \neq j$. It follows that $\Pi(a) \neq \Pi(b)$ and they are f-nonseparable.

Now it suffices to consider (4) to finish the proof of (i).

(ii) Let $x \in I$ and $\{\Pi(x)\}$ be a δ-scrambled set for f. Then (5) implies that $\{x\}$ is a δ-scrambled set for $F \setminus I$.

(iii) Let $x, y \in I$ and $\{\Pi(x), \Pi(y)\}$ be a scrambled set for function f. Then there is an increasing sequence $\{n_i\}_{i=1}^{\infty}$, $n_i \in N$ such that $\lim_{i\to\infty} |F^{n_i}(x) - F^{n_i}(y)| = k$ for some $k \in Z$ (see (2)).

Let us discuss the three cases (see (7)).

Let $deg(f) = 0$. Then $\lim_{i\to\infty} |F^{n_i+1}(x) - F^{n_i+1}(y)| = 0$ and now it is easy to see that $\{x; y\}$ is the scrambled set for $F \setminus I$.

Let $deg(f) = 1$. Then because I is greater than one there is $m \in N$ and $k_1, k_2 \in Z$ such that:

$$\begin{aligned} &F^m(x+k_1), F^m(y+k_2) \in I \qquad \text{and} \\ &\lim_{i\to\infty} |F^{n_i}(x+k_1) - F^{n_i}(y+k_2)| = 0. \end{aligned}$$

Then $\{F^m(x+k_1), F^m(y+k_2)\}$ is the scrambled set for function $F \setminus I$.

Let $deg(f) = -1$. Then $deg(f^2) = 1$ and $\{\Pi(x), \Pi(y)\}$ is the scrambled set for f^2 and F^2 is the lifting of f^2 and we are in the previous case. Now it suffices to consider Remark 2.1 to finish the proof of (iii).

(iv) Let $D \subset I$ be a nonempty perfect δ_F-scrambled set for function $F\setminus I$ ($\delta_F > 0$). Because $F \setminus I$ is continuous in the compact interval I we can show that:

* there exists $\delta_1 > 0$ such that for any $x, y \in D$, $x \neq y$ there is a limit point of the sequence $\{\|F^n(x), F^n(y)\|\}_{n=1}^{\infty}$ in the open interval $(\delta_1; 1-\delta_1)$;
* there exists $\delta_2 > 0$ such that for any $x \in D$, $p \in Per(F \setminus I)$ there is a limit point of $\{\|F^n(x), F^n(p)\|\}_{n=1}^{\infty}$ in the open interval $(k+\delta_2; k+1-\delta_2)$ for some $k \in Z$.

Now we can take $\delta_f = \min\{\delta_1; \delta_2\}$ and $\Pi(D)$ is the nonempty perfect δ_f-scrambled set for f. ■

PROOF of Theorem A: We can assume that $f \in C(S,S)$ (see Remark 1.1).

If $f \in W_1 \cup W_2$ then Theorem A follows from Remarks 1.1, 2.1 and Lemmas 2.3, 2.4.

If $f \in W_3$ then Theorem A follows from Remark 2.1 and Lemma 2.2. ■

Corollary 2.5. *The set W_1 contains only functions which have only trajectories which are convergent to periodic orbits.*

The set W_2 can be decomposed into two disjoint subsets:
Chaotic functions and functions that have only trajectories approximated by cycles.

The set W_3 contains only chaotic functions.

3. Maps without periodic points

Let $f \in C(S,S)$, $deg(f) = 1$ and F be a lifting of f. We shall call a set $A \subset S$ a **mat** set of f (minimal almost twist) if A is non-empty, closed, invariant ($f(A) = A$), minimal (for every $x \in A$; $\omega(f,x) = A$), almost twist ($F \setminus \Pi^{-1}(A)$ is nondecreasing) and there is $\rho(A) \in R$ such that $\rho(F,y) = \rho(A)$ for every $y \in \Pi^{-1}(A)$.

Remark 3.1. It is easy to see that if $f \in C(M,M)$ and $Per(f) = \emptyset$ then $f \in C(S,S)$ and $deg(f) = 1$.

Lemma 3.2. (Misiurewicz [9]). *Let $f \in C(S,S)$, $deg(f) = 1$ and F be a lifting of f. Then for every $a \in \rho(F)$ there exists $A \subset S$ a mat set of f such that $\rho(A) = a$.*

Lemma 3.3. (Misiurewicz [9]). *Let $f \in C(S,S)$, $deg(f) = 1$ and A be a mat set of f. Then:*
if $\rho(A)$ is rational then A is a finite set;
if $\rho(A)$ is irrational then:

(i) *either $A = S$ or A is homeomorphic to the Cantor set;*
(ii) *if $x \in A$ is an endpoint of an interval disjoint from A, then there exists a unique $y \in A$ with $f(y) = x$; this y is also an endpoint of an interval disjoint from A;*
(iii) *if $x \in A$, then either there is a unique $y \in A$ with $f(y) = x$, or there are two such points; in this case they are the endpoints of some interval disjoint from A.*

Corollary 3.4. *Let $f \in C(S,S)$, $deg(f) = 1$, F be a lifting of f and $A \subset S$ be a mat set of f. Then for all $x, y \in \Pi^{-1}(A)$ it holds:*

(i) *if $(x,y) \cap \Pi^{-1}(A) = \emptyset$ then $(F(x), F(y)) \cap \Pi^{-1}(A) = \emptyset$*
(ii) $[F(x), F(y)] \subset F([x,y])$
(iii) *if $A = S$ then F is increasing*

Lemma 3.5. *Let $f \in C(S,S)$, $deg(f) = 1$, F be a lifting of f and $A \subset S$ be a mat set of f homeomorphic to the Cantor set. Let, for all $x, y \in \Pi^{-1}(A)$, hold:*

$$\textit{if} \quad (x,y) \cap \Pi^{-1}(A) = \emptyset, \qquad \textit{then } F([x,y]) = [F(x), F(y)].$$

Then there is a semi-homeomorphism $h \in C(S,S)$ such that:

(i) *$h(x) = h(y)$ if and only if there is an interval $J \subset S$ such that $int(J) \cap A = \emptyset$ and $x, y \in J$;*

(ii) *$h \circ f = g \circ h$ where $g \in C(S,S)$ such that $deg(g) = 1$ and S is the mat set of function g.*

PROOF : Function h is a well known function of Cantor's type such that $h(A) = S$ and $deg(h) = 1$. ■

Lemma 3.6. *Let $f \in C(S,S)$, $deg(f) = 1$ and S be a mat set of f. Then f is topologically conjugated to the irrational rotation of circle.*

PROOF : Let F be a lifting of f. Then $\rho(F) = \{a\}$ where a is irrational (see Lemma 3.3) and there is $x \in R$ such that $F(x) = x + a$. We can assume that $F(k) = k + a$ for all $k \in Z$. Now we define:

$$H(F^n(k)) = na + k \qquad \text{for all } n, k \in Z \quad (F \text{ is homeomorphism}).$$

$$\begin{array}{lr}
\text{Let } \ n_1, n_2, z, r \in Z \quad \text{and} & F^{n_1}(z) > F^{n_2}(r) \\
\text{then } (deg(f) = 1) & F^{n_1 - n_2}(0) > r - z \\
\text{then } (F \text{ is increasing}) & F^{k(n_1 - n_2)}(0) > k(r - z) \\
\text{then } (k \text{ goes to infinity}) & \rho(F, 0)(n_1 - n_2) \geq r - z \\
\text{then } (\rho(F, 0) = a \text{ is irrational}) & an_1 + z > an_2 + r.
\end{array}$$

So we have that function H is increasing on the set $D \subset R$ which is dense in R (S is the mat set) and $H(D)$ is dense in R (a is irrational). Function H can be extended to a function from $C(R,R)$ and so we have $H \circ F = G \circ H$ where $G(x) = x + a$ for $x \in R$. Here H, G are the liftings of h, g respectively, where h is a homeomorphism, g is irrational rotation of circle and $h \circ f = g \circ h$. ■

Lemma 3.7. *Let $f \in W_0$, F be a lifting of f and $A \subset S$ be a mat set of f. Then for all $x, y \in \Pi^{-1}(A)$ it holds:*

$$\textit{If} \quad (x,y) \cap \Pi^{-1}(A) = \emptyset \qquad \textit{then} \quad F([x,y]) = [F(x), F(y)]$$

PROOF : Let $f_* \in C(S,S)$ be a function such that if F_* is a lifting of f_* then for all $x, y \in \Pi^{-1}(A)$ it holds:

(i) $F_*(x) = F(x)$;

(ii) if $(x,y) \cap \Pi^{-1}(A) = \emptyset$ then $F_*([x,y]) = [F_*(x), F_*(y)]$.

Since $\rho(A)$ is irrational ($f \in W_0$), from Lemmas 3.3, 3.5, 3.6 we obtain that for every closed intervals $J, K \subset S$ such that,

$$int(J) \cap A = \emptyset \quad \text{and} \quad int(K) \cap A \neq \emptyset$$

there is such $n \in N$ that $J \subset f_*^n(K)$ (A is the mat set of f_*).

If there is a closed interval $J \subset S$ such that:

$$int(J) \cap A = \emptyset \quad \text{and} \quad int(f(J)) \cap A \neq \emptyset,$$

then there is $n \in N$ such that $J \subset f_*^n(f(J))$ and from here we obtain that $J \subset f^{n+1}(J)$ which is impossible because $Per(f) = \emptyset$.

So we have $int(f(J)) \cap A = \emptyset$. ■

PROOF of Theorem B: From Lemmas 3.2, 3.5, 3.6, 3.7 it follows that f is topologically semi-conjugated to the irrational rotation of the circle. Let A be a mat set of f. The semihomeorphism h has the properties from Lemma 3.5 and $h \circ f = g \circ h$ where g is the rotation of the circle. Now we assume that

$$\liminf_{n\to\infty} \|f^n(x), f^n(y)\| = 0 \qquad \text{for some } x, y \in S.$$

So we have $\liminf_{n\to\infty} \|h(f^n(x)), h(f^n(y))\| = 0$, $\liminf_{n\to\infty} \|g^n(h(x)), g^n(h(y))\| = 0$ and because g is the rotation of the circle we have $h(x) = h(y)$. From here we obtain that there is an interval $J \subset S$ such that $x, y \in J$ and $int(J) \cap A = \emptyset$. From Lemma 3.7 we obtain that $int(J_n) \cap A = \emptyset$ $(J_n = f^n(J))$ for every $n \in N$. Because $f^n(J) \cap f^m(J) = \emptyset$ for $n \neq m$ $(Per(f) = \emptyset)$ we have:

$$\limsup_{n\to\infty} \|f^n(x), f^n(y)\| = 0.$$

Now we are completely done. ■

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