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Abstract. It is proved that Wiener's test of regularity provides a test for thinness of arbitrary sets. The result which is obtained in the context of harmonic spaces can be applied to a wide class of second order partial differential equations of elliptic or parabolic types.

Keywords: Wiener's test, regularity, thinness, capacity

Classification: 31D05, 35J25, 35K20

INTRODUCTION

Let $(X, \mathcal{H})$ be a $\mathcal{P}$–harmonic space with countable base such that points of X are polar; for definitions, see e.g. [**C–C**],[**Ba**]. Let $\hat{R}_u^E$ stand for the balayage of a hyperharmonic function u on X on a subset E of X. A subset E of X is said to be thin at a point $z \in X$ if

$$\hat{R}_p^E(z) < p(z)$$

for some strict potential p on X.

In this note we present a Wiener type test of thinness, if a suitable Wiener test for regularity is known.

We shall adopt notations of [**B–H2**].

SEMICAPACITY AND THINNESS

Let us denote by $\mathcal{P}(X)$ the collection of all subsets of X.

Definition. A set function $\gamma : \mathcal{P}(X) \to [0, \infty]$ is called semicapacity on X if the following conditions hold:

(i) $\gamma(A) \leq \gamma(B)$, whenever $A, B \in \mathcal{P}(X)$, $A \subset B$;
(ii) $\gamma(B) = \sup\{\gamma(K); K \subset B, K \text{ compact}\}$, whenever B is a Borel subset of X;
(iii) $\gamma(M) = \inf\{\gamma(U); M \subset U, U \text{ open}\}$, whenever M is a subset of X.

Remark. If c is a Choquet capacity on X, then the corresponding outer capacity c^* is a semicapacity on X; for definitions, see e.g. [**He**], [**Br1**].

Lemma 1. *Let E be a subset of X, let $(A_n)_{n=1}^{\infty}$ be a sequence of Borel subsets of X and let γ be a semicapacity on X. Then there exists a Borel subset B of X such that*

$$\gamma(E \cap A_n) = \gamma(B \cap A_n)$$

for every $n \in \mathbb{N}$.

PROOF : Proof follows [**Ha**]. For $n, k \in \mathbb{N}$ let $U_{n,k} \subset X$ be open sets such that $E \cap A_n \subset U_{n,k}$ and

$$\gamma(E \cap A_n) = \inf\{\gamma(U_{n,k}); k \in \mathbb{N}\}.$$

Let $B_n = \bigcap_{k=1}^{\infty} U_{n,k}$. Then B_n is a Borel set, $E \cap A_n \subset B_n$ and

$$(*) \qquad \gamma(E \cap A_n) = \gamma(B_n).$$

Let $B = \bigcap_{n=1}^{\infty}(B_n \cup (X \setminus A_n))$. Clearly, B is a Borel set and $E \subset B$. Consequently, $B \cap A_n \subset B_n$. Further,

$$\gamma(E \cap A_n) \leq \gamma(B \cap A_n) \leq \gamma(B_n).$$

In view of $(*)$, the assertion follows. ∎

Lemma 2. *Let A be a subset of X. Then there exists a G_δ set $A' \supset A$ such that*

$$\hat{R}_u^A = \hat{R}_u^{A'}$$

for every $u \in \mathcal{H}_+^*(X)$.

PROOF : See [**B–H2**], p.250. ∎

Lemma 3. *Let E be a subset of X, $z \in X$ and let E be thin at the point z. Then there exists a Borel subset B of X such that $E \subset B$ and B is thin at z.*

PROOF : By Lemma 2, there is a G_δ set $B \supset E$ such that

$$\hat{R}_u^E = \hat{R}_u^B$$

for all $u \in \mathcal{H}_+^*(X)$, thus also for potentials; now we can apply the assertion from [**C–C**], p.150.

∎

Lemma 4. *For an arbitrary set $E \subset X$ and $z \in X$, the following conditions are equivalent:*

(i) *A is thin at z;*
(ii) *$A \setminus \{z\}$ is thin at z;*
(iii) *$A \cup \{z\}$ is thin at z.*

PROOF : See [**C–C**], p.152. ∎

Lemma 5. *Let B be a Borel set which is not thin at a point $z \in B$. Then there exists a compact subset K of B such that K is not thin at z.*

PROOF : This is a special case of Lemma 5.1 from [**B–H1**]. For the convenience of the reader, we present a direct proof. Let p be a strict potential and let $(V_n)_{n=1}^{\infty}$ be a sequence of relatively compact open sets such that

$$\overline{V}_{n+1} \subset V_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} V_n = \{z\}.$$

Consider $n \in \mathbb{N}$. Then

$$p(z) = \hat{R}_p^B(z) = \hat{R}_p^{B \cap V_n}(z).$$

By [B–H2], p.248, there exists a compact subset K_n of $B \cap V_n$ such that

$$\hat{R}_p^{K_n}(z) > p(z) - \frac{1}{n} \quad .$$

Take $K = \bigcup_{n=1}^{\infty} K_n \cup \{z\}$. Clearly K is a compact subset of B and

$$\hat{R}_p^K(z) = p(z),$$

i.e., K is not thin at the point z.

■

Notation. For $z \in X$, $r \in]0,1]$, let $A^r(z)$ denote a compact set in X such that:

(i) $A^r(z) \subset A^s(z)$ for $r < s$;

(ii) $\bigcap_{o<r\leq 1} A^r(z) = \{z\}$.

For $r = 2^{-n}$ write A_n instead of A^r.

Theorem. *Let $z \in X$, let E be an arbitrary subset of X and let γ be a semicapacity on X. Suppose that the following condition holds.*

(P) *There exists a sequence of positive numbers $(c_k(z))_{k=1}^{\infty}$ such that the following statements are equivalent, whenever $F \subset X$ is compact:*

(i) *F is thin at z;*

(ii) $\sum_{k=1}^{\infty} c_k(z)\gamma(F \cap A_k(z)) < \infty$.

Then E is thin at z, if and only if the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z))$$

is convergent.

PROOF : Let E be not thin at z. By Lemma 4, we can assume that $z \in E$. According to Lemma 1, there exists a Borel set $B \supset E$ such that

$$\gamma(E \cap A_n(z)) = \gamma(B \cap A_n(z))$$

holds for every $n \in \mathbb{N}$. Clearly, B is not thin at z. By Lemma 5, there exists a compact set $K \subset B$ such that K is not thin at z, so according to the condition (P) the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z))$$

is divergent. Since $K \cap A_k(z) \subset B \cap A_k(z)$, we have

$$\gamma(K \cap A_k(z)) \leq \gamma(B \cap A_k(z)) = \gamma(E \cap A_k(z)),$$

hence
$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z)) \leq \sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z)).$$
Consequently, the series on the right hand side is also divergent.

Let now E be thin at z. According to Lemma 3, there exists a Borel set $B \supset E$ such that B is thin at z. Choose a sequence of strictly positive numbers $(\varepsilon_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} c_k(z)\varepsilon_k < \infty$. Since the set $B \cap A_k(z)$ is a Borel set, there exists , for every $k \in \mathbb{N}$, a compact set $K_k \subset B \cap A_k(z)$ such that
$$\gamma(B \cap A_k(z)) \leq \gamma(K_k) + \varepsilon_k.$$
Clearly, the set $K = \bigcup_{k=1}^{\infty} K_k \cup \{z\}$ is compact and $K \subset B \cup \{z\}$, i.e., the set K is thin at z. By the condition (P), the series
$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z))$$
is convergent. Since $E \subset B$, it follows
$$\gamma(E \cap A_k(z)) \leq \gamma(B \cap A_k(z)) \leq \gamma(K_k) + \varepsilon_k \leq \gamma(K \cap A_k(z)) + \varepsilon_k.$$
Thus
$$\sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z)) \leq \sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z)) + \sum_{k=1}^{\infty} c_k(z)\varepsilon_k,$$
and the series on the left hand side is convergent because both series on the right hand side are convergent. ∎

Remarks

The Wiener test for regularity in classical potential theory (i.e. for Laplace operator), was proved in 1924 by N.Wiener, see [**W**]. In 1944, M.Brelot proved the Wiener test of thinness in this case, see [**Br2**]. The way to an analogous criterion in the heat case took more than 50 years. In 1982, a heat analogy of the Wiener test for regularity was established in [**E–G**]. The Wiener test of thinness in the heat case was proved in [**Brz**].

If we apply Theorem proved above we get directly the corresponding criterions of thinness in the classical as well as in the heat case, because in these situations the condition (P) is fulfilled (the condition (P) is, as a matter of fact, a reformulation of the criterion of regularity). Thus immediately we get the corresponding assertions from [**Br2**] and [**Brz**].

Theorem can also be applied to parabolic equations with variable coefficients considered in [**G–L**].

In $\mathbb{R}^{n+1}$, $n \geq 1$, we consider the second order operator
$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}) - \frac{\partial u}{\partial t},$$

where $(a_{i,j}(x,t))_{i,j=1,\dots,n}$ is real symmetric, matrix-valued function on $\mathbf{R}^{n+1}$ with C^∞ entries. We assume that there exists $\nu \in]0,1]$ such that, for every $\xi \in \mathbf{R}^n$ and every $(x,t) \in \mathbf{R}^{n+1}$,

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \le \nu^{-1}|\xi|^2.$$

Let $\mathcal{H}^L$ be the sheaf of all continuously differentiable (twice with respect to $x_1,\dots,x_n$ and once with respect to t) solutions of the differential equation $Lu = 0$. According to [**Ba**], p.61, $(\mathbf{R}^{n+1}, \mathcal{H}^L)$ is a $\mathcal{P}$–harmonic space. It is easy to see that the points are polar. The capacity $\mathrm{cap_L}$ is defined in a usual way, see e.g.[**G–L**], $\mathrm{cap}^*_\mathrm{L}$ denotes the outer capacity deduced from the $\mathrm{cap_L}$. Let $\Gamma(x,t;y,s)$ denote the fundamental solution of L. Let us denote (for a given $z = (x,t) \in \mathbf{R}^{n+1}$, $k \in \mathbf{N}$, and $\lambda \in]0,1[$)

$$A(x,t;\lambda^k) = \{(y,s) \in \mathbf{R}^{n+1}; \left(4\pi\lambda^{k+1}\right)^{-n/2} \ge \Gamma(x,t;y,s) \ge \left(4\pi\lambda^{k}\right)^{-n/2}\} \cup \{(x,t)\}.$$

The validity of the condition (P) with $c_k(x,t) = \lambda^{-kn/2}$, $\lambda \in]0,1[$, is proved in [**G–L**]. We have now:

Theorem. *Let E be a subset of $\mathbf{R}^{n+1}$, let $\lambda \in]0,1[$ and let $z = (x,t) \in \mathbf{R}^{n+1}$. Then E is L–thin at z, if and only if the series*

$$\sum_{k=1}^{\infty} \lambda^{-kn/2}\, \mathrm{cap}^*_\mathrm{L}(E \cap A(x,t;\lambda^k))$$

is divergent.

Similarly, Theorem can be applied to a wide class of degenerate operators considered in [**N–S**].

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