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Representations in varieties of regular involution bands

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Abstract. A unary operation $^+$ on a semigroup S is called a regular involution if the following identities $x^{++} = x$, $xx^+x = x$ hold. We prove that the varieties of left normal bands with a regular involution, of right normal bands with a regular involution, and rectangular bands with a regular involution are universal.

Keywords: varieties of bands, a regular involution, a universal category

Classification: 18B10, 20M07, 20M30

1. Introduction.

A category K is *universally representative*, (or shortly *universal*) if any category of algebras A can be fully embedded into K. In case of concrete universal category K it may be often required that any category of alegebras A may be fully embedded into K in such a way that all finite algebras in A are carried to K-objects with finite support. In this case K is termed *finite–to–finite universal* . If we require slightly less namely than universality, that all one–object categories (i.e. monoids) be fully embeddable (i.e. representable as monoid endomorphism of suitable objects) in K, then K is said to be *monoid universal*. The universal representativeness of a category is directly related to those structural properties of its objects which enable the objects to control the morphism so as to represent any category of algebras. For this reason, much attention has been paid to varieties of algebras. Universality of a variety is an increasing property which can be lost when going down in the lattice of subvarieties. For example, the universal varieties of semigroups have been characterized [6] as follows: A variety $\underset{\sim}{V}$ of semigroups is universal if and only if contains all commutative semigroups and for no $n > 1$ the power law $x^n y^n = (xy)^n$ holds in $\underset{\sim}{V}$. By this characterization it is straightforward that no variety $\underset{\sim}{V}$ of *bands* (i.e. semigroups satisfying $x^2 = x$) is universal, the property is lost. However, and this is a motivation of our present work, we can expend the $\underset{\sim}{V}$-band structure by adding new operations so as to obtain again a universal variety, this time, however, in a different, extended type.

Some results have been obtained on expansion of bands by adding a small number of nullary operations [3]. A variety $\underset{\sim}{V}$ of bands has a universal expansion by nullary operations if and only if $\underset{\sim}{V}$ contains all left normal bands or all right normal bands – in this case it suffices to add three nullary operations. Further a variety $\underset{\sim}{V}$ of bands has a universal expansion by two nullary operations if and only if $\underset{\sim}{V}$ contains all semilattices of left zero–semigroups or all semilattices of right zero–semigroups.

Another possibility of a "moderate" expansions of bands is to add one unary operation. In this case the added operation $x \to x^+$ can even be tied to the band structure by the identities

(RI) $$x^{++} = x \text{ and } x = xx^+x\,,$$

qualifying the operation as what we call a *regular involution*, and the variety we thus obtain – the variety $RI\underset{\sim}{V}$ of regular involution $\underset{\sim}{V}$-bands – may still be universal.

The universality of the variety RIB with B the variety of all bands was proved in [**2**]. In this paper we want to give simple characterization of the universal varieties $RI\underset{\sim}{V}$ with $\underset{\sim}{V}$ a variety of bands.

We shall use the following abbreviations as names of the band varieties:

LZB – left zero–semigroups $[xy = x]$,
RZB – right zero–semigroups $[yx = x]$,
SL – semilattices $[xy = yx]$,
LNB – left normal bands $[xyz = xzy]$,
RNB – right normal bands $[yzx = zyx]$,
RB – rectangular bands $[xyx = x]$.

The result can be stated as follows:

Theorem 1.1. *A variety $RI\underset{\sim}{V}$ of regular involution $\underset{\sim}{V}$-bands is universal if and only if $\underset{\sim}{V}$ is a non–trivial variety distinct from each one of the atomic band varieties LZB, SL, or RZB.*

If we use facts concerning semigroups without any references then they are contained in [**1**].

2. Left normal bands.

We recall that the variety LNB of left normal bands is generated by the left zero–semigroups and the semilattices. The aim of this section is to prove that the expansion $RILNB$ of LNB by a regular involution is universal. For this reason we shall construct a full embedding from the variety $I(1,1)$ of algebras with two unary idempotent operations into $RILNB$. Since $I(1,1)$ is universal, see [**8**] or [**9**], the proof that $RILNB$ is universal will be complete.

First denote by L the meet semilattice over the set $\{a_i; i \in 7\}$ with $a_0 < a_1, a_2, a_3 < a_4 < a_5 < a_6$ and a_1, a_2, a_3 pairwise uncomparable. Assume that a unary algebra $X = (X, \varphi, \psi) \in I(1,1)$ is given. Let $\Psi' X$ be the subsemigroup of the product of L with the left zero–semigroup on the set $X \times 12 = \{(x,i); x \in X, i \in 12\}$ generated by the set $\{((x,2j),a_5); x \in X, j \in 6\} \cup \{((x,j),a_i); x \in X, j \in 12, i \in 5\} \cup \{((x,j),a_6); x \in X, j \in \{0,2\}\}$. Denote by $D'\colon \Psi' X \longrightarrow L$ the restriction of the projection. Then D' induces a decomposition of $\Psi' X$ into $\mathcal{D}$-classes. Let $\sim$ be the smallest congruence on $\Psi' X$ satisfying:

$((\varphi(x), 2i), a_1) \sim ((x, 2i+1), a_1)$ for every $x \in X$, $i \in 6$
$((\psi(x), 2i), a_2) \sim ((x, 2i+1), a_2)$ for every $x \in X$, $i \in 6$
$((x, 2i), a_3) \sim ((x, 2i+1), a_3)$ for every $x \in X$, $i \in 6$
$((x,j), a_0) \sim ((y,k), a_0)$ for every $x, y \in X$, $j, k \in 12$ such that there exists $i \in 6$ with $j, k \in \{2i, 2i+1\}$.

Set $\Psi X = \Psi' X/\sim$. Note that for every $i \in 12$, $x \in X$, and $j \in \{4,5,6\}$, the class of $\sim$ containing $((x,i),a_j)$ is a singleton, and, for every even $i \in 12$, $x \in X$, and $j \in \{1,2,3\}$, if $((y,i'),a_{j'}) \sim ((x,i),a_j)$ then $j = j'$ and either $(y,i') = (x,i)$ or $i' = i+1$. Whence, for every $x \in X$, $i \in 12$, $j \in \{1,2,3,4,5,6\}$ such that $((x,i),a_j) \in \Psi' X$ and either $j > 3$ or i is even we will denote the class of $\sim$ containing $((x,i),a_j)$ as (x,i,a_j). Further, for every even $i \in 12$, $x \in X$, $\{(y,i'),a_j); ((y,i'),a_j) \sim ((x,i),a_0)\} = \{((y,i'),a_0); y \in X, \ i' = i \text{ or } i' = i+1\}$, thus the class of $\sim$ containing $((x,i),a_0)$ will be denoted as (i,a_0). Note that $\Psi X = \{(i,a_0); i \in 12 \text{ is even}\} \cup \{(x,i,a_j); x \in X, \ i \in 12 \text{ is even}, j \in \{1,2,3,5\}\} \cup \{(x,i,a_4); x \in X, \ i \in 12\} \cup \{(x,i,a_6); x \in X, \ i \in \{0,2\}\}$.

The congruence $\sim$ is a refinement of the Green congruence $\mathcal{D}$ on $\Psi' X$ hence we can define a surjective homomorphism $D_X \colon \Psi X \longrightarrow L, D_X(x,i,a_j) = a_j$ for every $x \in X$, $i \in 12$, $j \in \{1,2,3,4,5,6\}$, $(x,i,a_j) \in \Psi X$, $D_X(i,a_0) = a_0$ for even $i \in 12$ inducing the decomposition of ΨX into $\mathcal{D}$-classes.

For every $j \in 7$ let μ_j be a bijection of the set 12 into itself without any fixed points such that μ_j^2 is the identity and

$$
\begin{array}{lll}
\mu_0(0) = 4\,, & \mu_0(2) = 6\,, & \mu_0(8) = 10\,,\\
\mu_1(0) = 2\,, & \mu_1(4) = 8\,, & \mu_1(6) = 10\,,\\
\mu_2(0) = 10\,, & \mu_2(2) = 4\,, & \mu_2(6) = 8\,,\\
\mu_3(0) = 8\,, & \mu_3(2) = 10\,, & \mu_3(4) = 6\,,\\
\mu_4(2i) = 2i+1 \quad \text{for every } i \in 6\,, & & \\
\mu_5(0) = 6\,, & \mu_5(2) = 8\,, & \mu_5(4) = 10\,,\\
\mu_6(0) = 2\,. & &
\end{array}
$$

Define a unary operation $^+$ on ΨX such that $(i,a_0)^+ = (\mu_0(i),a_0)$ for every $(i,a_0) \in \Psi X$, $(x,i,a_j) = (x,\mu_j(i),a_j)$ for every $x \in X$, $i \in 12$, $j \in 7$ such that $(x,i,a_j) \in \Psi X$. By a direct inspection we obtain that $^+$ is correctly defined and $^+$ is an involutory mapping preserving the $\mathcal{D}$-classes of ΨX. Since a unary operation α is a regular involution of a band S (i.e. it satisfies (RI)) if and only if α is an involutory mapping preserving the $\mathcal{D}$-classes of S, we conclude that ΨX with $^+$ belongs to the variety $RILBN$ being the expansion of LNB by a unary operation which is a regular involution.

For every semigroup S and every element $s \in S$ define $I(s) = \{z \in S; zs = s\}$. Then obviously, for every semigroup homomorphism $f \colon S \longrightarrow S'$ and every $s \in S$ we have $f(I(s)) \subseteqq I(f(s))$. The following lemma describes the basic properties of $I(s)$ for the semigroup $\Psi(X)$.

Lemma 2.1. *For every $X \in I(1,1)$ we have:*

a) $I(2i,a_0) = \{(2i,a_0)\} \cup \{(x,2i,a_j) \in \Psi X; \ j \in 7, \ x \in X\} \cup \{(x,2i+1,a_4); \ x \in X\}$ *for every* $i \in 6$;

b) $I(x,i,a_j) = \{(x,i,a_k) \in \Psi X; \ k \in 7$ *and either* $k = j$ *or* $k \geq 4\} \cup \{(y,i+1,a_4); \eta(y) = x\}$ *for every* $x \in X$, $i \in 12$, i *is even*, $j \in \{1,2,3\}$ *where* $\eta = \varphi$ *if* $j = 1$, $\eta = \psi$ *if* $j = 2$, η *is the identity if* $j = 3$;

c) $I(x,i,a_j) = \{(x,i,a_k) \in \Psi X;\quad k \ge j\}$ *for every* $x \in X$, $i \in 12$ *which is even*, $j \in 7$, $j \ge 4$;
d) $I(x,i,a_4) = \{(x,i,a_4)\}$ *for every* $x \in X$, $i \in 12$ *which is odd;*
e) *for every* $i \in 6$, $\{z \in I(2i,a_0);\quad z^+ \in I(2i,a_0)\} = I(2i,a_0) \cap D_X^{-1}(a_4)$.

Proof is straightforward.

Denote by ΦX the semigroup ΨX with the unary operation $^+$. For a homomorphism $f\colon X_0 \longrightarrow X_1$ where $X_i = (X_i,\varphi,\psi) \in I(1,1)$ for $i \in 2$ define $\Phi f\colon \Phi X_0 \longrightarrow \Phi X_1$ such that $\Phi f(i,a_0) = (i,a_0)$ for every $(i,a_0) \in \Phi X_0$, $\Phi f(x,i,a_j) = (f(x),i,a_j)$ for every $x \in X_0$, $i \in 12$, $j \in 7$ with $(x,i,a_j) \in \Phi X_0$. We check easily that ΦX is correctly defined. We can summarize:

Proposition 2.2. *Φ is an embedding from $I(1,1)$ into $RILNB$.*

To prove that Φ is a full embedding consider a homomorphism $f\colon \Phi X_0 \longrightarrow \Phi X_1$ where $X_i = (X_i,\varphi,\psi) \in I(1,1)$ for $i \in 2$. Since every homomorphism preserves the $\mathcal{D}$-classes there exists an endomorphism $g\colon L \longrightarrow L$ with $D_{X_1} \circ f = g \circ D_{X_0}$.

Lemma 2.3. *The restriction $g \restriction \{a_1,a_2,a_3\}$ is one–to–one.*

PROOF : Assume that $g(a_i) = g(a_j)$ for distinct $i,j \in \{1,2,3\}$. Since g is a semilattice homomorphism we conclude that $g(a_0) = g(a_i)$. Assume that $f(0,a_0) = z$, $D(z) = g(a_0)$ and $f(x,0,a_j) = z$ for every $x \in X_0$ and $j \in \{1,2,3\}$ with $g(a_j) = g(a_0)$. Since f preserves $^+$ we observe that $f(4,a_0) = z^+$ and $f(x,4,a_j) = z^+$ for every $x \in X_0$ and $j \in \{1,2,3\}$ with $g(a_j) = g(a_0)$. Assume that $g(a_1) = g(a_0)$ then for every $x \in X_0$ we have $f(x,0,a_1) = z$ and $f(x,4,a_1) = z^+$ whence $f(x,2,a_1) = z^+$ and $f(x,8,a_1) = z$ because f preserves $^+$. Thus $f(2,a_0) = z^+$, $f(8,a_0) = z$. If $g(a_j) = g(a_0)$ for $j \in \{2,3\}$ then we obtain $f(x,0,a_j) = z^+$ and hence in both cases $f(0,a_0) = z^+$. Since $^+$ has not any fixed point we obtain a contradiction. Thus $g(a_1) \ne g(a_0)$ and hence $g(a_2) = g(a_3) = g(a_0)$. Then $f(x,4,a_2) = f(x,4,a_3) = z^+$, for every $x \in X_0$. Whence $f(x,2,a_2) = z = f(x,6,a_3)$ and thus $f(2,a_0) = f(6,a_0) = z$ this contradicts to $(2,a_0)^+ = (6,a_0)$. Therefore $g \restriction \{a_1,a_2,a_3\}$ is one–to–one. ∎

Corollary 2.4. *$g(a_0) = a_0$, $g(\{a_1,a_2,a_3\}) \subseteq \{a_j; j \in 4\}$, $g(a_4) = a_4$ and $g(a_5) \ge a_5$.*

PROOF : If $g(a_0) \ne a_0$ then by the definition of L there exist distinct $i,j \in \{1,2,3\}$ such that $g(a_i) = g(a_j) = g(a_0)$ – a contradiction with Lemma 2.3. If $g(a_i) = a_j$ for $i \in \{1,2,3\}$, $j \ge 4$, then either $g(a_0) \ne a_0$ or $g(a_k) = a_0$ for every $k \in \{1,2,3\} \setminus \{i\}$ – again a contradiction. Furthermore by e) of Lemma 2.1 we obtain that $g(a_4) = a_4$. If $g(a_5) = a_4$ then for every $x \in X_0$ $f(x,6,a_5) \in I(f(0,a_0))$, $f(x,8,a_5)^+ \in I(f(2,a_0))$, $f(x,10,a_5)^+ \in I(f(4,a_0))$ because f preserves $^+$. Thus $f(0,a_0) = f(6,a_0)$, $f(2,a_0) = f(8,a_0)$, $f(4,a_0) = f(10,a_0)$, and thus $f(2,a_0) = f(4,a_0)$ because $(6,a_0)^+ = (2,a_0)$ – this is a contradiction because $f(8,a_0) = f(10,a_0)$ but $(8,a_0)^+ = (10,a_0)$. Therefore $g(a_5) \ge a_5$. ∎

Lemma 2.5. *For every $i \in 6$, $f(2i,a_0) = (2i,a)$. Moreover g is an identity.*

PROOF : Set $A = \{2i;\quad i \in 6\}$ and define $h\colon A \longrightarrow A$ such that for every $i \in A$, $f(i,a_0) = (h(i),a_0)$. Since f preserves the operation $^+$ we obtain for every $i \in 7$, if

$g(a_i) = a_j$ then $h \circ \mu_i = \mu_j \circ h$. Further for every $j \in A$ there exists a permutation μ in the permutation group generated by μ_0 and μ_1 with $\mu(0) = j$. According to Corollary 2.4 we have $g(a_0) = a_0$, whence h is uniquely determined by the values $h(0)$ and $g(a_1)$. Therefore we say that f satisfies condition (i, j) if $f(0, a_0) = (i, a_0)$ and $g(a_1) = a_j$. We conclude that if f satisfies (i, j) then h has to be of the form given in the following table:

	(0,0)	(0,1)	(0,2)	(0,3)	(2,0)	(2,1)	(2,2)	(2,3)	(4,0)	(4,1)
0	0	0	0	0	2	2	2	2	4	4
2	4	2	10	8	6	0	4	10	0	8
4	4	4	4	4	6	6	6	6	0	0
6	0	6	8	10	2	4	0	8	4	10
8	0	8	2	6	2	10	8	4	4	2
10	4	10	6	2	6	8	10	0	0	6

	(4,2)	(4,3)	(6,0)	(6,1)	(6,2)	(6,3)	(8,0)	(8,1)	(8,2)	(8,3)
0	4	4	6	6	6	6	8	8	8	8
2	2	6	2	10	8	4	10	4	6	0
4	0	0	2	2	2	2	10	10	10	10
6	6	2	6	8	10	0	8	0	2	4
8	10	8	6	0	4	10	8	6	0	2
10	8	10	2	4	0	8	10	2	4	6

	(10,0)	(10,1)	(10,2)	(10,3)
0	10	10	10	10
2	8	6	0	2
4	8	8	8	8
6	10	2	4	6
8	10	4	6	0
10	8	0	2	4

If $g(a_1) = a_0$ then $h(0) = h(8)$ and since $\mu_3(0) = 8$ we obtain that μ_j has a fixed point where j is determined by $g(a_3) = a_j$ (by Corollary 2.4, $j \leq 3$) – this is a contradiction with the definition of μ_i. Hence $g(a_1) \neq a_0$ and h is a bijection.

If $g(a_6) = g(a_5) = a_5$ then h must preserve the sets $\{0, 6\}$, $\{2, 8\}$, $\{4, 10\}$ and it must map the set $\{0, 2\}$ into one of these sets, thus h is not one–to–one, this is a contradiction. Analogously if $g(a_5) = g(a_6) = a_6$ then we obtain that h is not one–to–one. By Corollary 2.4 we conclude that $g(a_6) = a_6$, $g(a_5) = a_5$. As a consequence we obtain $h(\{0, 2\}) = \{0, 2\}$. Since $\mu_1(0) = 2$ but $\mu_j(0) \neq 2$ for every $j \in \{2, 3\}$ we prove by Corollary 2.4 that $g(a_1) = a_1$. Thus f can satisfy only (0,1) or (2,1). If f satisfies (2,1) then $h(\{0, 10\}) = \{2, 8\}$. Since $\mu_2(0) = 10$ we obtain that $h \circ \mu_2 = \mu_j \circ h$ implies $j = 5$ – this is a contradiction with Corollary 2.4. Hence h is an identity and thus $f(i, a_0) = (i, a_0)$ for every $i \in A$. Now the identity $h \circ \mu_i = \mu_j \circ h$ whenever $g(a_i) = a_j$ implies that g is an identity. ∎

Lemma 2.6. *There exists a mapping $k\colon X_0 \longrightarrow X_1$ such that for every $(x,j,a_k) \in \Psi X_0$ we have $f(x,j,a_k) = (k(x),j,a_k)$.*

PROOF : By Lemmas 2.1 and 2.5 if $(x,j,a_k) \in \Psi X_0$ and j is even then there exists $y \in X_1$ such that $f(x,j,a_k) = (y,j,a_k)$ whenever $k \neq 4$ and $f(x,j,a_4) = (y,j',a_4)$ where $j' \in \{j,j+1\}$. Since $I(x,2i,a_4) = \{(x,2i,a_k);\quad k \in \{4,5,6\}\}$ and $I(x,2i+1,a_4) = \{(x,2i+1,a_4)\}$ for every $x \in X_k$, $k \in 2$, $i \in 6$ we conclude that $f(x,j,a_4) \in (X_1 \times \{j,a_4\})$ for every $x \in X_0$, $j \in 12$.

Let $k\colon X_0 \longrightarrow X_1$ be a mapping with $f(x,0,a_6) = (k(x),0,a_6)$ for every $x \in X_0$. Then by Lemma 2.1 we obtain that $f(x,0,a_i) = (k(x),0,a_i)$ for every $i \in 7$, $i \neq 0$. Since f preserves $^+$ we obtain that $f(x,j,a_i) = (k(x),j,a_i)$ for any pairs (i,j) from the set $\{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(1,6),(2,6),(3,6),(4,6),(5,6),(1,8),(2,8),(3,8),(4,8),(5,8),(1,4),(1,10),(2,4),(2,10),(3,4),(3,10),(4,1),(4,3),(4,7),(4,9)\}$. If we again apply Lemma 2.1 we obtain that $f(x,j,a_k) = (k(x),j,a_k)$ for every pair (i,j) such $(x,i,a_j) \in \Psi X_0$. ∎

Theorem 2.7. *The variety RILNB is universal and Φ is a full embedding.*

PROOF : We prove that Φ is full embedding. By Lemmas 2.5, 2.6 and 2.7 it suffices to show that k is a homomorphism from X_0 to X_1. By the definition of $\sim$ we have $I(x,0,a_1) \cap \{(y,1,a_4);\quad y \in X_k\} = \{(y,1,a_4);\quad y \in X_k,\quad \varphi(y) = x\}$ for $k \in 2$. Whence k commutes with φ because φ is idempotent. Since $I(x,2,a_2) \cap \{(y,3,a_4);\quad y \in X_k\} = \{(y,3,a_4);\quad y \in X_k,\quad x = \psi(y)\}$ we obtain that k commutes also with ψ and the proof is complete. ∎

Corollary 2.8. *The variety RILNB is finite–to–finite universal.*

PROOF : The proof follows from the fact that $I(1,1)$ is finite–to–finite universal and Φ preserves finiteness. ∎

3. Rectangular bands.

The aim of this section is to prove that the variety *RIRB* of rectangular bands with an added unary operation $^+$ satisfying (*RI*) is universal. Since the variety *RB* satisfies the identity $x = xyx$ we obtain that for every unary operation $^+$ on a rectangular band the identity $xx^+x = x$ holds. Thus *RIRB* is given by the identities of rectangular bands with the identity $x^{++} = x$.

The proof of universality of *RIRB* is divided into two steps. We first define rectangular bands with a partial involution and their homomorphisms. We prove that the category of rectangular bands with a partial involution and their homomorphisms is universal. In the second step we show that every rectangular band with a partial involution has a free completion in *RIRB* and as a consequence we obtain that *RIRB* is universal. An analogous method of universality proof was used by Pigozzi and Sichler [7] for the variety of quasigroups.

Definition. A rectangular band B with a partial mapping α is called a *rectangular band with a partial involution* if for every $b \in B$ such that $\alpha(b)$ is defined also $\alpha(\alpha(b))$ is defined and equal to b. If (B,α), (B',α') are rectangular bands with a partial involution then a semigroup homomorphism $f\colon B \longrightarrow B'$ is said to be a homomorphism from (B,α) to (B',α') if for every $b \in B$, if $\alpha(b)$ is defined then

$\alpha'(f(b))$ is also defined and $\alpha'(f(b)) = f(\alpha(b))$. Obviously, rectangular bands with a partial involution and their homomorphisms form a concrete category, denote it by $PIRB$.

Denote the category of all undirected graphs without loops and isolated points and compatible mappings by GRA. The following is well known, see [**9**]

Theorem 3.1[**9**]. *The category GRA is universal.*

To prove that $PIRB$ is universal it suffices to construct a full embedding from GRA into $PIRB$. For a graph $(V, E) \in GRA$ define $\Lambda(V, E) = (X, \alpha)$ as a rectangular band with a partial involution on the set $X = V \times E$ where the band multiplication is defined by:

$$(v, e)(w, f) = (v, f) \qquad \text{for every } v, w \in V, \quad e, f \in E,$$

and the partial operation α is defined for the pairs $(v, e) \in V \times E$ satisfying $v \in e$ by $\alpha(v, e) = (w, e)$, where $e = \{v, w\}$. The following lemma is straightforward.

Lemma 3.2. *For every graph $(V, E) \in GRA$, $\Lambda(V, E)$ is a rectangular band with a partial involution such that for every $x \in \Lambda(V, E)$ there exist $y, z \in \Lambda(V, E)$ such that $\alpha(xy)$ and $\alpha(zx)$ are defined and $xy\alpha(xy) = xy$, $zx\alpha(zy) = zx$.*

For a compatible mapping $f\colon (V, E) \longrightarrow (V', E')$ define $\Lambda f(v, e) = (f(v), f(e))$ for every $v \in V$, $e \in E$ where $f(e) = \{f(u), f(t)\} \in E'$ if $e = \{u, t\}$.

Theorem 3.3. *Λ is a full embedding from GRA into PIRB and thus the category PIRB is universal.*

PROOF : It is easy to see that Λ is an embedding from GRA into $PIRB$. To prove that Λ is full consider $f\colon \Lambda(V, E) \longrightarrow \Lambda(V', E')$. Since f is a homomorphism of rectangular bands there exist mappings $g\colon V \longrightarrow V'$, $h\colon E \longrightarrow E'$ such that $f(v, e) = (g(v), h(e))$ for every $v \in V$, $e \in E$. Since $\alpha(v, e)$ is defined if and only if $v \in e$ and then $e = \{v, w\}$ where $(w, e) = \alpha(v, e)$ we conclude that $v \in e$ implies $g(v) \in h(e)$. Thus g is a compatible mapping from (V, E) into (V', E') and for every $e \in E$, $h(e) = g(e)$, whence $\Lambda g = f$ and Λ is full. ■

In the following we shall investigate a free completion of rectangular bands with a partial involution. Clearly, $RIRB$ is a full subcategory of $PIRB$ and we say that an injective homomorphism $f\colon (B, \alpha) \longrightarrow (B', {}^+)$ is a *free completion* if $(B', {}^+) \in RIRB$ and for every homomorphism $g\colon (B, \alpha) \longrightarrow (B'', {}^+)$ into a rectangular band with a regular involution there exists exactly one homomorphism $h\colon (B', {}^+) \longrightarrow (B'', {}^+)$ with $h \circ f = g$.

To prove that every rectangular band with a partial involution has a free completion we show the following technical lemma which is a modification of well known constructions of completion, see e.g. [**5**] or [**4**]:

Lemma 3.4. *For every $(B, \alpha) \in PIRB$ there exist $(B_1, \alpha_1) \in PIRB$ and an injective homomorphism $f\colon (B, \alpha) \longrightarrow (B_1, \alpha_1)$ satisfying:*

a) *for every homomorphism $g\colon (B, \alpha) \longrightarrow (B', \alpha')$ where $(B', \alpha') \in RIRB$ there exists exactly one homomorphism $h\colon (B_1, \alpha_1) \longrightarrow (B', \alpha')$ with $h \circ f = g$,*

b) *for every* $b \in B$, $\alpha_1(f(b))$ *is defined,*

c) *for every* $b \in B_1$ *if there exist* $c, d \in B_1$ *such that* $\alpha_1(bc)$ *and* $\alpha_1(db)$ *are defined and* $\alpha_1(bc)bc = bc$, $\alpha_1(db)db = db$ *then* $b \in Im(f)$,

d) *for every* $b \in B$, *if* $\alpha_1(f(b)) \in Im(f)$ *then* $\alpha(b)$ *is defined.*

PROOF : Denote by $X = \{b \in B; \quad \alpha(b) \quad \text{is not defined}\}$ and assume that $B = C \times D$ such that $(c,d)(c',d') = (c,d')$ for every $c, c' \in C$, $d, d' \in D$. Let B_1 be a rectangular band on the set $(C \cup X) \times (D \cup X)$ with the operation $(c,d)(c',d') = (c,d')$ for every $c, c' \in C \cup X$, $d, d' \in D \cup X$. Then B is a subband of B_1; denote by f the inclusion from B to B_1. The operation α_1 is an extension of α such that $\alpha_1(b) = (b,b)$, $\alpha_1(b,b) = b$ for every $b \in X$. It is obvious that $(B_1, \alpha_1) \in PIRB$ and that f is a homomorphism satisfying b), d), and moreover, if $x \in B_1$ with $\alpha_1(x)$ defined so that $\alpha_1(x)x = x$ then $x \in B$. Hence we immediately obtain c) because B and B_1 are rectangular bands. We prove a). If $g \colon (B, \alpha) \longrightarrow (B', \alpha')$ is a homomorphism and (B', α') is a rectangular band with a regular involution then $B' = C' \times D'$ and there exist mappings $g_1 \colon C \longrightarrow C'$, $g_2 \colon D \longrightarrow D'$ such that $g(c,d) = (g_1(c), g_2(d))$ for every $c \in C$, $d \in D$. Define $h_1 \colon C \cup X \longrightarrow C'$, $h_2 \colon D \cup X \longrightarrow D'$ as follows: if $c \in C$ then $h_1(c) = g_1(c)$, if $d \in D$ then $h_2(d) = g_2(d)$ if $b \in X$ then $h_1(b)$, $h_2(b)$ are defined by the expression $\alpha'(g(b)) = (h_1(b), h_2(b))$. Define $h = h_1 \times h_2$ then obviously h is a semigroup homomorphism and from the definition of α_1 it preserves also α_1, thus h is a homomorphism from (B_1, α_1) into (B', α'). Obviously, $g = f \circ h$, the unicity of h follows from the fact that (B, α) generates (B_1, α_1). ∎

Theorem 3.5. *Every rectangular band* (B, α) *with a partial involution has a free completion* $f \colon (B, \alpha) \longrightarrow (B', {}^+)$. *Moreover* $(B', {}^+)$ *is unique up to isomorphism and it satisfies*

a) *if for* $b \in B'$ *there exist* $c, d \in B'$ *such that* $(cb)^+cb = cb$ *and* $(bd)^+bd = bd$ *then* $b \in Im(f)$,

b) *for every* $b \in B$, $f(b)^+ \in Im(f)$ *if and only if* $\alpha(b)$ *is defined.*

PROOF : For every natural number i define $(B_i, \alpha_i) \in PIRB$ such that the following hold:

(1) for every $i \in \mathbb{N}$, $B_i \subseteq B_{i+1}$ and the inclusion is a homomorphism from (B_i, α_i) into (B_{i+1}, α_{i+1}),

(2) for every $b \in B_i$, $\alpha_{i+1}(b)$ is defined,

(3) $(B_0, \alpha_0) = (B, \alpha)$.

We shall construct inductively (B_{i+1}, α_{i+1}) from (B_i, α_i) by Lemma 3.4. Then 1), 2), 3) are satisfied. Set $B' = \bigcup\{B_i; \quad i \in \mathbb{N}\}$. By standard calculation we obtain that B' is a rectangular band and ${}^+ = \bigcup\{\alpha_i; \quad i \in \mathbb{N}\}$ is a regular involution. Moreover, the inclusion $f \colon (B, \alpha) \longrightarrow (B', {}^+)$ is a homomorphism. By a) of Lemma 3.4 we easily obtain that it is a free completion and by standard categorical calculus we obtain that it is determined uniquely up to isomorphism. The properties a) and b) are direct consequences of c) and d) in Lemma 3.4. ∎

We can now define a reflection functor $\Psi \colon PIRB \longrightarrow RIRB$. For a rectangular band with a partial involution (B, α) , if $f \colon (B, \alpha) \longrightarrow (B', {}^+)$ is a free completion then define $\Psi(B,\alpha) = (B',^+)$. For a homomorphism $f\colon (B_0,\alpha_0) \longrightarrow (B_1,\alpha_1)$ denote by $h_i\colon (B_i,\alpha_i) \longrightarrow (C_i,^+)$ a free completion of (B_i,α_i), $i \in 2$ and let $\Psi f\colon (C_0,^+) \longrightarrow (C_1,^+)$ be the homomorphism satisfying $\Psi f \circ h_0 = h_1 \circ f$. It is well known that Ψ is an embedding.

Theorem 3.6. *$\Phi = \Psi \circ \Lambda\colon GRA \longrightarrow RIRB$ is a full embedding, thus $RIRB$ is universal.*

PROOF : Since the composition of two embeddings is an embedding it suffices to show that Φ is full. Let $f\colon \Phi(V_0,E_0) \longrightarrow \Phi(V_1,E_1)$ be a homomorphism where $(V_i,E_i) \in GRA$ for $i \in 2$. Let $h_i\colon \Lambda(V_i,E_i) \longrightarrow \Phi(V_i,E_i)$ be a free completion for $i \in 2$. According to Lemma 3.2, for every $z \in \Lambda(V_0,E_0)$ there exist $x, y \in \Phi(V_0,E_0)$ such that $zx(zx)^+ = zx$ and $yz(yz)^+ = yz$ whence the same property is enjoyed by $f(z)$ and $f(x)$, $f(y)$. By a) of Theorem 3.5 we conclude that $f(z) \in \Lambda(V_1,E_1)$ and thus $f(Im(h_0)) \subseteq Im(h_1)$. By b) of Theorem 3.5 we conclude that the restriction g of f to $\Lambda(V_i,E_i)$ is a homomorphism and by Theorem 3.3 there exists a compatible mapping $g'\colon (V_0,E_0) \longrightarrow (V_1,E_1)$ with $g = \Lambda g'$. Since $V_0 \times E_0$ generates $\Phi(V_0,E_0)$ and $\Phi g'$ and f coincide on $V_0 \times E_0$ we conclude that $\Phi g' = f$. Thus Φ is full. ■

4. Conclusions.

By a well known description of the lattice L of subvarieties of bands B we know that LZB, RZB, and SL are the atoms of L. Moreover, LNB, RNB, and RB are the varieties covering an atom of L and every non–trivial variety $\underset{\sim}{V}$ of bands which is not an atom in L contains one of them. By Theorem 2.7, the dual of Theorem 2.7 and Theorem 3.6 we obtain that $RILNB$, $RIRBN$, and $RIRB$ are universal, thus $RI\underset{\sim}{V}$ is universal whenever $\underset{\sim}{V}$ is a non–trivial variety of bands which is not an atom in L. Denote by IN the variety of all unary algebras with one operation f satisfying the identity $f^2(x) = x$. The varieties IN and SL are not universal, see [9], and because IN is isomorphic to the variety $RILZB$ and also to $RIRZB$ and $RISL$ is isomorphic to SL (because the identities $x^{++} = x$, $x = xx^+x$, $xy = yx$, $x^2 = x$ imply $x^+ = x$) we conclude that $RILZB$, $RIRZB$, and $RISL$ are not universal and the proof of Theorem 1.1. is complete.

The following two questions remain open:

a) determine the minimal universal subvarieties of RIB,
b) is there at all a universal subvariety $\underset{\sim}{V}$ of RIB such that for every band there exists its expansion into $\underset{\sim}{V}$?

We can only give to these questions a partial answer. We use the following notations: the variety of all semigroups is denoted by SEM. For a variety $\underset{\sim}{V}$ of semigroups, $RI\underset{\sim}{V}[\alpha = \beta]$ denotes the variety of all semigroups from $\underset{\sim}{V}$ with an added unary operation $^+$ satisfying (RI) and the semigroup identity $\alpha = \beta$. In the following we show

Theorem 4.1. *$RISEM[xx^+yy^+ = yy^+xx^+]$, $RIB[(xy)^+ = y^+x^+]$, $RIB[xx^+ = x]$, and $RIB[x^+x = x]$ are neither universal nor monoid universal.*

Proof : First, note that the variety $RISEM[xx^+ = yy^+]$ is the variety of all groups.

Since the variety $RISEM[xx^+yy^+ = yy^+xx^+]$ is the variety of all inverse semigroups, see [1], we obtain for every $x \in S \in RISEM[xx^+yy^+ = yy^+xx^+]$ that xx^+ is an idempotent with $(xx^+)^+ = xx^+$. Whence the constant mapping of S onto xx^+ is an endomorphism of S being a left zero of $End(S)$. Thus either S is a group or $End(S)$ has at least two left zeros, consequently $RISEM[xx^+yy^+ = yy^+xx^+]$ is neither universal nor monoid universal.

Analogously for $x \in S \in RIB[(xy)^+ = y^+x^+]$ we have that xx^+ is an idempotent with $(xx^+)^+ = x^{++}x^+ = xx^+$, thus the constant mapping of S onto xx^+ is an endomorphism of S and a left zero of $End(S)$. Since only a trivial group is in $RIB[(xy)^+ = y^+x^+]$ we have that either S is a singleton or $End(S)$ has at least two left zeros, consequently $RIB[(xy)^+ = y^+x^+]$ is neither universal nor monoid universal. Consider $S \in RIB[xx^+ = x]$. Then for every $s \in S$ either $s = s^+$ or $\{s, s^+\}$ is a subalgebra of S. In the first case the constant mapping of S onto s is an endomorphism of S and a left zero of $End(S)$. If $s \neq s^+$ for every $s \in S$ then for every pair $p = \{s, s^+\}$, $s \in S$ choose an element $e_p \in p$ and define a mapping $f\colon S \longrightarrow S$ such that $f(e_p) = s$, $f(e_p^+) = s^+$, for every pair p. By direct inspection we obtain that f is an endomorphism of S belonging to the smallest $\mathcal{J}$-class of $End(S)$. Hence $End(S)$ has the smallest $\mathcal{J}$-class and thus $RIB[xx^+ = x]$ is neither universal nor monoid universal.

The proof for $RIB[x^+x = x]$ is dual. ■

Even stronger results can be proved, for example:

Proposition 4.2. *For every algebra $A \in RIB[(xy)^+ = y^+x^+]$ and for every chain C in the semilattice L of $\mathcal{D}$-classes of A there exists an idempotent endomorphism f of A such that $Im(f)$ is contained in the union of $\mathcal{D}$-classes in C and the meet of $Im(f)$ with every $\mathcal{D}$-class in C is a singleton.*

Proof : We give only an outline of proof. First, if B is a $\mathcal{D}$-class of S then $B = X \times X$ such that for every $x, y, v, w \in X$ we have $(x,y)(v,w) = (x,w)$. For every $x \in X$ then $(x,x)^+ = (x,x)$ and we can choose (x_l, x_l) in every $\mathcal{D}$-class $l \in C$ such that $A = \{(x_l, x_l);\ \ l \in C\}$ is a subalgebra of S. Then by a standard way we can construct an idempotent endomorphism f of S with $Im(f) = A$. ■

It is an open problem whether the expansion $RIRB$ of rectangular bands by a regular involution is finite–to–finite universal. GRA is known to be finite–to–finite universal, see [9], and Λ preserves finiteness, thus also $PIRB$ is finite–to–finite universal. But the reflection functor Ψ does not preserve finiteness (moreover $\Psi(B,\alpha)$ is finite if and only if B is finite and $(B,\alpha) \in RIRB$.

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