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Derivations on the restricted Nijenhuis-Schouten bracket algebra

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Dedicated to the memory of Zdeněk Frolík

Abstract. In this paper, we describe all derivations on the restricted Nijenhuis-Schouten bracket algebra. This is a graded Lie algebra associated with every C^∞ - manifold. We show that with the one exception, all these derivations are inner.

Keywords: Nijenhuis-Schouten bracket, graded Lie algebra, derivation

Classification: 17B40, 17B70, 57R25

This paper represents a direct continuation of my previous paper [4]. We shall investigate here derivations on the restricted Nijenhuis-Schouten bracket algebra $L^{\geqq 0} = \sum_{i=0}^{m-1} L_i$, which is a subalgebra in the Nijenhuis-Schouten bracket algebra $L = \sum_{i=-1}^{m-1} L_i$. The absence of the -1-st component in the restricted algebra requires an application of methods different from those used in [4].

All structures appearing in this paper are of class C^∞. We shall consider a connected paracompact manifold $M, \dim M = m$. In contrast to [4], we do not assume that M is orientable. Let TM denote the tangent bundle of M, $\Lambda^i TM$ its i-th exterior power, and let us set

$$L_i = \Gamma\Lambda^{i+1}TM, \quad -1 \leqq i \leqq m-1,$$

where Γ is the functor of sections over M. Obviously, L_i is a real vector space. Further, we set

$$L_i = 0 \quad \text{for} \quad i < -1 \quad \text{or} \quad i > m-1.$$

We define

$$L = \sum_{i=-\infty}^{\infty} L_i.$$

Provided with the Nijenhuis-Schouten bracket $[,] : L \times L \to L$, L is a graded Lie algebra. We call it *Nijenhuis-Schouten bracket algebra*. (For its basic properties see e.g. [3], [4]). Now we set

$$L_i^{\geq 0} = L_i \quad \text{for} \quad i \neq -1,$$
$$L_{-1}^{\geq 0} = 0,$$
$$L^{\geq 0} = \sum_{i=-\infty}^{\infty} L_i^{\geq 0}.$$

Obviously $L^{\geq 0} \subset L$ is a subalgebra. We shall call $L^{\geq 0}$ the *restricted Nijenhuis-Schouten bracket algebra*. An element $\alpha \in L_i^{\geq 0}$ will be called homogeneous, and we shall write $|\alpha| = i$.

A derivation of degree $k \in \mathbb{Z}$ on $L^{\geq 0}$ is a linear mapping $D : L^{\geq 0} \to L^{\geq 0}$ such that

(i) $DL_i^{\geq 0} \subset L_{i+k}^{\geq 0}$ for every $i \in \mathbb{Z}$

(ii) $D[\alpha, \beta] = [D\alpha, \beta] + (-1)^{k.|\alpha|}\, [\alpha, D\beta]$
for any two homogeneous elements $\alpha, \beta \in L$.

A derivation D is called *local*, if it satisfies the following condition: If $\alpha \in L_i^{\geq 0}$, $U \subset M$ is an open subset, and $\alpha|U = 0$, then $D\alpha|U = 0$. We shall denote by $\mathrm{Der}_k^{\geq 0}$ the vector space of all local derivations of degree k on $L^{\geq 0}$. We set

$$\mathrm{Der}^{\geq 0} = \sum_{k=-\infty}^{\infty} \mathrm{Der}_k^{\geq 0}.$$

As usual, for $D_1 \in \mathrm{Der}_k^{\geq 0}$ and $D_2 \in \mathrm{Der}_l^{\geq 0}$, we define $[D_1, D_2] \in \mathrm{Der}_{k+l}^{\geq 0}$ by the formula

$$[D_1, D_2] = D_1 D_2 - (-1)^{kl} D_2 D_1.$$

With this operation $\mathrm{Der}^{\geq 0}$ is a graded Lie algebra. The goal of this paper is to describe the Lie algebra $\mathrm{Der}^{\geq 0}$.

We notice first that any derivation $D \in \mathrm{Der}_k^{\geq 0}$ is local. Therefore, by virtue of the Peetreś theorem, D is a linear differential operator.

Proposition 1. $\mathrm{Der}_k^{\geq 0} = 0$ *for* $k < 0$.

PROOF : Let $D \in \mathrm{Der}_k^{\geq 0}$, where $k < 0$, and let $\alpha \in L_i$, $0 \leqq i \leqq m-1$ be arbitrary. For any vector field $X \in L_o$ we have

$$D[X, \alpha] = [DX, \alpha] + [X, D\alpha] = [X, D\alpha]$$
$$(\mathcal{L}_X D - D\mathcal{L}_X)\alpha = 0,$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X. In other words, the differential operator

$$D : \Gamma\Lambda^{i+1}TM \to \Gamma\Lambda^{i+k+1}TM$$

commutes with the Lie derivative with respect to arbitrary $X \in L_0$. (We take $\Gamma\Lambda^j TM = 0$ for $j < 0$.) Using [2], we can find easily that $D = 0$. ∎

Proposition 2. *For any derivation* $D \in \mathrm{Der}_0^{\geq 0}$ *there exist a unique* $X_D \in L_0$ *and* $c \in \mathbf{R}$ *such that*

$$D\alpha = \mathcal{L}_{X_D}\alpha + ic\alpha \quad for \quad \alpha \in L_i, \quad 0 \leqq i \leqq m-1.$$

Conversely for any $X \in L_0$ *and* $c \in \mathbf{R}$ *the formula*

$$D\alpha = \mathcal{L}_X\alpha + ic\alpha \quad for \quad \alpha \in L_i, \quad 0 \leqq i \leqq m-1$$

defines a derivation $D \in \mathrm{Der}_0^{\geq 0}$.

PROOF : Let $D \in \mathrm{Der}_0^{\geq 0}$. Obviously $D|L_0$ is a derivation on L_0. It is well known that any such derivation is inner, i.e. there exists a unique $X_D \in L_0$ such that for any $X \in L_0$ there is $DX = [X_D, X]$. We denote $D' = D - \mathcal{L}_{X_D}$. There is $D' \in \mathrm{Der}_0^{\geq 0}$ and $D'|L_0 = 0$.

For any $X \in L_0$ and $\alpha \in L_i$, $0 \leqq i \leqq m-1$ we have

$$D'[X,\alpha] = [X, D'\alpha],$$
$$(\mathcal{L}_X D' - D'\mathcal{L}_X)\alpha = 0.$$

Therefore, by virtue of [2], there exist $d_i \in \mathbf{R}$, $0 \leqq i \leqq m-1$ such that for any $\alpha \in L_i$, $0 \leqq i \leqq m-1$ there is

$$D'\alpha = d_i\alpha.$$

Considering for any $\alpha \in L_i$, $\beta \in L_j$, $0 \leqq i, j, i+j \leqq m-1$ the equation

$$D'[\alpha,\beta] = [D'\alpha,\beta] + [\alpha, D'\beta],$$

we find $d_i + d_j = d_{i+j}$. Now we can easily see that there exists $c \in \mathbf{R}$ such that for any $0 \leqq i \leqq m-1$ there is $d_i = ic$. The rest of the proof is obvious. ∎

We shall now start to study a derivation $D \in \mathrm{Der}_k^{\geq 0}$, where $k > 0$. We take any L_i with $0 \leqq i \leqq m-1$. For any $\alpha \in L_i$ and $X \in L_0$, we get

$$D[X,\alpha] = [DX,\alpha] + [X, D\alpha],$$
$$D\mathcal{L}_X\alpha = (ad(DX))\alpha + \mathcal{L}_X D\alpha,$$
$$(\mathcal{L}_X D - D\mathcal{L}_X)\alpha = -(ad(DX))\alpha.$$

The last equality shows that the commutator $\mathcal{L}_X D - D\mathcal{L}_X$ is a linear differential operator of order $\leqq 1$.

Lemma 3. *Let* $D : \Gamma\Lambda^{i+1}TM \to \Gamma\Lambda^{i+k+1}TM$, *where* $0 \leqq i \leqq m-1$, $k > 0$, *be a linear differential operator such that for any* $X \in L_0$ *there is ord* $(\mathcal{L}_X D - D\mathcal{L}_X) \leqq 1$. *Then ord* $D \leqq 1$.

PROOF : Because our considerations have local character we may assume that there is $r > 1$ such that ord $D \leqq r$. We shall denote by σ_E the r-*th* order symbol of a linear differential operator E. ∎

The formula

$$\sigma_{\mathcal{L}_X D - D\mathcal{L}_X} = \mathcal{L}_X \sigma_D$$

together with the assumption ord $(\mathcal{L}_X D - D\mathcal{L}_X) \leqq 1$ shows that for any $X \in L_0$ there is $\mathcal{L}_X \sigma_D = 0$. Obviously

$$\sigma_D \in (S^r TM \otimes \Lambda^{i+1} T^* M \otimes \Lambda^{i+k+1} TM).$$

Let us define a 0-th order linear differential operator

$$K : \Gamma \Lambda^0 TM \to \Gamma(S^r TM \otimes \Lambda^{i+1} T^* M \otimes \Lambda^{i+k+1} TM)$$

by the formula

$$Kf = f \cdot \sigma_D.$$

We find easily that $\mathcal{L}_X \sigma_D = 0$ is equivalent with the equality

$$\mathcal{L}_X K - K \mathcal{L}_X = 0$$

Using again [2], we get $K = 0$, and consequently $\sigma_D = 0$. We have thus shown that ord $D \leqq r - 1$. Now we can easily see that ord $D \leqq 1$.

So far we have proved that every derivation $D \in \mathrm{Der}_k^{\geqq 0}$, $k > 0$ is a linear differential operator of order $\leqq 1$. We shall now denote by σ_D the first symbol of the first order linear differential operator $D|L_0 : L_0 \to L_k$. Let $x \in M$, $v, w \in T_x M$ and ξ, $\eta \in T_x^* M$ be arbitrary. Let $X, Y \in L_0$, and $f, g \in L_{-1}$ be such that

$$X_x = v, \quad Y_x = w, \qquad f(x) = 0 \quad g(x) = 0,$$
$$\mathrm{d} f_x = \xi, \quad \mathrm{d} g_x = \eta.$$

We have

$$(*) \qquad (D[fX, gY])_x = [D(fX), gY]_x + [fX, D(gY)]_x.$$

Using the formula

$$\mathcal{L}_{fX} \alpha = f \mathcal{L}_X \alpha - X \wedge \iota_{\mathrm{d}f} \alpha,$$

where ι denotes the inner product operator, and which holds for any $f \in L_{-1}$, $X \in L_0$, and $\alpha \in L_k$, we shall calculate both sides of the above equality.

$$\begin{aligned}
(D[fX, gY])_x &= (D(f.Xg.Y - g.Yf.X + fg[X,Y]))_x = \\
&= \sigma_D(\xi)(\eta(v).w) - (\sigma_D(\eta)(\xi(w).v) = \\
&= \eta(v).\sigma_D(\xi)(w) - \xi(w).\sigma_D(\eta)(v), \\
[D(fX), gY]_x &= -[gY, D(fX)]_x = -(\mathcal{L}_{gY}(D(fX)))_x = \\
&= -(g\mathcal{L}_Y(D(fX)) - Y \wedge \iota_{\mathrm{d}g}(D(fX)))_x = w \wedge \iota_\eta(\sigma_D(\xi)(v)), \\
[fX, D(gY)]_x &= -v \wedge \iota_\xi(\sigma_D(\eta)(w)).
\end{aligned}$$

Using (*), we obtain the equality

$$(**)\qquad \begin{aligned} &v \wedge \iota_\xi(\sigma_D(\eta)(w)) - w \wedge \iota_\eta(\sigma_D(\xi)(v)) \\ &-\xi(w).\sigma_D(\eta)(v) + \eta(v).\sigma_D(\xi)(w) = 0 \end{aligned}$$

To understand properly this equality, let us consider cochains on the Lie algebra $\mathrm{gl}(V) = V^* \otimes V$, where $V = T_xM$, with coefficients in the $\mathrm{gl}(V)$-module $\wedge^{k+1}V$. The bilinear mapping

$$\sigma_D : V^* \times V \to \wedge^{k+1}V$$

defined by the formula $\sigma_D(\xi, v) = \sigma_D(\xi)(v)$, induces a linear mapping (which we denote by the same symbol)

$$\sigma_D : \mathrm{gl}(V) \to \wedge^{k+1}V.$$

This shows that we can consider σ_D as an element from the vector space of cochains $C^1(\mathrm{gl}(V), \wedge^{k+1}V)$. We shall now compute the coboundary $\delta\sigma_D$.

$$\begin{aligned} (\delta\sigma_D)(\xi \otimes v, \eta \otimes w) = (\xi \otimes v)\sigma_D(\eta \otimes w) - (\eta \otimes w)\sigma_D(\xi \otimes v)- \\ -\sigma_D([\xi \otimes v, \eta \otimes w]) = v \wedge \iota_\xi(\sigma_D(\eta \otimes w)) - w \wedge \iota_\eta(\sigma_D(\xi \otimes v)) \\ -\sigma_D(\xi(w).v \otimes \eta - \eta(v).w \otimes \xi) = v \wedge \iota_\xi(\sigma_D(\eta)(w))- \\ -w \wedge \iota_\eta(\sigma_D(\xi)(v)) - \xi(w).\sigma_D(\eta)(v) + \eta(v).\sigma_D(\xi)(w). \end{aligned}$$

We can now see that (**) can be written in a simpler from

$$(\delta\sigma_D)(\xi \otimes v, \quad \eta \otimes w) = 0.$$

The $\mathrm{gl}(V)$-module $\wedge^{k+1}V$ is irreducible, which implies that $\mathrm{Inv}\wedge^{k+1}V = \{a \in \wedge^{k+1}V;\ (\forall l \in \mathrm{gl}(V))(la = 0\} = 0$. Consequently (see [1]), there is $H^1(\mathrm{gl}(V);\ \wedge^{k+1}V) = 0$. Moreover, because $\mathrm{Inv}\wedge^{k+1}V = 0$, the coboundary operator $\delta : C^0(\mathrm{gl}(V);\ \wedge^{k+1}V) = \wedge^{k+1}V \to C^1(\mathrm{gl}(V);\ \wedge^{k+1}V)$ is injective. Therefore there exists a unique element $a_x \in \wedge^{k+1}V = \wedge^{k+1}T_xM$ such that for any $v \in T_xM$, $\xi \in T_x^*M$ there is

$$\sigma_D(\xi)(v) = v \wedge \iota_\xi a_x.$$

It can be easily verified that the family $\{a_x\}_{x\in M}$ determines an element $\alpha_D \in \Gamma\wedge^{k+1}TM = L_k$. We have thus proved the following lemma.

Lemma 4. *For every $D \in \mathrm{Der}_k^{\geq 0}$, $k > 0$, there exists a unique element $\alpha_D \in L_k$ such that for any $x \in M$, $v \in T_xM$, $\xi \in T_x^*M$ there is*

$$\sigma_D(\xi)(v) = v \wedge \iota_\xi \alpha_D.$$

Let us consider now the inner derivation $ad\,\alpha_D \in \mathrm{Der}_k^{\geq 0}$, $k > 0$. Obviously the restriction

$$ad\,\alpha_D|L_0 : L_0 \to L_k$$

is a linear differential operator of order $\leqq 1$. For ist first symbol we find easily

$$\sigma_{ad\alpha_D|L_0}(\xi)(v) = v \wedge \iota_\xi \alpha_D.$$

By virtue of the preceding lemma, we can immediately see that $\mathrm{ord}((D - ad\alpha_D)|L_0) = 0$.

Lemma 5. *Let* $D' \in \mathrm{Der}_k^{\geq 0}$, $k > 0$ *be such that* $\mathrm{ord}(D'|L_0) = 0$. *Then* $D'|L_0 = 0$.

PROOF : Let $X \in L_0$, and let $g \in L_{-1}$, $Y \in L_0$ be arbitrary. We get

$$\begin{aligned} 0 =& D'[X, gY] - [D'X, gY] - [X, D'(gY)] = \\ =& D'(Xg.Y + g[X, Y]) + [gY, D'X] - [X, gD'Y] = \\ =& Xg.D'Y + gD'[X, Y] + \mathcal{L}_{gY}(D'X) - \mathcal{L}_X(gD'Y) = \\ =& Xg.D'Y + gD'[X, Y] + g\mathcal{L}_Y(D'X) - Y \wedge \iota_{dg}(D'X) - \\ &-Xg.D'Y - g\mathcal{L}_X(D'Y) = \\ =& gD'[X, Y] + g[Y, D'X] - Y \wedge \iota_{dg}(D'X) - g[X, D'Y] = \\ =& -Y \wedge \iota_{dg}(D'X). \end{aligned}$$

But this equality implies $D'X = 0$. ∎

Using the last lemma, we find that for any $D \in \mathrm{Der}_k^{\geq} 0$, $k > 0$ there is

$$(D - ad\alpha_D)|L_0 = 0.$$

Lemma 6. *Let* $D' \in \mathrm{Der}_k^{\geq 0}$, $k > 0$ *be such that* $D'|L_0 = 0$. *Then* $D' = 0$.

PROOF : For arbitrary $i > 0$ let us consider the linear differential operator $D'|L_i : L_i \to L_{i+k}$. For $X \in L_0$ and $\alpha \in L_i$, we get

$$D'[X, \alpha] = [D'X, \alpha] + [X, D'\alpha] = [X, D'\alpha],$$
$$(D'\mathcal{L}_X - \mathcal{L}_X D')\alpha = 0.$$

Now [2] again gives $D' = 0$. ∎

Applying this lemma we come to the following proposition.

Proposition 7. *Every derivation* $D \in \mathrm{Der}_k^{\geq 0}$, $k > 0$ *is inner.*

Proposition 2 shows that

$$\mathrm{Der}_0^{\geq 0} = \mathrm{Der}_0^{\geq 0'} \oplus \mathrm{Der}_0^{\geq 0''},$$

where

$$\mathrm{Der}_0^{\geq 0'} = \{D = adX; X \in L_0\},$$
$$\mathrm{Der}_0^{\geq 0''} = \{D; D\alpha = ic\alpha \quad \text{for} \quad \alpha \in L_i, \quad 0 \leqq i \leqq m-1, c \in \mathbf{R}\}.$$

The decomposition $\mathrm{Der}^{\geq 0} = \mathrm{Der}_0^{\geq 0'} \oplus \mathrm{Der}_0^{\geq 0''} \oplus \sum_{k=1}^{m-1} \mathrm{Der}_k^{\geq 0}$ induces a projection $\pi : \mathrm{Der}^{\geq 0} \to \mathrm{Der}_0^{\geq 0''}$. We can easily see that $\mathrm{Der}_0^{\geq 0''}$ is a one-dimensional commutative Lie algebra which may be naturally identified with $\mathbf{R}$, and π is a Lie algebra homomorphism. Thus we come to the following proposition.

Proposition 8. *The sequence*

$$0 \to L^{\geq 0} \xrightarrow{ad} \mathrm{Der}^{\geq 0} \xrightarrow{\pi} \mathbf{R} \longrightarrow 0$$

is an exact sequence of Lie algebras.

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