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Dedicated to the memory of Zdeněk Frolík

Abstract. For every triple of natural numbers a, b, c there is a metric space X, the m-th power and the n-th power of which are

homeomorphic	iff $m \equiv n \mod a$
uniformly homeomorphic	iff $m \equiv n \mod ab$
isometric	iff $m \equiv n \mod abc$.

This is a consequence of the Main Theorem proved in the present paper, where simultaneous representations of commutative semigroups by the products of metric, uniform and topological spaces are investigated.

Keywords: products, commutative semigroups

Classification: primary 54B10, 20M30, secondary 54G15, 18A99

I. Introduction and the Main Theorem.

The forming of products of mathematical structures is an important operation on the isomorphism types of the structures. The properties of this operation have been investigated in various parts of mathematics. From the older references, we recall at least the Ulam problem [U] about the validity of the square root property with respect to it in the class of topological spaces and the monograph [Ta] with a large appendix by A. Tarski and B. Jónsson devoted to the investigation of this operation on the isomorphism types of some algebraic and combinatorial structures. From the recent references, we mention at least the monograph [MMT], where Chapter 5 deals essentially with products of algebraic structures, and [H], chapter 21, where products and coproducts of isomorphism types of countable Boolean algebras are investigated.

The language of category theory is suitable for the formulation of general ideas in this field of problems. Let $\mathcal{K}$ be a category with finite products (= with products of finite collections of objects of $\mathcal{K}$). Then the isomorphism types of objects of $\mathcal{K}$ endowed with the binary operation of forming product form a (possibly large) commutative semigroup, let us denote it by $\Pi\mathcal{K}$. One of the natural question is, given a category $\mathcal{K}$, which abstract commutative semigroups can be embedded into $\Pi\mathcal{K}$. This is still unknown for $\mathcal{K}$ being the category of Boolean algebras (though every countable commutative semigroup and every Abelian group can be embedded in it, by [K] and [AKT]) or for $\mathcal{K}$ being the category of compact Hausdorff spaces (though some partial results are also known, see [OR], [T_4], [TK]). On the other hand, it was proved in [T_2] that every commutative semigroup (on a set) can be embedded in ΠMetr, in ΠUnif and in ΠTop, where

Metr is the category of all metric spaces of diameter ≤ 1 and all non–expanding maps,

Unif is the category of all uniform spaces and all uniformly continuous maps and

Top is the category of all topological spaces and all continuous maps.

(We recall that products in Metr are the products with the metric of uniform convergence, isomorphisms in Metr are isometries.)

Let us mention that the forgetful functors F_1, F_2 in the diagram

$$\text{Metr} \xrightarrow{F_1} \text{Unif} \xrightarrow{F_2} \text{Top}$$

preserve finite products (though F_1 and $F_2 \circ F_1$ do not preserve all products), so that they determine homomorphisms ΠF_1, ΠF_2 in the diagram

(1) $$\Pi\text{Metr} \xrightarrow{\Pi F_1} \Pi\text{Unif} \xrightarrow{\Pi F_2} \Pi\text{Top}$$

of (large) commutative semigroups by the rule

$$\Pi F_i(\overline{o}) = \overline{F_i(o))}\ ,$$

where the strip denotes the isomorphism type of an object o (or $F_i(o)$) in the category in question.

In the present paper, we investigate which diagrams of abstract commutative semigroups

(2) $$S_0 \xrightarrow{h_1} S_1 \xrightarrow{h_2} S_2$$

(i.e. S_0, S_1, S_2 are arbitrary commutative semigroups [on sets] and h_1, h_2 are arbitrary homomorphism, situated as indicated in (2)) can be embedded in the diagram (1) (in the sense that there exist semigroup monomorphism φ_0, φ_1, φ_2 such that the diagram below commutes.

$$\begin{array}{ccccc} S_0 & \xrightarrow{h_1} & S_1 & \xrightarrow{h_2} & S_2 \\ {\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow \\ \Pi\text{Metr} & \xrightarrow{\Pi F_1} & \Pi\text{Unif} & \xrightarrow{\Pi F_2} & \Pi\text{Top} \end{array}\ ).$$

The aim of the present paper is to prove following

Main Theorem. *Every diagram (2) can be embedded in the diagram (1).*

Let us show some consequences of the Main Theorem.

a) There exists a metrizable topological space X and a sequence $\rho_1, \rho_2, \rho_3, \ldots$ of metrics on it, each of them metrizing it, such that, for every n, the metric space

(X, ρ_n) is isometric to its $(n+1)$-th power but it is not uniformly homeomorphic to its k-th power for all k with $1 < k \leq n$.

This is obtained immediately from the Main Theorem by the choice $S_0 = S_1 = \coprod_{n=1}^{\infty} c_n$ (= the coproduct [in the category of commutative semigroups] of all finite cyclic groups c_n of order n), h_1 is the identity, $S_2 = \{e\}$ and h_2 maps the whole $S_0 = S_1$ on the unique (idempotent) element of S_2. If $\{\varphi_0, \varphi_1, \varphi_2\}$ is an embedding of this diagram into the diagram (1) and g_n is the generator of $c_n \subseteq S_0 = S_1$, then $\varphi_0(g_n)$ is the isomorphism type of a metric space isometric to its $(n+1)$-th power but not uniformly homeomorphic to its k-th power with $1 < k \leq n$; the underlying topological spaces of all these metric spaces belong to the isomorphism type $\varphi_2(e)$.

b) The result formulated in the abstract is obtained from the Main Theorem by the choice $S_0 = c_{abc}$, $S_1 = c_{ab}$, $S_2 = c_a$, where h_1, h_2 are the corresponding quotient homomorphisms. If $\{\varphi_0, \varphi_1, \varphi_2\}$ is an embedding of this diagram in the diagram (1) and g is the generator of the group c_{abc}, then any metric space X of the isomorphism type $\varphi_0(g)$ has the property formulated in the abstract.

c) Many other "strange" situations can be obtained by other choices of semigroups S_0, S_1, S_2 and the homomorphisms h_1 and h_2 (e.g. $S_0 = R$, $S_1 = S_2 = \{e\}$ or $S_0 = S_1 = R$, $S_2 = \{e\}$, where R is the additive group of real numbers). On the other hand, these "strange" situations are not too exceptional: in the last part of the present paper, we show that all the spaces can contain a prescribed metric space as a retract. Hence every metric space (of diameter ≤ 1) is a retract of a space X with the property described in the abstract or in a).

The Main Theorem was announced in [T_7], where a general notion of simultaneous productive representation was formulated and a general categorial method for the constructing of simultaneous productive representations was described. By means of this general method, every diagram of commutative semigroups

$$S_0 \xrightarrow{h} S_1$$

was embedded in the diagram

$$\Pi\mathrm{Metr} \xrightarrow{\Pi c} \Pi\mathrm{Metr} \ ,$$

where c denotes the completion functor, and every diagram of commutative semigroups

$$S_0 \xrightarrow{h_1} S_1 \qquad S_0 \xrightarrow{h_2} S_2$$

was embedded in the diagram

$$\Pi\mathrm{Bitop} \xrightarrow{\Pi F_1} \Pi\mathrm{Top}, \qquad \Pi\mathrm{Bitop} \xrightarrow{\Pi F_2} \Pi\mathrm{Top}$$

and in the diagram

$$\Pi\mathrm{Alg}(1,1) \xrightarrow{\Pi F_1} \Pi\mathrm{Alg}(1), \qquad \Pi\mathrm{Alg}(1,1) \xrightarrow{\Pi F_2} \Pi\mathrm{Alg}(1)$$

where Bitop denotes the category of bitopological spaces and Alg(1, 1) (or Alg(1)) denotes the category of universal algebras with two (or one) unary operations, F_1 and F_2 denote the corresponding pair of the forgetful functors (see [$\mathbf{T}_7$]).

The general categorial method of [$\mathbf{T}_7$] cannot be used to prove the Main Theorem here: the method requires that all the functors which appear in the investigated diagrams, preserve all products. This is not true for the forgetful functors Metr $\longrightarrow$ Unif and Metr $\longrightarrow$ Top. Though the basic idea of [$\mathbf{T}_7$] is also used here, some (not quite immediate) modifications are necessary. The proof of the Main Theorem is presented in the next part II. The last part III contains two possible strengthenings of the Main Theorem.

II. The proof of the Main Theorem.

1. Let ω be the additive semigroup of all noon-negative integers. If M is a set, let us denote by ω^M the set of all functions on M with values in ω. This is a commutative semigroup with the operation + defined by the usual rule

$$(f+g)(m) = f(m) + g(m) \text{ for all } m \in M .$$

The set $\exp \omega^M$ of all subsets of ω^M is also a commutative semigroup: its operation + is defined by the usual rule

$$A + B = \{f + g \mid f \in A, \quad g \in B\} .$$

Let M be infinite. Then all $A \in \exp \omega^M$ such that

$$\operatorname{card} A = \operatorname{card} M \text{ and } const_0 \notin A ,$$

where $const_0 \colon M \longrightarrow \omega$ is the constant zero, form a subsemigroup of $\exp \omega^M$. Denote it by $\mathsf{U}_\varkappa$, where $\varkappa = \operatorname{card} M$. By [$\mathbf{T}_3$], every commutative semigroup S with $\operatorname{card} S \leq \omega$ can be embedded into $\mathsf{U}_\varkappa$.

2. Let a poset (= partially ordered set) $P = (P, \leq)$ and an infinite cardinal number $\varkappa$ be given. Let us define a diagram $\mathcal{D} = \mathcal{D}(P, \varkappa)$ over P in the category of commutative semigroups as follows: for every $p \in P$,

$$\mathcal{D}(p) = \prod_{q \geq p} (\mathsf{U}_\varkappa)_q ,$$

where $(\mathsf{U}_\varkappa)_q$ is a copy of the semigroup $\mathsf{U}_\varkappa$ described above, i.e. $\mathcal{D}(p)$ is a direct product of the family $\{(\mathsf{U}_\varkappa)_q \mid q \in P, q \geq p\}$ of commutative semigroups; if $p \leq p'$, then the homomorphism

$$\mathcal{D}\binom{p'}{p} : \mathcal{D} \longrightarrow \mathcal{D}(p')$$

is just the projection of $\prod_{q \geq p}(\mathsf{U}_\varkappa)_q$ onto $\prod_{q \geq p'}(\mathsf{U}_\varkappa)_q$.

Lemma. *Let $P = (P, \leq)$ be a poset, let $\mathcal{C}$ be a diagram over P in the category of commutative semigroups. Let $\varkappa$ be a cardinal number with*

$$\varkappa \geq \aleph_0 \cdot \max_{p \in P} \operatorname{card} \mathcal{C}(p) .$$

Then there is an embedding (= monotransformation) $\mu = \{\mu_p \mid p \in P\}$ of $\mathcal{C}$ into the diagram $\mathcal{D} = \mathcal{D}(P, \varkappa)$.

PROOF : For every $p \in P$, choose a monomorphism $h_p : \mathcal{C}(p) \longrightarrow \mathsf{U}_\varkappa$; this is possible by [**T**$_3$]. Define $\mu_p : \mathcal{C}(p) \longrightarrow \mathcal{D}(p)$ by

$$\mu_p(x) = \{h_q(\mathcal{C}\tbinom{q}{p}(x)) \mid q \in P, q \geq p\}$$

for all $x \in \mathcal{C}(p)$. The $\mu = \{\mu_p \mid p \in P\}$ has the required properties.

3. By the above lemma, it is sufficient to construct, for every infinite cardinal number $\varkappa$, an embedding of the diagram

$$\mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \xrightarrow{\pi_1} \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \xrightarrow{\pi_2} \mathsf{U}_\varkappa ,$$

where π_1 and π_2 both forget the first coordinate, into the diagram (1). We construct such an embedding $\mu = \{\mu_1, \mu_2, \mu_3\}$ by means of a collection

$$\mathcal{X} = \{A_i, B_i, C_i, D_i \mid i \in \varkappa\}$$

of connected metric spaces of the diameter 1 (we denote the metric of A_i by α_i, the metric of B_i, C_i, D_i by β_i, γ_i, δ_i if it is necessary to name explicitly the metric of some of the spaces A_i, B_i, C_i, D_i) such that the collection $\mathcal{X}$ fulfils all the statements (a), ... ,(f) below (a collection with these properties will be constructed in II.8–II.12).

(a) For every $i \in \varkappa$, the spaces A_i, B_i, D_i have a set $\mathcal{O}_i$ in common, their metrics coincide on $\mathcal{O}_i$ (i.e. $\alpha_i(x,y) = \beta_i(x,y) = \delta_i(x,y)$ for all $x, y \in \mathcal{O}_i$); moreover, there is a point $a_i \in \mathcal{O}_i$ such that $\mathcal{O}_i$ contains its $\frac{1}{2}$-ball in all the spaces A_i, B_i, D_i. Also, in each of the spaces C_i, a distinguished point is given, we denote it also by a_i (it can be the same point as in A_i).

(b) For every $i \in \varkappa$, there is a homeomorphism of B_i onto D_i which is identical on $\mathcal{O}_i$ (let us denote it by ζ_i).

(c) For every $i \in \varkappa$ there is a bijection ξ_i of A_i onto D_i, identical on $\mathcal{O}_i$, such that both ξ_i and ξ_i^{-1} are Lipschitz maps with the constant 2, i.e.

$$\delta_i(\xi_i(x), \xi_i(y)) \leq 2\alpha_i(x, y) \quad \text{and } \alpha_i(x, y) \leq 2\delta_i(\xi_i(x), \xi_i(y))$$

for all $x, y \in A_i$.

(d) The collection $\{C_i, D_i \mid i \in \varkappa\}$ is stiff in Top.

(e) The collection $\{B_i, C_i, D_i \mid i \in \varkappa\}$ is stiff in Unif.

(f) The collection $\{A_i, B_i, C_i, D_i \mid i \in \varkappa\}$ is stiff in Metr.

(Let us recall that a collection $\{o_j \mid j \in J\}$ of objects of a concrete category $\mathcal{K}$ is called stiff if it has the following property: if there is a nonconstant morphism $F\colon o_j \longrightarrow o_{j'}$, then necessarily $j = j'$ and f is the identity. In (d), we investigate the nonconstant continuous maps [the exact formulation is that

$$\{F_2(F_1(C_i)), F_2(F_1(D_i)) \mid i \in \varkappa\} \text{ is stiff in Top]};$$

in (e) we investigate the nonconstant uniformly continuous maps, in (f) the nonconstant non-expanding maps.)

4. We shall form products of collections consisting of some spaces of the collection $\mathcal{X} = \{A_i, B_i, C_i, D_i \mid i \in \varkappa\}$. (The products will be formed in the category Metr; we use the symbol $\prod$ or $\times$.) Since, in each of the spaces A_i, B_i, C_i, D_i, the distinguished point a_i is given, there is a distinguished point in each of such products, namely the point with all the coordinates equal to the corresponding a_i. The (metric !) subspace of the product consisting of the points which differ from the distinguished point only in finitely many coordinates, is called a reduced product and denoted by $\dot{\times}$ or $\dot{\prod}$. For every $f = (f_0, f_1, f_2) \in \omega^\varkappa \times \omega^\varkappa \times \omega^\varkappa$, let us put

$$A(f_0, f_1, f_2) = \dot{\prod_{i\in\varkappa}} A_i^{f_0(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} B_i^{f_1(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} C_i^{f_2(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} D_i^{\aleph_0}$$
$$B(f_1, f_2) = \dot{\prod_{i\in\varkappa}} B_i^{f_1(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} C_i^{f_2(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} D_i^{\aleph_0}$$
$$C(f_2) = \dot{\prod_{i\in\varkappa}} C_i^{f_2(i)} \dot{\times} \dot{\prod_{i\in\varkappa}} D_i^{\aleph_0} \quad ,$$

where $A_i^{f_0(i)}$ is the (reduced) product of $f_0(i)$ copies of the space A_i (if $f_0(i) = 0$, then it is a one-point space), $B_i^{f_1(i)}$ (or $C_i^{f_2(i)}$) is a (reduced) product of $f_1(i)$ (or $f_2(i)$) copies of the space B_i (or C_i) and $D_i^{\aleph_0}$ is the reduced product of $\aleph_0$ copies of the space D_i.

By (c), the bijection

$$\dot{\prod_{i\in\varkappa}} \xi_i^{f_0(i)}\colon \dot{\prod_{i\in\varkappa}} A_i^{f_0(i)} \longrightarrow \dot{\prod_{i\in\varkappa}} D_i^{f_0(i)}$$

is a uniform homeomorphism so that $\prod_{i\in\varkappa} A_i^{f_0(i)} \dot{\times} \prod_{i\in\varkappa} D_i^{\aleph_0}$ is uniformly homeomorphic to $\prod_{i\in\varkappa} D_i^{\aleph_0}$, hence $A(f_0,f_1,f_2)$ is uniformly homeomorphic to $B(f_1,f_2)$.

By (b), the bijection

$$\prod_{i\in\varkappa} \zeta^{f_1(i)}\colon \prod_{i\in\varkappa} B_i^{f_1(i)} \longrightarrow \prod_{i\in\varkappa} D_i^{f_1(i)}$$

is a homeomorphism so that $\prod_{i\in\varkappa} B_i^{f_1(i)} \dot{\times} \prod_{i\in\varkappa} D_i^{\aleph_0}$ is homeomorphic to $\prod_{i\in\varkappa} D_i^{\aleph_0}$, hence $B(f_1,f_2)$ is homeomorphic to $C(f_2)$.

Lemma C. *Let $f_2, g_2 \in \omega^\varkappa$. If $C(f_2)$ is homeomorphic to $C(g_2)$, then $f_2 = g_2$.*

PROOF : Let $h\colon C(f_2) \longrightarrow C(g_2)$ be a homeomorphism. We show that $f_2 \leq g_2$. Choose $j \in \varkappa$ and an embedding $e\colon C_j^{f_2(j)} \longrightarrow C(f_2)$ such that the composition of e with any projection of $C(f_2)$ onto D_i or onto any C_i with $i \neq j$ is constant. By (d), $h \circ e$ composed with any projection of $C(f_2)$ onto D_i or onto any C_i with $i \neq j$ is also constant. Hence h determines a homeomorphism of $C_j^{f_2(j)}$ into $C_j^{g_2(j)}$. This implies (see e.g. [T1] that $f_2(j) \leq g_2(j)$. ■

Lemma B. *Let $(f_1,f_2), (g_1,g_2) \in \omega^\varkappa \times \omega^\varkappa$. If $B(f_1,f_2)$ is uniformly homeomorphic to $B(g_1,g_2)$, then $(f_1,f_2) = (g_1,g_2)$.*

PROOF : This is quite analogous to the previous proof, we only use (e) and the fact that the projections are uniformly continuous. ■

Lemma A. *Let $(f_0,f_1,f_2), (g_0,g_1,g_2) \in \omega^\varkappa \times \omega^\varkappa \times \omega^\varkappa$. If $A(f_0,f_1,f_2)$ is isometric to $A(g_0,g_1.g_2)$, then $(f_0,f_1,f_2) = (g_0,g_1,g_2)$.*

PROOF : This is quite analogous to the previous proofs, we only use (f) and the fact that the projections are non–expanding maps.

6. Now, we define

$$\mu_0\colon \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \longrightarrow \Pi\mathrm{Metr}$$

as follows:

$$\mu_0(V_0,V_1,V_2) = \coprod_{j\in\varkappa}\Big(\coprod_{f_i\in V_i,\, i=0,1,2} A(f_0,f_1,f_2)\Big)_j \quad ,$$

i.e. $\mu_0(V_0,V_1,V_2)$ is a coproduct of $\varkappa$ copies of the coproduct

$$\coprod_{f_0\in V_0, f_1\in V_1, f_2\in V_2} A(f_0,f_1,f_2)$$

(the coproduct is in Metr, i.e. a disjoint union of the metric spaces in question with the distance equal to 1 for pairs of points of distinct spaces).

Lemma. *μ_0 is a monomorphism.*

PROOF : a) First, we show that μ_0 is a homomorphism. Let (V_0,V_1,V_2) and (W_0,W_1,W_2) in $\mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa$ be given. Since $\mu_0(V_0,V_1,V_2)$ (or $\mu_0(W_0,W_1,W_2)$) is a coproduct of $\varkappa$ copies of a coproduct of the system $\{A(f_0,f_1,f_2) \mid f_i \in V_i, i =$

$0,1,2\}$) the product (in Metr !) $\mu_0(V_0,V_1,V_2) \times \mu_0(W_0,W_1,W_2)$ is isometric to a coproduct of $\varkappa^2 = \varkappa$ copies of a coproduct of the system $\{A(f_0,f_1,f_2) \times A(g_0,g_1,g_2) \mid f_i \in V_i, g_i \in W_i, i = 0,1,2\}$. Since $\operatorname{card} V_i = \operatorname{card} W_i = \varkappa$ and $A(f_0,f_1,f_2) \times A(g_0,g_1,g_2)$ is isometric to $A(f_0+g_0, f_1+g_1, f_2+g_2)$, the coproduct of $\varkappa$ copies of the coproduct of the last system is isometric to the coproduct of $\varkappa$ copies of the coproduct of the system $\{A(h_0,h_1,h_2) \mid h_i \in V_i + W_i, i = 0,1,2\}$, i.e. to $\mu_0(V_0+W_0, V_1+W_1, V_2+W_2)$.

b) Now, we show that μ_0 is one–to–one. Let $(V_0,V_1,V_2) \neq (W_0,W_1,W_2)$, i.e. $V_0 \times V_1 \times V_2 \neq W_0 \times W_1 \times W_2$, and let us suppose that there is $(f_0,f_1,f_2) \in V_0 \times V_1 \times V_2 \setminus W_0 \times W_1 \times W_2$. Then $\mu_0(V_0,V_1,V_2)$ has a component isometric to $A(f_0,f_1,f_2)$, but there is no such component in $\mu_0(W_0,W_1,W_2)$, by Lemma A in II.5. ∎

7. We define

$$\mu_1 \colon \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \longrightarrow \Pi\ \text{Unif}$$
$$\mu_2 \colon \mathsf{U}_\varkappa \longrightarrow \Pi\ \text{Top}$$

by

$$\mu_1(V_1,V_2) = F_1\Big(\coprod_{j\in\varkappa}\Big(\coprod_{f_i\in V_i, i=1,2} B(f_1,f_2)\Big)_j\Big)\,,$$
$$\mu_2(V_2) = F_2 \circ F_1\Big(\coprod_{j\in\varkappa}\Big(\coprod_{f_2\in V_2} C(f_2)\Big)_j\Big)\,.$$

Since both the forgetful functors F_1, F_2 preserve coproducts, we obtain that $\mu_1(V_1,V_2)$ is a coproduct of $\varkappa$ copies of a coproduct (coproducts in Unif !) of the system $\{F_1(B(f_1,f_2)) \mid f_i \in V_i, i = 1,2\}$ and $\mu_2(V_2)$ is a coproduct of $\varkappa$ copies of a coproduct (coproducts in Top !) of the system $\{F_2 \circ F_1(C(f_2)) \mid f_2 \in V_2\}$. Since both the functors F_1, F_2 preserve finite products, the proof that μ_1 and μ_2 are homomorphisms is quite analogous to the proof that μ_0 is a homomorphism. The fact that μ_1 is one–to–one follows from the Lemma B in II.5, μ_2 is one–to–one by Lemma C in II.5. We conclude that the diagram

$$\begin{array}{ccccc}
\mathsf{U}_\varkappa \times \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa & \xrightarrow{\pi_1} & \mathsf{U}_\varkappa \times \mathsf{U}_\varkappa & \xrightarrow{\pi_2} & \mathsf{U}_\varkappa \\
{\scriptstyle \mu_0}\downarrow & & {\scriptstyle \mu_1}\downarrow & & {\scriptstyle \mu_2}\downarrow \\
\Pi\ \text{Metr} & \xrightarrow{F_1} & \Pi\ \text{Unif} & \xrightarrow{F_2} & \Pi\ \text{Top}
\end{array}$$

commutes, by II.4, so that $\mu = (\mu_0,\mu_1,\mu_2)$ is a required embedding.

8. It remains to show that a system

$$\mathcal{X} = \{A_i, B_i, C_i, D_i \mid i \in \varkappa\}$$

with the properties (a) – (f) in II.3 really does exist. The construction of such a system which we present in II.8 – II.12 below, is a simplified modification of the construction in $[\mathbf{T}_6]$; however, in $[\mathbf{T}_6]$, the properties (a) – (f) are not explicitly stated, so we present here briefly this modification, and point them out. First, we construct a triangle space T with three distinguished points t_0, t_1, t_2 by means of a countable disjoint system $\mathcal{S}$ of subcontinua of a Cook continuum [C] as in $[\mathbf{T}_6]$ (in another setting, the construction is also described in [PT], p. 223–4). We recall that T is a compact metric space of the diameter 1 and the distance of t_i and t_j is equal to 1 for $i \neq j$. We need here two triangle spaces, say T_1 and T_2, their distinguished points are denoted by $t_{i,0}$, $t_{i,1}$, $t_{i,2}$, $i = 1,2$, constructed by means of countable disjoint systems $\mathcal{S}_1$, $\mathcal{S}_2$ of a Cook continuum such that $\mathcal{S}_1 \cup \mathcal{S}_2$ is also disjoint. Then, by $[\mathbf{T}_6]$, the spaces T_1 and T_2 have following property ($c\ell$ denotes the closure in the corresponding topological space):

(*) Put $M_1 = T_1$, $N_1 = \{t_{1,0}, t_{1,1}, t_{1,2}\}$, $M_2 = T_2 \setminus \{t_{2,2}\}$, $N_2 = \{t_{2,0}, t_{2,1}\}$. Let $i \in \{1,2\}$, let Y be a metrizable space containing the [topological] space $M_i \cap c\ell(Y \setminus M_i) \subseteq N_i$. If $j \in \{1,2\}$ and $f\colon M_j \longrightarrow Y$ is a nonconstant continuous map, then either $f(M_j) \subseteq c\ell(Y \setminus M_i)$ or $i = j$ and f is the inclusion, i.e. $f(m) = m$ for all $m \in M_j$.

We form a topological space S from a disjoint union of T_1 and $T_2 \setminus \{t_{2,2}\}$, identifying $t_{1,0}$ with $t_{2,0}$ and $t_{1,1}$ with $t_{2,1}$ (for simplicity, we suppose that $S = T_1 \cup (T_2 \setminus \{t_{2,2}\})$ and $T_1 \cap (T_2 \setminus \{t_{2,2}\}) = \{t_0, t_1\}$, $t_i = t_{1,i} = t_{2,i}$ for $i = 0,1$; moreover, we denote $t_{1,2}$ by t_2. We investigate the following three metrics σ_0, σ_1, σ_2 on S: we start from the metric τ of T_1 and μ of T_2 and define the following metrics on $T_2 \setminus \{t_{2,2}\}$:

$$\mu_0 = \mu \qquad \mu_1 = \min(1, 2\mu)$$
$$\mu_2(x,y) = \min(1, 2\mu(x,y) + |\mu(x,t_{2,2})^{-1} - \mu(y,t_{2,2})^{-1}|) \,.$$

Then we put

$$\sigma_i(x,y) = \begin{cases} \tau(x,y) \text{ if } x,y \in T_1 \\ \mu_i(x,y) \text{ if } x,y \in T_2 \setminus \{t_{2,2}\} \\ \min(1, \tau(x,t_0) + \mu_i(t_0,y), \tau(x,t_1) + \mu_i(t_1,y) \text{ if } x \in T_1, y \in T_2 \setminus \{t_{2,2}\} \end{cases}$$

i.e. (S, σ_i) is a quotient in the category Metr of a coproduct (coproduct also in Metr) $(T_1, \tau) \coprod (T_2 \setminus \{t_{2,2}\}, \mu_i)$, given by the identification $t_{1,0} = t_{2,0}$ and $t_{1,1} = t_{2,1}$.

In the diagram

$$(S, \sigma_2) \xrightarrow{\ \imath_2\ } (S, \sigma_1) \xrightarrow{\ \imath_1\ } (S, \sigma_0)\,,$$

both $\imath_2$ and $\imath_1$ denote the identity map of the set S onto itself. The construction of the metrics σ_0, σ_1, σ_2 implies immediately that $\imath_1$ is a non–expanding map and $\imath_1^{-1}$ is not non–expanding but it is a Lipschitz map with the constant 2, $\imath_2$ is uniformly continuous and $\imath_2^{-1}$ is continuous but not uniformly continuous and all the maps

$\imath_2$, $\imath_2^{-1}$, $\imath_1$, $\imath_1^{-1}$ are isometries on the subspace T_1 of all the three spaces (S,σ_i), $i = 0,1,2$.

9. Let $G = (V; R_0, R_1, R_2)$ be a quadruple, where V is a nonempty set (of vertices) and R_0, R_1, R_2 are three binary relations on it such that

(V, R_2) is a connected directed graph without loops (i.e. $R_2 \subseteq V \times V$, never $(v,v) \in R_2$ and for every $v, w \in V$, [not necessarily distinct] there exist $v_0 = v, v_1, \ldots, v_n = w$ in V such that $(v_i, v_{i+1}) \in R_2 \cup R_2^{-1}$ for $i = 0, \ldots, n-1$)

$R_0 \subseteq R_1 \subseteq R_2$.

Depending on these data, we create a metric space $M(G)$ as follows (see [$\mathbf{T}_6$]):

(1) for each $r \in R_0$, we denote by Z^r a copy of (S, σ_0),

(2) for each $r \in R_1 \setminus R_0$, we denote by Z^r a copy of (S, σ_1),

(3) for each $r \in R_2 \setminus R_1$, we denote by Z^r a copy of (S, σ_2).

The distinguished points t_0, t_1, t_2 of the copy Z^r are denoted by t_0^r, t_1^r, t_2^r. The metric space $M(G)$ is the quotient (the quotient formed in the category Metr !) of a coproduct (the coproduct also in Metr)

$$\coprod_{r \in R_2} Z^r$$

given by the following identifications (where $\pi_0(x,y) = x$, $\pi_1(x,y) = y$):

t_0^r with $t_0^{r'}$ whenever $\pi_0(r) = \pi_0(r')$

t_0^r with $t_1^{r'}$ whenever $\pi_0(r) = \pi_1(r')$

t_1^r with $t_1^{r'}$ whenever $\pi_1(r) = \pi_1(r')$

t_2^r with $t_2^{r'}$ for all r, $r' \in R_2$; let us denote by a_G the point of $M(G)$ obtained by this last identification.

Let us denote by $T(G)$ the subspace of $M(G)$ obtained by the above identifications from $\coprod_{r \in R_2} T_1^r$ (where T_1^r is the subspace of Z^r corresponding to the subspace T_1 of (S, σ_i)). Then $T(G)$ is a neighbourhood of a_G in $M(G)$, it contains the open 1-ball of a_G in $M(G)$.

10. <u>Observation:</u> Let $G = (V; R_0, R_1, R_2)$ and $G' = (V'; R_0', R_1', R_2')$ be two quadruples as above and $M(G)$, $M(G')$ be the metric spaces constructed by means of them as described in II.9. If

$$V = V' \text{ and } R_2 = R_2' ,$$

then the spaces $M(G)$ and $M(G')$ have the same underlying set and the identity map

$$\imath \colon M(G) \longrightarrow M(G')$$

is a homeomorphism which is an isometry on $T(G) = T(G')$. If, moreover,

$$R_1 = R_1' ,$$

then $\imath$ is a uniform homeomorphism such that both $\imath$ and $\imath^{-1}$ are Lipschitz maps with the constant 2.

11. Let $G = (V; R_0, R_1, R_2)$ and $G' = (V'; R_0', R_1', R_2')$ be two quadruples as above. Then any map

$$h: V \longrightarrow V'$$

which is R_2R_2'-compatible (i.e. $(x,y) \in R_2 \implies (h(x), h(y)) \in R_2'$) determines a nonconstant continuous map

$$M(h): M(G) \longrightarrow M(G')$$

such that it sends each copy Z^r in $M(G)$ with $r = (v, w)$ "identically" onto the copy $Z^{r'}$ in $M(G')$, where $r' = (h(v), h(w))$ (since the graph (V, R_2) is connected, this rule really gives a nonconstant continuous map of $M(G)$ into $M(G')$). If, moreover, h is R_1R_1'-compatible, then $M(h)$ is uniformly continuous. If, moreover, h is also R_0R_0'-compatible, then $M(h)$ is non-expanding. The property (*) of the triangle spaces T_1 and T_2 (see II.8) implies that the converse is also true (see [T_6]): if $f: M(G) \longrightarrow M(G')$ is a nonconstant continuous map, then there exists a (unique !) map $h: V \longrightarrow V'$ which is R_2R_2'-compatible such that $f = M(h)$; if f is uniformly continuous, then h is also R_1R_1'-compatible; and if f is non-expanding, then h is also R_0R_0'-compatible.

12. Now, it is already easy to see that the system $\mathcal{X} = \{A_i, B_i, C_i, D_i \mid i \in \varkappa\}$ of metric spaces with the properties (a) – (f) in II.3 can be obtained as the system of the spaces $M(G)$ for suitable quadruples G. Let us denote by

$$G(X_i) = (V(X_i); R_0(X_i), R_1(X_i), R_2(X_i)) ,$$

where $i \in \varkappa$ and X denotes some of the symbols A, B, C, D, the quadruples which "should create" the metric space X_i. We obtain the system $\mathcal{X}$ of metric spaces with the properties (a) – (f) whenever the system of quadruples has all the following properties (i), (ii), (iii):

(i) $V(A_i) = V(B_i) = V(D_i)$, $R_2(A_i) = R_2(B_i) = R_2(D_i)$ and the system $\{(V(X_i); R_2(X_i)) \mid i \in \varkappa, X \in \{C, D\}\}$ of directed graphs (connected, without loops) is rigid in the category $\mathfrak{G}_2$ of all directed connected graphs without loops, and all compatible maps. (Let us recall that a category is called discrete if it has no morphisms except the unities; and $C \subseteq \operatorname{obj} \mathcal{K}$ is called rigid in a category $\mathcal{K}$ if the full subcategory of $\mathcal{K}$ with C as its class of objects is discrete.)

(ii) $R_1(A_i) = R_1(D_i)$ and the system $\{(V(X_i); R_1(X_i), R_2(X_i)) \mid i \in \varkappa, X \in \{B, C, D\}\}$ is rigid in the category $\mathfrak{G}_1$ of all triples $(V; R_1, R_2)$ where $(V, R_2) \in \operatorname{obj} \mathfrak{G}_2$ and $R_1 \subseteq R_2$ and all maps which are compatible with respect to the both relations.

(iii) The system $\{(V(X_i); R_0(X_i), R_1(X_i), R_2(X_i)) \mid i \in \varkappa, X \in \{A, B, C, D\}\}$ is rigid in the category $\mathfrak{G}_0$ of all quadruples $(V; R_0, R_1, R_2)$, where $(V; R_1, R_2) \in \operatorname{obj} \mathfrak{G}_1$, $R_0 \subseteq R_1$ and all maps which are compatible with respect to all the three relations.

However, this is also ready. A special case of Lemma 2.4 in [T$_5$] gives the following result: For arbitrary diagram of small categories and faithful functors

$$k_0 \xrightarrow{\Psi_0} k_1 \xrightarrow{\Psi_1} k_2$$

(i.e. k_0, k_1, k_2 are small categories, Ψ_0, Ψ_1 are faithful functors), there exists its simultaneous representation in the diagram

$$\mathfrak{S}_0 \xrightarrow{\Phi_0} \mathfrak{S}_1 \xrightarrow{\Phi_1} \mathfrak{S}_2$$

(where Φ_0, Φ_1 are the forgetful functors (i.e. $\Phi_0(V; R_0, R_1, R_2) = (V; R_1, R_2)$, $\Phi_1(V; R_1, R_2) = (V; R_2)$), i.e. there exist full faithful functors $\Lambda_j\colon k_j \longrightarrow \mathfrak{S}_j$, $j = 0, 1, 2$, such that the diagram below commutes:

$$\begin{array}{ccccc} k_0 & \xrightarrow{\Psi_0} & k_1 & \xrightarrow{\Psi_1} & k_2 \\ \downarrow{\scriptstyle \Lambda_0} & & \downarrow{\scriptstyle \Lambda_1} & & \downarrow{\scriptstyle \Lambda_2} \\ \mathfrak{S}_0 & \xrightarrow{\Phi_0} & \mathfrak{S}_1 & \xrightarrow{\Phi_1} & \mathfrak{S}_2 \end{array}$$

To obtain the system of quadruples which fulfills (i), (ii), (iii), it suffices to choose all categories k_0, k_1, k_2 discrete and

$$\begin{aligned} \mathrm{obj}k_2 &= \{o(X_i) \mid i \in \varkappa,\ X \in \{C, D\}\} \\ \mathrm{obj}k_1 &= \{o(X_i) \mid i \in \varkappa,\ X \in \{B, C, D\}\} \\ \mathrm{obj}k_0 &= \{o(X_i) \mid i \in \varkappa,\ X \in \{A, B, C, D\}\} \end{aligned}$$

and

$$\begin{aligned} \Psi_0(o(A_i)) &= \Psi_0(o(D_i))\ , \quad \Psi_0(o) = o \text{ else} \\ \Psi_1(o(B_i)) &= \Psi_0(o(D_i))\ , \quad \Psi_1(o) = o \text{ else.} \end{aligned}$$

III. Concluding remarks.

A) The following strengthening of the Main Theorem can be obtained by an easy modification of its proof.

Given a metric space M of diameter ≤ 1, every diagram (2) can be embedded in the diagram (1) by an embedding $\psi = (\psi_0, \psi_1, \psi_2)$ such that each space of the isomorphism type $\psi_0(s)$, $s \in S_0$, contains M as a retract in Metr (i.e. if $X \in \psi_0(s)$, then there are non-expanding maps $M \xrightarrow{c} X$ and $X \xrightarrow{r} M$ with $r \circ c$ identity).

We sketch briefly the needed modifications (this trick was already used many times): we construct the embedding $\varphi = (\varphi_0, \varphi_1, \varphi_2)$ precisely as in II, only the system $\mathcal{X}$ in II.3 (with the properties (a) – (f) is supposed to be a subsystem of a system $\mathcal{X} \cup \{Z\}$, where Z is a metric space such that

$$\begin{aligned} &\mathcal{X} \cup Z \text{ is stiff in Metr,} \\ &\{B_i, C_i, D_i \mid i \in \varkappa\} \cup \{Z\} \text{ is stiff in Unif,} \\ &\{C_i, D_i \mid i \in \varkappa\} \cup \{Z\} \text{ is stiff in Top.} \end{aligned}$$

Such larger system $\mathcal{X} \cup \{Z\}$ can be constructed precisely as in II.8 – II.12, only the small categories k_0, k_1, k_2 in II.12 are selected to have one object $o(Z)$ more.

For every $s \in S_0$, we choose a metric space X_s of the isomorphism type $\varphi_0(s)$ and put $\psi_0(s)$ to be the isomorphism type of the metric space

$$Y_s = \coprod_{n=0}^{\infty} X_s \times (M \times Z)^n$$

(all products and coproducts are in Metr). Then

α) M is a retract of $M \times Z$, hence of Y_s;

β) $Y_{s+s'}$ is isometric to $Y_s \times Y_{s'}$ (because $\varkappa \geq \aleph_0$);

γ) the subspace $X_s = X_s \times (M \times Z)^0$ of Y_s can be recognized from Y_s: it consists of all its components C such that each non-expanding map $Z \longrightarrow C$ is constant; hence if $s \neq s'$, Y_s is not isometric to $Y_{s'}$, so that ψ_0 is one-to-one.

We proceed similarly for ψ_1 and ψ_2.

Thus, the consequences of the Main Theorem, described in I (and in the abstract) can be enriched by the requirement that any metric space of diameter ≤ 1 can be embedded as a retract in a space X with the curious properties.

B) In [**AK**], Adámek and Koubek investigate sum-productive representations of ordered commutative semigroups. This setting can be transformed to the present "simultaneous" approach without any difficulty and the Main Theorem can be strengthened as follows: we investigate ΠMetr, ΠUnif, ΠTop (partially) ordered by the rule

$\overline{o} \leq \overline{o}'$ iff o' is a coproduct of o and of an object (possibly empty) of the category in question

(where o, o' are objects of Metr or Unif or Top and the strip denotes the isomorphism type). Then

> any diagram (2) of ordered commutative semigroups S_0, S_1, S_2 and order-preserving homomorphisms h_0, h_1 can be embedded in the diagram (1) by an embedding $\varphi = (\varphi_0, \varphi_1, \varphi_2)$, where φ_i are order-preserving monomorphisms.

To obtain this strengthening of the Main Theorem, quite easy and straightforward modifications can be done: by [**AK**], every ordered commutative semigroup S can be embedded into $\mathsf{U}_\varkappa$ (with $\varkappa \geq \aleph_0 \cdot \operatorname{card} S$), ordered by inclusion, by an order-preserving monomorphism. We modify Lemma in II.2 in the evident way and the rest of the proof is practically the same (only some formal changes are necessary in the part b) of the proof of the Lemma in II.6 and II.7).

C) The strengthenings of the Main Theorem, described in III.A) and in III.B) can both be investigated simultaneously. The formulation is left to the reader.

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