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Measure–determined enlargements of Boolean σ-algebras

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Dedicated to the memory of Zdeněk Frolík

Abstract. Let B be a Boolean σ-algebra and let $\mathcal{M}(B)$ denote the set of all probability measures on B. We say that B admits measure–determined enlargements if B fulfils the following condition: If C is a Boolean σ-algebra, then B can be embedded into a Boolean σ-algebra D such that $\mathcal{M}(C) = \mathcal{M}(D)$. We show in this note that B admits measure–determined enlargements if and only if it possesses a two–valued measure.

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Notions and results.

Definition 1. Let B be a Boolean σ-algebra. A mapping $m: B \longrightarrow \langle 0,1\rangle$ is called a measure on B if

(i) $m(1) = 1$,
(ii) whenever $\{b_i \mid i \in N\}$ is a countable subset of B such that $b_i \wedge b_j = 0$ for distinct $i, j \in N$, then $m(\bigvee_{i\in N} b_i) = \sum_{i\in N} m(b_i)$.

Let us denote by $\mathcal{M}(B)$ the set of all measures on B. As usual, we put $\mathcal{M}(B) = \mathcal{M}(D)$ if $\mathcal{M}(B)$ is affinely isomorphic to $\mathcal{M}(D)$. The question we ask (and answer) here is if (when) a given Boolean σ-algebra admits enlargements to Boolean σ-algebras with preassigned measure spaces. (This question is relevant to the axiomatics of quantum theories ([2]). In its orthomodular setup it has been analyzed in [3], [4] and [5]. In the realm of Boolean σ-algebras it presents quite a different problem, of course.)

We shall need a few lemmas. In their proofs we sometimes omit technicalities assuming that the reader is well acquainted with standard methods of Boolean algebras (see [6]). We call an injective σ-homomorphism $e: B_1 \longrightarrow B$ an embedding; in this case we say that B_1 is a subalgebra of B or that B is an enlargement of B_1.

Lemma 1. *Let B_1, B_2 be Boolean σ-algebras. Then there exists a Boolean σ-algebra B such that the following properties are satisfied:*

(i) *There are embeddings $e_1: B_1 \longrightarrow B$ and $e_2: B_2 \longrightarrow B$ such that the set $e_1(B_1) \cup e_2(B_2)$ generates B and moreover, $e_1(a_1) \wedge e_2(a_2) = 0$ if and only if either $a_1 = 0$ or $a_2 = 0$,*
(ii) *if $m_1 \in \mathcal{M}(B_1)$ and $m_2 \in \mathcal{M}(B_2)$, then there exists exactly one measure $m \in \mathcal{M}(B)$ such that $m(e_1(a_1) \wedge e_2(a_2)) = m_1(a_1) \cdot m_2(a_2)$ for any $a_1 \in B_1$ and $a_2 \in B_2$.*

PROOF : We could take for B the maximal σ-product of B_1 and B_2 and apply [6], Theorem 38.14 (it contains our Lemma 1 as a special case). To maintain the self-containedness of this note as well as diminish the reader's inconvenience, let us sketch the argument. Let $C(T_1)$ (resp. $C(T_2)$) be the topological representation of B_1 (resp. B_2). Thus, $C(T_1)$ (resp. $C(T_2)$) is the Boolean algebra of all clopen subsets of a compact space T_1 (resp. T_2) and there is a Boolean isomorphism $i_1: B_1 \longrightarrow C(T_1)$ (resp. $i_2: B_2 \longrightarrow C(T_2)$). As known (Loomis–Sikorski theorem), if we denote by (T_1, Σ_1) (resp. (T_2, Σ_2)) the σ-field generated by the set $S_1 = \{i_1(b) \mid b \in B_1\}$ (resp. $S_2 = \{i_2(b) \mid b \in B_2\}$), then the σ-algebra B_1 is σ-isomorphic to Σ_1/I_1 (resp. Σ_2/I_2), where I_1 (resp. I_2) is the σ-ideal of meagre sets in T_1 (resp. T_2). Suppose that $(T_1 \times T_2, \Sigma_1 \times \Sigma_2)$ is the usual σ-product of the σ-fields Σ_1 and Σ_2, and denote by I the σ-ideal of subsets of $T_1 \times T_2$ generated by the set $\{A_1 \times T_2 \mid A_1 \in I_1\} \cup \{T_1 \times A_2 \mid A_2 \in I_2\}$. Then if we put $B = \Sigma_1 \times \Sigma_2/I$ and take the natural embeddings $e_1: B_1 \longrightarrow B$, $e_2: B_2 \longrightarrow B$, we find that the required properties (i), (ii) are satisfied. (The property (ii) follows by transferring the standard extension construction from the σ-field $\Sigma_1 \times \Sigma_2$ to B.) ∎

Prior to the following lemmas, let us call B *measureless* if $\mathcal{M}(B) = \emptyset$, and let us call it *poor* if $\mathcal{M}(B)$ is a singleton.

Lemma 2. *Every Boolean σ-algebra can be embedded into a measureless one.*

PROOF : Suppose that B_1 is a measureless Boolean σ-algebra (it is known that such a σ-algebra does exist, see e.g. [1]). If we are given an arbitrary Boolean σ-algebra, say B_2, then both B_1, B_2 can be embedded into a Boolean σ-algebra B (Lemma 1). Since B_1 is measureless, B has to be measureless, too, and therefore B_2 can be embedded into a measureless algebra. The proof is complete. ∎

Lemma 3. *If B is a measureless Boolean σ-algebra and $\{0,1\}$ denotes the two point Boolean algebra, then $B \times \{0,1\}$ is a poor Boolean σ-algebra.*

PROOF : Suppose that $m \in \mathcal{M}(B \times \{0,1\})$. Since $(1,0) \vee (0,1) = (1,1)$, we see that $m(1,0) + m(0,1) = 1$. If $m(1,0)$ is positive then we could easily construct a measure on B. This cannot be done. It follows that $m(1,0) = 0$ and therefore $m(b,0) = 0$ for any $b \in B$. Consequently, $m(0,1) = 1$ and therefore $m(b,1) = 1$ for any $b \in B$. We have obtained necessary values for the measure $m \in \mathcal{M}(B \times \{0,1\})$ to attain. It remains to be verified that these values really define a measure, but this is easy. We infer that $B \times \{0,1\}$ has exactly one measure and the proof of Lemma 3 is complete. ∎

Lemma 4. *If B is poor and $m \in \mathcal{M}(B)$, then m is two–valued.*

PROOF : Suppose that there is an element $b \in B$ such that $0 < m(b) < 1$. If we put $m_1(a) = m(b)^{-1} \cdot m(a \wedge b)$ $(a \in B)$ and $m_2(a) = (1 - m(b))^{-1} \cdot m(a \wedge b)$ $(a \in B)$, then m_1, m_2 are two distinct measures on B. This contradicts our assumption. ∎

Lemma 5. *Let B posses a two–valued measure. Then B can be embedded into a poor σ-algebra.*

PROOF : By Lemma 2, B can be embedded into a measureless σ-algebra. Thus, we have an embedding $e: B \longrightarrow C$, where $\mathcal{M}(C) = \emptyset$. By Lemma 3, the σ-algebra

$C \times \{0,1\}$ is poor. Let $m \in \mathcal{M}(B)$ be the two–valued measure which is guaranteed by the assumption. Define now a mapping $f: B \longrightarrow C \times \{0,1\}$ as follows: If $m(b) = 1$, we put $f(b) = (e(b), 1)$; if $m(b) = 0$, we put $f(b) = (e(b), 0)$. We have to verify that f is an embedding. Obviously, f is injective and $f(0) = 0$. Since m is two–valued, we have $f(b') = (f(b))'$. Finally, if $\{b_i \mid i \in N\}$ is a subset of mutually disjoint elements in B, then $m(b_i) = 1$ for at most one index $i \in N$. Thus, $f(\bigvee_{i \in N} b_i) = \bigvee_{i \in N} f(b_i)$ and we conclude that f is indeed an embedding. Since $C \times \{0,1\}$ is poor, the prof of Lemma 5 is complete. ∎

We are now ready to state our result.

Theorem. *Let B be a Boolean σ-algebra. Then B admits measure-determined enlargements if and only if B possesses a two-valued measure.*

PROOF : Let us first show that the assumption on B is necessary. If $\mathcal{M}(B) = \emptyset$, then B cannot be embedded into any D with $\mathcal{M}(D) \neq \emptyset$. If $\mathcal{M}(B) \neq \emptyset$ and no measure $m \in \mathcal{M}(B)$ is two–valued, then B cannot be embedded into a poor σ-algebra (Lemma 4). Thus, our assumption is necessary.

Let us now take up the sufficiency. Suppose that there is a two–valued measure in $\mathcal{M}(B)$ and suppose that we are given a Boolean σ-algebra, C. We have to find a Boolean σ-algebra D such that D contains B and $\mathcal{M}(D) = \mathcal{M}(C)$. By Lemma 5, B can be embedded into a poor σ-algebra. Thus, we have an embedding $f: B \longrightarrow E$, where $\operatorname{card} \mathcal{M}(E) = 1$. Moreover, the only measure $t \in \mathcal{M}(E)$ is two–valued (Lemma 4). Let us now apply Lemma 1 to σ-algebras E and C. We obtain a σ-algebra D and embeddings $e_1: E \longrightarrow D$, $e_2: C \longrightarrow D$ such that e_1, e_2 satisfy the conditions (i), (ii) of Lemma 1. We are going to show that $\mathcal{M}(D) = \mathcal{M}(C)$. According to the condition (ii), it suffices to check that for every measure $m \in \mathcal{M}(D)$ there exists a measure $k \in \mathcal{M}(C)$ such that $m(e_1(a_1) \wedge e_2(a_2)) = t(a_1)k(a_2)$ $(a_1 \in E$, $a_2 \in C)$. But this is obvious because t is two–valued and we can therefore put $k(a) = m(e_2(a))$ $(a \in C)$. It follows that $\mathcal{M}(D) = \mathcal{M}(C)$. Finally, if we put $g = e_1 \circ f$, we obtain an embedding of B into D. The proof is complete. ∎

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