Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 1, 95--98

Persistent URL: http://dml.cz/dmlcz/106822

On the simplicity of some categories of closure spaces

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Dedicated to the memory of Zdeněk Frolík

Abstract. For the category Pretop of closure spaces, we investigate the simplicity of its epireflective subcategories defined by some separation properties . The subcategories defined by the T_1, T_2 and T_3 properties are non-simple as follows easily from the corresponding results for Top.

For the T_0 property, however, the result for Pretop is quite the opposite of what happens in Top. It is well known that T_0Top being the epireflective hull of the Sierpiński space is simple. We show that T_0Pretop on the other hand is not simple. In fact our main theorem states the much stronger result that a closure space Y is T_0 if and only if there exists a T_0 closure space X (with at least two points) such that every continuous function from X in Y is constant.

Keywords: closure space, simple subcategory

Classification: 54A05, 54B30, 18B99

A closure space consists of a set structured by a (not necessarily idempotent closure operator [2]. The category of closure spaces and continuous maps is isomorphic to the category of neighborhoodspaces in the sense of [4] and to the category of principal or Pretopological spaces in the sense of [3] and [5]. It is a topological category which we shall denote by Pretop. Top is a full and bireflective subcategory of Pretop. Separation axioms as T_0, T_1, T_2 and T_3 are defined quite analogously as in Top. See e.g. [5] for detailed definitions.

Subcategories will be assumed to be full and isomorphism closed. In general, when $\mathcal{H}$ is a topological category, every subcategory $\mathcal{E}$ of $\mathcal{H}$ is contained in a smallest epireflective subcategory, the epireflective hull in $\mathcal{H}$, which is denoted by H$\mathcal{E}$. An object A of $\mathcal{H}$ belongs to H$\mathcal{E}$ if and only if it is a subobject of a product of objects in $\mathcal{E}$. In this context "Y is a subobject of X" means that there exists an embedding from Y to X, so Y is an extremal subobject in the categorical sense. For further details on these notions we refer to [8].

A subcategory $\mathcal{L}$ of $\mathcal{H}$ is called simple (in $\mathcal{H}$) if there exists a single object E of $\mathcal{L}$ such that $\mathcal{L}$ is the epireflective hull of $\{E\}$ i.e. $\mathcal{L}$ =H$\{E\}$. It is well known that Pretop itself is simple, since Pretop = H$\{E\}$, where E is the closure space on the set $\{0,1,2\}$ with smallest neighborhoods $V_0 = \{0,1\}$, $V_1 = \{0,1,2\}$, $V_2 = \{0,1,2\}$ for $0, 1, 2$ respectively.

In this paper, we want to investigate the simplicity of the epireflective subcategories T_0Pretop, T_1Pretop, T_2Pretop and T_3Pretop consisting of closure spaces having the T_0, T_1, T_2 or T_3 property respectively. The parallel questions in Top have been answered for quite some time. Top itself and T_0Top are simple. On the other hand T_1Top, T_2Top and T_3Top are not simple [6] and these negative results all follow from the next very deep result of Herrlich [7].

Theorem 1 (Herrlich). *If Y is a topological space, then the following are equivalent:*

(a) *Y is T_1.*
(b) *There exists a T_3 topological space X (with at least two points) such that every continuous map from X to Y is constant.*

Since the bireflector from Pretop to Top preserves the T_1 property, the previous theorem remains valid when Y is supposed to be a closure space. As a consequence we immediately obtain the following situation which is quite parallel to the topological case.

Corollary 1. *Whenever $\mathcal{H}$ is an epireflective subcategory of Pretop such that T_3Top $\subset \mathcal{H} \subset T_1$ Pretop, then $\mathcal{H}$ is not simple. In particular T_1Pretop, T_2Pretop and T_3Pretop are not simple.*

We will show that for the T_0 case, however, the parallelism between the Top and Pretop case does not go through, as was conjectured by E. Giuli. We prove that T_0Pretop is not simple. The proof of this fact will follow from an stronger result proved in Theorem (2) which can be seen as a counterpart of Theorem (1) for T_0 closure spaces.

Construction.

Let X_0 be the closure space containing only one point. If X_α is already defined for some ordinal α, then construct $X_{\alpha+1}$ as follows. For the underlying set we take:

$$X_{\alpha+1} = X_\alpha \cup \{x_0^\alpha, x_1^\alpha\} \quad \text{where} \quad x_0^\alpha, x_1^\alpha \notin X_\alpha.$$

The closure structure of $X_{\alpha+1}$ is defined by means of the neighborhoodfilters:

$$\begin{aligned} \mathcal{W}_{\alpha+1}(x_0^\alpha) &= \langle\{x_0^\alpha, x_1^\alpha\}\rangle \\ \mathcal{W}_{\alpha+1}(x_1^\alpha) &= \langle X_\alpha \cup \{x_1^\alpha\}\rangle \\ \mathcal{W}_{\alpha+1}(x) &= \langle\{V \cup \{x_0^\alpha\} \mid V \in \mathcal{W}_\alpha(x)\}\rangle \quad \text{when} \quad x \in X_\alpha \end{aligned}$$

If α is a limit ordinal and X_β has been defined for every $\beta < \alpha$, then we take $X_\alpha = \cup\{X_\beta \mid \beta < \alpha\}$, as an underlying set for X_α.

Let $i_\beta : X_\beta \to X_\alpha$ be the canonical injection and consider the sink $(X_\beta \xrightarrow{i_\beta} X_\alpha)_{\beta<\alpha}$. Then the closure structure of X_α is the final Pretop structure of the sink.

Lemma 1. *The closure spaces X_α all have the T_0 property.*

PROOF : First remark that X_0 clearly is T_0. Using transfinite induction, assume the spaces X_β, for $\beta < \alpha$, all have the T_0 property. If α is a successor ordinal, $\alpha = \gamma + 1$, then clearly the T_0 property of X_γ implies the T_0 property of X_α. If α is a limit ordinal then take $x \neq x'$ in X_α.

Further let $\beta < \alpha$ be an ordinal such that x and x' both belong to X_β and suppose there exists a $U \in \mathcal{W}_\beta(x)$ such that $x' \notin U$. Then, for any $\beta \leqslant \gamma < \alpha$ there is a $U_\gamma \in \mathcal{W}_\gamma(x)$ such that $U_\gamma \cap X_\beta = U$. Now let $U_\alpha = \cup\{U_\gamma | \beta \leqslant \gamma < \alpha\}$, then clearly $U_\alpha \in \mathcal{W}_\alpha(x)$ and $U_\alpha \cap X_\beta = U$. Hence $x' \notin U_\alpha$. ∎

Lemma 2. *Let Y be any T_0 closure space, α any ordinal and f any continuous map $f : X_\alpha \to Y$. If $x \neq x'$ both belong to X_α and $f(x) = f(x')$, then $f|_{X_\lambda}$ is constant, where the ordinal λ is defined as :*

$$\lambda = \min\{\xi | \xi \leqslant \alpha, \quad x \quad \textit{and} \quad x' \in X_\xi\}.$$

PROOF : The statement is obviously true for $\alpha = 0$. Using transfinite induction, assume the statement holds for any ordinal $\beta < \alpha$. If α is a successor ordinal, $\alpha = \gamma + 1$ then suppose $f : X_\alpha \to Y$ is continuous, $x \neq x'$, x and x' belong to X_α and $f(x) = f(x')$. ∎

The following cases can occur:

(a) $x = x_0^\gamma$ and $x' = x_1^\gamma$.
Put $y = f(x) = f(x')$ and let $z \in X_\gamma$ be arbitrary. For any neighborhood U of y we have $f^{-1}(U) \in \mathcal{W}_{\gamma+1}(x_1^\gamma)$, hence $X_\gamma \subset f^{-1}(U)$ and $f(z) \in U$. On the other hand, if V is a neighborhood of $f(z)$, then we have $x_0^\gamma \in f^{-1}(V)$ and $y \in V$. The T_0 property then implies $f(z) = y$. So we can conclude that f is constant on X_α.

(b) $x \in X_\gamma$ and $x' = x_0^\gamma$.
Put $y = f(x) = f(x')$. For any neighborhood U of y we have $f^{-1}(U) \in \mathcal{W}_{\gamma+1}(x_0^\gamma)$ and then $f(x_1^\gamma) \in U$. On the other hand, for any neighborhood V of $f(x_1^\gamma)$ we have $X_\alpha \subset f^{-1}(V)$ and then $y \in V$. So we can conclude that $f(x_0^\gamma) = f(x_1^\gamma)$ and then we can apply the previous case.

(c) $x \in X_\gamma$ and $x' = x_1^\gamma$.
Put $y = f(x) = f(x')$. For any neighborhood U of y we have $f^{-1}(U) \in \mathcal{W}_{\gamma+1}(x)$ and then $f(x_0^\gamma) \in U$. On the other hand, if V is any neighborhood of $f(x_0^\gamma)$,then $y = f(x_1^\gamma) \in V$. So we again can conclude that $f(x_0^\gamma) = f(x_1^\gamma)$ and then we can apply case a).

(d) x and $x' \in X_\gamma$.
Clearly $\lambda = \min\{\xi | \xi \leqslant \alpha \; x \text{ and } x' \in X_\xi\} \leqslant \gamma$.
Apply the induction hypothesis on the function $f|_{X_\gamma}$ to conclude that $f|_{X_\lambda}$ is constant.

Finally, if α is a limit ordinal, again suppose $f : X_\alpha \to Y$ is continuous, $x \neq x'$, x and x' both belong to X_α and $f(x) = f(x')$.

Then there is an ordinal $\gamma < \alpha$ such that x and x' both belong to X_γ. Moreover as in case d) $\lambda = \min\{\xi | \xi \leqslant \alpha, \; x \text{ and } x' \in X_\xi\} \leqslant \gamma$. Applying the induction hypothesis to $f|_{X_\gamma}$ we can conclude that $f|_{X_\lambda}$ is constant.

Theorem 2. *If Y is a closure space, then the following statements are equivalent:*

(a) *Y is a T_0 space.*

(b) *There exists a T_0 closure space X (with at least two points) such that every continuous map from X in Y is constant.*

PROOF :

(b) $\Rightarrow$ (a) If Y is not T_0, then take $y \neq y'$ in Y such that every neighborhood of y contains y' and vice versa. Then the subspace $\{y, y'\}$ is indiscrete. Hence b) does not hold.

(a) $\Rightarrow$ (b) Let Y be a T_0 space. For X we take a closure space of the type constructed above. If Y is finite then take $\lambda = \omega$ and if Y is infinite and has cardinality κ, then take $\lambda = \kappa^+$ and put $X = X_\lambda$. In both cases the cardinality of X is exactly λ and $cf\lambda = \lambda$. So if f is continuous from X in Y, then there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ has cardinality λ. Hence, for every $\alpha < \lambda$ there is a point $x_\alpha \in X \setminus X_\alpha$ such that $f(x_\alpha) = y$. It follows that for every $\alpha < \lambda$ the function $f|_{X_\alpha}$ is constant and then f itself is constant, too. ■

Corollary 2. *T_0Pretop is not simple.*

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(*Received* September 9,1989)