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Václav Koubek

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Finite–to–finite universal varieties of distributive double p–algebras

V.KOUBEK

Dedicated to the memory of Zdeněk Frolík

Abstract. A concrete category K is called finite–to–finite universal if there exists a full embedding from the category of graphs in K preserving finiteness. It is shown that a variety $\underset{\sim}{V}$ of distributive double p–algebras is finite–to–finite universal if and only if every finite monoid M is isomorphic to an endomorphism monoid of an finite algebra in $\underset{\sim}{V}$ and this is equivalent with the existence of special finite algebras in $\underset{\sim}{V}$. As a consequence we obtain that a variety $\underset{\sim}{V}$ of distributive double p–algebras is finite–to–finite universal just when $\underset{\sim}{V}$ contains a finite–to–finite universal, finitely generated subvariety.

Keywords: a distributive double p–algebra, a finite–to–finite universal category, a finite monoid universal category, a variety.

Classification: 18B10, 06D15, 20M30

Introduction.

An algebra $(L; \vee, \wedge, ^*, ^+, 0, 1)$ of signature $(2,2,1,1,0,0)$ is a *distributive double p–algebra* (shortly dp–algebra) provided $(L; \vee, \wedge, 0, 1)$ is a distributive bounded lattice and * is a unary operation of pseudocomplementation (i.e. $x \leqslant a^*$ if and only if $x \wedge a = 0$), $^+$ is a unary operation of dual pseudocomplementation (i.e. $x \geqslant a^+$ if and only if $x \vee a = 1$).

A concrete category K is *representative for the category G of graphs and compatible mappings*, or shortly *universal*, if there is a full embedding functor $F\colon G \longrightarrow K$. If, moreover, F takes the finite graphs to finitely underlied K–objects then K is termed *finite–to–finite universal*. As explained in **[17]**, the term "universal" is due to the fact that besides graphs, all other concrete categories can be represented as full subcategories in a universal category (if the set axiom (M) holds). For example every monoid (i.e. one–object category) can be represented as the endomorphism monoid *End*(A) of a suitable representing K–object A (i.e. as a full one–object subcategory of K). If a category enjoys this weaker property then we call it a category *representative for monoids*, or shortly *monoid universal*. In this paper we shall be interested in another property: A concrete category K is said to be *finite monoid universal* if every finite monoid M can be represented as the endomorphism monoid of a suitable K–object A with a finite underlying set. Thus, any finite–to–finite universal category is finite monoid universal (because graphs are finite monoid universal, see **[17]**), while obviously the converse is false.

The investigation of dp–algebras started in [2] where it was proved that the variety of all dp–algebras is finite monoid universal. In [10] it was shown that there exists a finitely generated variety of dp–algebras which is finite–to–finite universal. This result was strengthened in [12] where all universal finitely generated varieties of dp–algebras were characterized. The aim of this paper is to characterize finite–to–finite universal varieties of dp–algebras. We show that for varieties of dp–algebras the notions of finite–to–finite universality and finite monoid universality coincide. On the other hand we show that there exists a finitely generated universal variety of dp–algebras which is not finite–to–finite universal.

1. Preliminaries and results.

The proofs make extensive use of the topological duality introduced by H.A. Priestley [13]. A basic outline follows; for further information, see the survey papers B.A. Davey and D. Duffus [5] or H.A. Priestley [13]. A triple $(X, \leqslant, \tau)$ is called a *Priestley space* if X is a set, $\leqslant$ is an ordering on X, τ is a compact topology on X such that for every pair x, y of elements of X with $x \nleqslant y$ there exists a clopen decreasing set $U \subseteq X$ with $y \in U$, $x \notin U$. Clopen decreasing sets in a Priestley space form a distributive $(0,1)$–lattice and the inverse image map f^{-1} of any continuous and order preserving mapping is a $(0,1)$–homomorphism of the corresponding lattices. This gives rise to a contravariant functor D from the category of all Priestley spaces and continuous, order preserving mappings into the category of all distributive $(0,1)$–lattices and $(0,1)$–homomorphisms.

Conversely, for a distributive $(0,1)$–lattice L the triple $P(L) = (F(L), \leqslant, \tau)$ forms a Priestley space where $F(L)$ is the set of all prime filters, $\leqslant$ is the reversed inclusion and the topology τ is given by an open subbasis $\{\{x \in F(L); a \notin x\}; a \in L\}$. For a $(0,1)$–homomorphism f denote by $P(f)$ the inverse image map, then $P(f)$ is continuous, order preserving mapping between the corresponding Priestley spaces.

Theorem 1.1[13,14]: *The composite functors $P \circ D$ and $D \circ P$ are naturally equivalent to the identity functor of their domains.*

For a finite distributive lattice L, $P(L)$ can be alternatively defined as a poset of all join irreducible elements with the discrete topology. This fact is used for investigation of finite distributive lattices.

For a subset U of a Priestley space X denote by $(U]$ the smallest decreasing subset of X containing U, $[U)$ the smallest increasing subset of X containing U, $Min(U)$ the set of all minimal elements in $(U]$, $Max(U)$ the set of all maximal elements in $[U)$, $Ext(U) = Max(U) \cup Min(U)$. For an element x we shall write $(x]$, $[x)$, $Min(x)$, $Max(x)$, $Ext(x)$ instead of $(\{x\}]$, $[\{x\})$, $Min(\{x\})$, $Max(\{x\})$, $Ext(\{x\})$. Further denote by $Mid(X) = X \setminus Ext(X)$. For a finite distributive lattice L denote by $Max(L)$ the set of all maximal join irreducible elements of L, $Min(L)$ the set of all minimal join irreducible elements of L, $Ext(L) = Min(L) \cup Max(L)$, and $Mid(L)$ the set of all join irreducible elements of L which do not belong to $Ext(L)$. The following result describes the restriction of Priestley duality to the variety of dp–algebras:

Theorem 1.2[15] or [4]: *For a Priestley space X, DX is a dp–algebra if and only* *if $[A)$ is clopen for every clopen decreasing set $A \subseteq X$, $(A]$ is clopen for every clopen increasing set $A \subseteq X$.*

For a continuous order preserving mapping $f: X \longrightarrow X'$ between duals of dp–algebras, Df is a homomorphism of dp–algebras if and only if $f(Min(x)) = Min(f(x))$, $f(Max(x)) = Max(f(x))$ for every $x \in X$.

A Priestley space X such that DX is a dp–algebra is called a *dp–space*, an order preserving continuous mapping f between dp–spaces X and X' is called a *dp–map* if Df is a homomorphism of dp–algebras. For a variety $\underset{\sim}{V}$ of dp–algebras denote by $P(\underset{\sim}{V})$ the category of all dp–spaces X with $DX \in \underset{\sim}{V}$ and all dp–maps.

A sequence $x = x_0, x_1, \ldots, x_n = y$ in a poset $(X, \leqslant)$ is called a *path* from x to y of *length* n if for every $i \in \{0, 1, \ldots, n-1\}$, x_i is comparable with x_{i+1}. Then we say that x and y are *connected*. Any maximal subset of a poset with every pair of elements in it connected is called a *component* of $(X, \leqslant)$. We say that $(X, \leqslant)$ is *connected* if it has exactly one component. An ordered pair (x, y) is called an *arc* of $(X, \leqslant)$ if $x < y$. For a dp–space X, C is a component of X just when $Ext(C)$ is a component of $Ext(X)$. Then we have

Proposition 1.3[11,12]: *If $f: X \longrightarrow Y$ is a dp–map between two dp–spaces X, Y then for every component C of X there exists a component C' of Y such that $f(C) \subseteq C'$ and $f(Ext(C)) = Ext(C')$.*

We recall several notions and facts proved in [**12**]. For any dp–algebra A denote by $Sk(A)$ the least subalgebra of A containing the set $\{x^*; x \in A\} \cup \{x^+; x \in A\}$ and closed under relative complementation. $Sk(A)$ is called a *skeleton* of A. If $Sk(A) = A$ then we say then A is *skeletal* and, moreover, if it is directly indecomposable then we say that A is a *frame*. Obviously, any finite frame is uniquely determined by the poset of all its join–irreducible elements. For a finite frame A denote by $F(A)$ the frame generated by the poset $(X, \leqslant)$ where X is the set of all join irreducible elements of A and $x \leqslant y$ if and only if $x \leqslant y$ in A and either $x \in Ext(A)$ or $y \in Ext(A)$.

Proposition 1.4[12]: *For a dp–algebra A and the inclusion morphism $f: Sk(A) \longrightarrow A$ we have:*

a) *for the dual dp–map h of f we have $h(x) = h(y)$ if and only if $Ext(x) = Ext(y)$;*
b) *the order in the Priestley dual of $Sk(A)$ is the partial order containing all pairs $\{h(x), h(y)\}$ for which $x \leqslant y$ in the dual of A;*
c) *A is skeletal if and only if any pair of distinct elements x, y of the dual of A satisfies $Ext(x) \neq Ext(y)$, moreover, A is a frame if its dual is a connected poset;*
d) *if $Sk(A)$ is finite then for an arbitrary variety $\underset{\sim}{V}$ of dp–algebras we have $A \in \underset{\sim}{V}$ if and only if $Sk(A) \in \underset{\sim}{V}$;*
e) *any endomorphism of a finite frame is invertible;*
f) *for every homomorphism $f: A \longrightarrow A'$ we have $f(Sk(A)) \subseteq Sk(A')$;*

g) *if A is a finite frame then for an arbitrary variety $\underset{\sim}{V}$ of dp–algebras we have $A \in \underset{\sim}{V}$ if and only if $F(A) \in \underset{\sim}{V}$.*

The dp–map h from a) is called *skeletal*. Since we shall work with dp–spaces rather than the algebras themselves, we extend all algebraic terminology to corresponding dp–spaces.

The aim of this paper is to prove the following

Theorem 1.5: *For any variety $\underset{\sim}{V}$ of dp–algebras the following are equivalent:*

a) *$\underset{\sim}{V}$ is finite–to–finite universal;*
b) *$\underset{\sim}{V}$ is finite monoid universal;*
c) *$\underset{\sim}{V}$ contains a finite frame F such that the poset $Mid(F)$ has an order component with at least three distinct arcs, and such that the only endomorphism of F whose fixed points include $Mid(F)$ is the identity;*
d) *$\underset{\sim}{V}$ contains a finite frame G such that $Mid(G)$ has an order component C containing exactly three distinct arcs and at most four other elements, and such that the only endomorphism of G whose fixed points include C is the identity.*

For comparison we recall the main result from [**12**]:

Theorem 1.6[**12**]: *For a finitely generated variety $\underset{\sim}{V}$ of dp–algebras the following are equivalent:*

a) *$\underset{\sim}{V}$ is universal;*
b) *$\underset{\sim}{V}$ contains a proper class of non–isomorphic rigid algebras;*
c) *$\underset{\sim}{V}$ contains an infinite rigid algebra;*
d) *$\underset{\sim}{V}$ contains a rigid algebra which is not skeletal;*
e) *every finite monoid is isomorphic to $End(A)$ for some $A \in \underset{\sim}{V}$;*
f) *every cyclic group C_p of prime order p is isomorphic to $End(A)$ for some $A \in \underset{\sim}{V}$;*
g) *$\underset{\sim}{V}$ contains a finite frame F such that the poset $Mid(F)$ has an order component with at least three elements, and such that the only endomorphism of F whose fixed points include $Mid(F)$ is the identity;*
h) *$\underset{\sim}{V}$ contains a finite frame G such that $Mid(G)$ has an order component C containing exactly three elements and at most three other elements, and such that the only endomorphism of G whose fixed points include C is the identity.*

Following Beazer [**3**], for a dp–algebra A denote by Φ_A the *determination congruence* of A, defined by $(a,b) \in \Phi_A$ if and only if $a^* = b^*$ and $a^+ = b^+$. For any directly indecomposable algebra A of finite range, the algebra A/Φ_A is simple [**3**]. By Davey's [**4**] description of finitely subdirectly irreducible algebras, Φ_A is the least nontrivial congruence of any finite non-simple subdirectly irreducible algebra. Analogously as in [12] we obtain

Corollary 1.7: *If $\underset{\sim}{V}$ is a finite-to-finite universal variety of dp-algebras then*

a) *$\underset{\sim}{V}$ contains a finitely generated finite-to-finite universal subvariety $\underset{\sim}{W}$ generated by a set of no more than eight finite subdirectly irreducible algebras A with the same quotient A/Φ_A;*

b) *$\underset{\sim}{V}$ must have at least two finite non-isomorphic subdirectly irreducible algebras A which are not simple and have the same quotient A/Φ_A.*

We return to the proof of Theorem 1.5. Any finite-to-finite universal category is finite monoid universal, see [17], thus a)⇒b). The proof of the implication b)⇒c) is given in the second section. The third section is devoted to the proof of c)⇒d). The four section contains the proof of d)⇒a). The last section contains some examples and concluding remarks.

2. Necessity.

Denote by S the semigroup given by the table:

S	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_0	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_1	a_2	a_3	a_4	a_5	a_6
a_2	a_2	a_2	a_3	a_1	a_5	a_6	a_4
a_3	a_3	a_3	a_1	a_2	a_6	a_4	a_5
a_4	a_4	a_4	a_5	a_6	a_4	a_5	a_6
a_5	a_5	a_5	a_6	a_4	a_5	a_6	a_4
a_6	a_6	a_6	a_4	a_5	a_6	a_4	a_5

The aim of this section is to prove that every finite dp-space X with $End(X) \cong S$ satisfies

(X1) there exists an order component C of $Mid(Sk(X))$ having at least three arcs;

(X2) if $f\colon X \longrightarrow X$ is a dp-map such that $f(x) = x$ for every $x \in Mid(X)$ then f is the identity.

Note that the component of $Sk(X)$ containing C is a frame whose dual satisfies c). Hence if $\underset{\sim}{V}$ is a finite monoid universal variety of dp-algebras then the skeleton of an algebra A with $End(A) \cong S$ has a direct indecomposable quotient satisfying c) (since S is commutative we have $End(A) \cong S$ whenever $End(P(A)) \cong S$). In this way the implication b)⇒c) in Theorem 1.5 will have been proved.

In the following assume that X is a finite dp-space with $End(X) \cong S$. The dp-map corresponding to a_i is denoted by f_i. Since X is finite every order preserving map $g\colon X \longrightarrow X$ satisfying $g(Max(x)) = Max(g(x))$, $g(Min(x)) = Min(g(x))$ for every $x \in X$ is a dp-map. We shall exploit this fact without any reference. For a mapping $f\colon Z \longrightarrow Y$ denote by $Im(f) = \{y \in Y; \exists z \in Z \text{ with } f(z) = y\}$. First, we immediately have

Lemma 2.1: *f_0 is the identity, $Im(f_0) \supseteq Im(f_1) = Im(f_2) = Im(f_3) \supseteq Im(f_4) = Im(f_5) = Im(f_6)$.*

We prove

Lemma 2.2: *There exists an order component C of X with $End(C) \cong S$.*

PROOF : First we show that for every order component C and every $i \in 6$, $f_i(C) \subseteq C$ whenever $f_i^3(C) \subseteq C$. Assume that $f_i(C)$ is a subset of an order component $C' \neq C$ and define $g: X \longrightarrow X$ such that $g(x) = x$ for every $x \in X \setminus (C \cup C')$, $g(x) = f_i(x)$ for $x \in C$, $g(x) = f_i^2(x)$ for every $x \in C'$. Then g is a dp–map of X with $g^2, g^4 \neq g$ – a contradiction. Whence for every $i \in 6$ and for every order component C we have that $f_i^2(C) \subseteq f_i(C)$. Assume that there exist an order component C and $i \in 6$ with $f_i(C) \cap C = \emptyset$. By Lemma 2.1 we can assume that $i = 4$. If $f_4(x) = x$ for every $x \in X \setminus C$ then define $g: X \longrightarrow X$ such that $g(x) = f_5(x)$ for $x \in X \setminus C$, $g(x) = x$ for $x \in C$. In this case g is an invertible dp–map and since f_5 is not idempotent we conclude that g is not an identity – a contradiction. Therefore there exist an order component $C' \neq C$ and $x \in C'$ with $f_4(x) \neq x$. Define $g_0, g_1: X \longrightarrow X$ such that $g_0(x) = g_1(x) = x$ for $x \in X \setminus (C \cup C')$, $g_0(x) = x$, $g_1(x) = f_4(x)$ for $x \in C$, $g_0(x) = f_4(x)$, $g_1(x) = x$ for $x \in C'$. Both g_0 and g_1 are idempotent non–identical dp–maps of X distinct from f_4 – a contradiction. Thus for every order component C, and every $i \in 6$ we have $f_i(C) \subseteq C$. Hence $End(X) \cong \prod\{End(C);$ C is a component of $X\} \cong S$ and therefore there exists a component C of X with $End(C) \cong S$. ∎

Let Z be a finite dp–space with a skeletal mapping $h: Z \longrightarrow Y$. An idempotent order preserving mapping $g: Z \longrightarrow Z$ is called *contracting* if the following hold

a) for every $x \in Z$, $g(x) = x$ whenever $x \in Ext(Z)$ or $h(x)$ belongs to a component of $Mid(Z)$ with at least three arcs;
b) for every $x \in Z$, $g(x) \in h^{-1}(h(x))$;
c) if $y \in Y$ such that either $y \in Ext(Y)$ or a component of $Mid(Y)$ containing y has at most two elements then $g(h^{-1}(y))$ is a singleton;
d) if $\{y_0 < y_1 > y_2\}$ or $\{y_0 > y_1 < y_2\}$ is an order component of $Mid(Y)$, then $g^{-1}(h(y_i))$ is a singleton for $i \in \{0, 2\}$ and if there exists no order component C of $Mid(Z)$ with $C \cap h^{-1}(y_i) \neq \emptyset$ then $|g^{-1}(h(y_1))| = 2$, else $g(h^{-1}(\{y_i; i \in 3\}))$ is a shortest path connecting $g(h^{-1}(y_0))$ and $g(h^{-1}(y_2))$;

Lemma 2.3: *For every finite poset Z there exists a contracting mapping.*

PROOF : It is easy to construct an idempotent order preserving mapping satisfying a), b), and c). To fulfil d) it suffices to apply Lemma 3.10 from [9]. ∎

A subset Z' of a finite order connected dp–space Z is called a *block* if $B = h^{-1}(x)$ for an $x \in Ext(Y)$ or $B = h^{-1}(C)$ for a component of $Mid(Y)$ where $h: Z \longrightarrow Y$ is a skeletal dp–map of Z. A block B is said to be *contractible* if either $B = h^{-1}(x)$ for $x \in Ext(Y)$ or $B = h^{-1}(C)$ for an order component C of $Mid(Y)$ containing at most two arcs.

Lemma 2.4: *If g is a contracting mapping of a connected finite dp–space Z then for every idempotent dp–map $f: Z \longrightarrow Z$ with $Im(f) \subseteq Im(g)$ we have that*

$f \restriction B = g \restriction B$ for every contractible block.

PROOF : By a) and b) in the definition of a contracting mapping we obtain that $g(Max(x)) = Max(g(x))$ and $g(Min(x)) = Min(g(x))$ for every $x \in Z$. If f is an idempotent dp–map then $f \restriction Ext(Z)$ is the identity, thus f preserves each set $h^{-1}(y)$ for $y \in Mid(Y)$. The rest is clear. ∎

By Lemma 2.2 we can assume in the following that X is order connected. Assume that $h\colon X \longrightarrow Y$ is a skeletal dp–map.

Lemma 2.5: *Every contracting mapping of X is the identity.*

PROOF : Assume that g is a non–identical contracting mapping. Then there exists a contractible block B such that $g \restriction B$ is not the identity. First we prove that B is unique. Assume that there exists a contractible block B' distinct from B such that g is not identical on B'. Define two mappings $g_0, g_1\colon X \longrightarrow X$ as follows: $g_0(x) = g_1(x) = x$ for every $x \in X \setminus (B \cup B')$, $g_0(x) = g(x)$, $g_1(x) = x$ for $x \in B$, $g_0(x) = x$, $g_1(x) = g(x)$ for $x \in B'$. Since g is idempotent we obtain that g, g_0, g_1 are three distinct idempotent nonidentical mappings. Obviously, g_0, g_1 are dp–maps – a contradiction with $End(X) \cong S$. Thus B is a unique contractible block of X on which g is not the identity. Assume $g = f_1$. Then by Lemmas 2.1 and 2.4 there exists a block B' (it is not contractible) such that $f_4 \restriction B$ is not the identity. Define $g_0\colon X \longrightarrow X$ such that $g_0(x) = x$ for all $x \in X \setminus B'$, $g_0(x) = f_4(x)$ for $x \in B'$. Since f_4 is idempotent we obtain by a direct inspection that g_0 is an idempotent non–identical dp–map distinct from g and f_4 – a contradiction. Thus $g = f_4$. If $f_5 \restriction g(B)$ is the identity then $g_0\colon X \longrightarrow X$ defined by $g_0(x) = f_5(x)$ for $x \in X \setminus B$, $g_0(x) = x$ for $x \in B$ is an invertible non–identical dp–map because f_5 is one–to–one on the set $Im(g)$ and $f_5 \neq g$ – a contradiction. Thus $f_5(g(B)) \cap g(B) = \emptyset$, moreover, $B' = g^{-1}(f_5(g(B)))$ is a contractible block of X. Since $f_2 \circ f_4 = f_4 \circ f_2 = f_5$ we conclude that $f_2(B) \subseteq B'$ and $f_2 \circ f_1 = f_2$ implies that f_2 is one–to–one on $f_1(B)$. Since $f_1 \neq f_4$ we conclude that f_4 is one–to–one neither on $f_1(B)$ nor on $f_2(B)$. Thus g is not one–to–one on the contractible block B' distinct from B – a contradiction. ∎

Corollary 2.6: *There exists a component of $Mid(Y)$ having at least three arcs.*

PROOF : If every component of Y has at most two arcs then by Lemmas 2.4 and 2.5 every idempotent dp–map of X is the identity – a contradiction. ∎

We have shown that X satisfies (X1). The following lemma gives the proof of (X2).

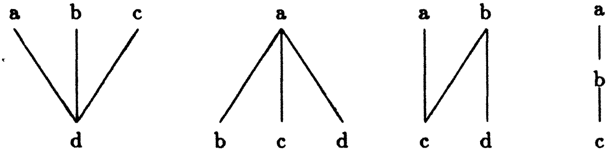
Lemma 2.7: *Every dp–map f of Y into itself such that $f(y) = y$ for every $y \in Mid(Y)$ is the identity.*

PROOF : Define a mapping $f_0\colon X \longrightarrow X$ such that $f_0(x) = x$ for every $x \in Mid(X)$, and for $x \in Ext(X)$, $f_0(x)$ is an element of $Ext(X)$ satisfying $h(f_0(x)) = (f(h(x)))$. Since $f \restriction Ext(X)$ and $h \restriction Ext(X)$ are one–to–one the definition of f_0 is correct. Moreover, f_0 is a dp–map of X into itself because f is a dp–map of Y and $f(y) = y$ for every $y \in Mid(Y)$. Since $f_0(x) = x$ for every $x \in Mid(X)$ we conclude that f_0 is invertible, hence f_0 is the identity and so is f. ∎

Thus any finite dp–space X with $End(X) \cong S$ fulfils (X1) and (X2). Whence the implication b)⇒c) in Theorem 1.5.

3. Smaller frames.

The aim of this section is to prove of the implication c)⇒d) in Theorem 1.4. The proof is analogous to the proof in **[12]** and therefore we give only a brief proof. Clearly, if a poset P has at least three arcs then one of the following posets is a subposet of P:



S_0 S_1 S_2 S_3

We prove

Proposition 3.1: *If A is a finite frame algebra satisfying*

(P1) *$Mid(A)$ contains an order component having at least three arcs,*

(P2) *every endomorphism f of A such that $f(x) = x$ for every join irreducible element in $Mid(A)$ is the identity,*

then there exists a subalgebra B of a quotient of $F(A)$ which is a frame and satisfies

(Y1) *one of the order components of $Mid(B)$ is isomorphic to one of the posets S_0, S_1, S_2, S_3 and $Mid(B)$ has at most four other components all being singletons,*

(Y2) *every endomorphism f of B satisfying $f(x) = x$ for every join irreducible element x in the more–element component of $Mid(B)$ is the identity.*

Obviously, the implication c)⇒d) immediately follows from Proposition 3.1. If A is a finite frame satisfying (P1) and (P2) then clearly there exists a subalgebra B' of A being a frame and satisfying (Y1) or a subalgebra of B'' of A satisfying (Y1) and (Y2) (but it can not be a frame).

For an element $a \in A$ denote by $\bar{a}$ the greatest element with $\bar{a} \not\geq a$. We say that an element $a \in Mid(A)$ is min–defective if $Max(v) = Max(a)$ for every $v \in Min(a)$ and $a \in Mid(A)$ is max–defective if $Min(v) = Min(a)$ for every $v \in Max(a)$. We recall two auxiliary lemmas proved in **[12]**

Lemma 3.2[12]: *Let A be a finite frame such that $|Mid(A)| \geq 2$. Then the subalgebra B of A generated by the set $T(A) = Mid(A) \cup \{\bar{a}; a \in Mid(A)\}$ satisfies $Mid(B) \cong Mid(A)$ and for every pair $x, y \in Mid(A)$ if $Min(x) \setminus Min(y) \neq \emptyset$ in A then so is in B, if $Max(x) \setminus Max(y) \neq \emptyset$ in A then so is in B.*

Moreover the algebra B is a frame whenever

(1) *for every min–defective a which is minimal in $Mid(A)$ there is some $y \in Mid(A)$ such that $Min(a) \cap Min(y)$ and $Min(a) \setminus Min(y)$ are both nonvoid,*

(2) *for every max–defective a which is maximal in $Mid(A)$ there is some $y \in Mid(A)$ such that $Max(a) \cap Max(y)$ and $Max(a) \setminus Max(y)$ are both non-void.*

For $a \in Mid(A)$ define

$$M(a) = \{y \in Mid(A); Min(a) \cap Min(y) \neq \emptyset \neq Min(a) \setminus Min(y)\},$$
$$N(a) = \{y \in Mid(A); Max(a) \cap Max(y) \neq \emptyset \neq Max(a) \setminus Max(y)\}.$$

Lemma 3.3[12]: *For every $a \in Mid(A)$ which is min–defective and minimal in $Mid(A)$ we have that $M(a) \neq \emptyset$, and moreover, every $x \in M(a)$ is not max–defective and either is not min–defective or $a \in M(x)$.*

For every $a \in Mid(A)$ which is max–defective and maximal in $Mid(A)$ we have that $N(a) \neq \emptyset$, and moreover, every $x \in N(a)$ is not min–defective and either is not max–defective or $a \in N(x)$.

We prove Proposition 3.1. $X = P(A)$ is a poset of all join irreducible elements of A. Since A satisfies (P1) there exists a subposet Y of X on the same set satisfying

(i) $Ext(Y) = Ext(X)$,
(ii) there exists exactly one more–element order component of Mid(Y) which is isomorphic to one of the following posets S_0, S_1, S_2, S_3.

By Theorem 1.2, $B' = D(Y)$ is a dp–algebra with $F(A) = F(B')$. Denote by $Z = D(F(A))$. Clearly, the identity is a dp–map from Z onto Y, hence B' is a quotient algebra of $F(A)$, see [13]. By a direct inspection we obtain that B' satisfies (P1) and (P2). By Lemma 3.3 there exists a subset Y' of Y such that the restriction to Y' of the order of Y satisfies

1) $Ext(Y') = Ext(Y)$,
2) one of the order components of $Mid(Y')$ is isomorphic to one of the posets S_0, S_1, S_2, S_3 and $Mid(Y')$ has at most four other components all being singletons,
3) for every min–defective a which is minimal in $Mid(Y')$ there is some $y \in Mid(Y')$ such that $Min(a) \cap Min(y)$ and $Min(a) \setminus Min(y)$ are both non–void,
4) for every max–defective a which is maximal in $Mid(Y')$ there is some $y \in Mid(Y')$ such that $Max(a) \cap Max(y)$ and $Max(a) \setminus Max(y)$ are both non–void,
5) for every pair $a, b \in Mid(Y')$ such that $\{a\}$ and $\{b\}$ are order components of $id(Y')$ there exists an element c in the non–singleton component of $Mid(Y')$ such that one of the following conditions holds:

A) $Min(a) \cap Min(c) \neq \emptyset \neq Min(c) \setminus Min(a)$ and either
• $Min(b) \supseteq Min(c)$ or $Min(b) \cap Min(c) = \emptyset$,
B) $Min(b) \cap Min(c) \neq \emptyset \neq Min(c) \setminus Min(b)$ and either
$Min(a) \supseteq Min(c)$ or $Min(a) \cap Min(c) = \emptyset$,
C) $Max(a) \cap Max(c) \neq \emptyset \neq Max(c) \setminus Max(a)$ and either
$Max(b) \supseteq Max(c)$ or $Max(b) \cap Max(c) = \emptyset$,

D) $Max(b) \cap Max(c) \neq \emptyset \neq Max(c) \setminus Max(b)$ and either
$Max(a) \supseteq Max(c)$ or $Max(a) \cap Max(c) = \emptyset$.

By Proposition 1.4, $D(Y')$ is a frame and since the inclusion of Y' into Y is a dp–map we obtain that $B'' = D(Y')$ is a quotient of B'. We apply Lemma 3.2 to B'' and we obtain an algebra B. By Lemma 3.2, B is a frame satisfying (Y1) (because B' satisfies (Y1) by 2)). Note that the elements of Y' can be considered as elements of B'. Let f be an endomorphism of B such that $f(x) = x$ for every x belonging to the non–singleton order component C of $Mid(B)$. By Proposition 1.4, f is an automorphism of B, hence according to 5), Lemma 3.2, and Theorem 1.2 we conclude that $f(x) = x$ for every $x \in Mid(Y')$ and thus $f(\bar{x}) = \bar{x}$ for every $x \in Mid(Y')$. We show that f fixes the generators of B and thus f is the identity of B and thus B satisfies (Y2). Proposition 3.1 is proved. ■

4. Finite–to–finite universality.

In this section we prove that every variety $\underset{\sim}{V}$ of dp–algebras containing a finite frame A satisfying (Y1) and (Y2) is finite–to–finite universal. This will complete the proof of Theorem 1.5. The proof is based on an idea in [10] and [12] and we will work only with dp–spaces and dp–maps. We shall substitute suitable Priestley spaces instead of several elements of the dual Y of the frame A. The following two technical lemmas formalize this idea.

Lemma 4.1: *Let X be a frame. Assume that a family $\{Z_y; y \in X'\}$ of non–empty Priestley spaces and a relation R satisfying*

(*) *if $(u,v) \in R$ then $u \in Z_x$, $v \in Z_y$ for some distinct $x, y \in X'$ with $x < y$,*

(**) *if $x, y \in X'$ and y covers x in X' then there exist $u \in Z_x$, $v \in Z_y$ with $(u,v) \in R$,*

(***) *for every $x \leqslant y \leqslant z$, $x, y, z \in X'$ and for every closed set $U \subseteq Z_y$ the sets $\{v \in Z_x;$ there exists $u \in U$ with $(v,u) \in R\}$ and $\{v \in Z_z;$ there exists $u \in U$ with $(u,v) \in R\}$ are closed,*

are given where $X' \subseteq Mid(X)$. Define $(Z, \leqslant, \sigma)$ as follows:

1) $Z = (X \setminus X') \cup (\bigcup\{Z_y; y \in X'\})$,
2) *$\leqslant$ is the smallest ordering such that $u \leqslant v$ whenever*
   either $u, v \in X \setminus X'$ and $u \leqslant v$ in X
   or $u, v \in Z_y$ for some $y \in X'$ and $u \leqslant v$ in Z_y
   or $u \in X \setminus X'$, $v \in Z_y$ for some $y \in X'$ with $u \leqslant y$ in X
   or $v \in X \setminus X'$, $u \in Z_y$ for some $y \in X'$ with $y \leqslant v$ in X
   or $(u,v) \in R$.
3) *the topology σ is the union of topologies of Z_y, $y \in X'$ and the discrete topology on $X \setminus X'$.*

Further, let $\psi \colon Z \longrightarrow X$ be the mapping with $\psi(x) = x$ for every $x \in X \setminus X'$, $\psi(x) = y$ for $x \in Z_y$, $y \in X'$.

Then $Z = (Z, \leqslant, \sigma)$ is an order connected dp–space with $X \cong Sk(Z)$ and ψ is a skeletal dp–map from Z onto X.

PROOF : The topology σ is compact being a finite union of compact topologies. Furthermore, for every $U \subseteq X$, the set $\psi^{-1}(U)$ is clopen in σ. We prove that Z is a Priestley space. Let $u \not\leqslant v$ be distinct elements of Z. Set $[\psi(u)) \cap (\psi(v)] = T$ in X. For every $t \in T$ the set $U_t = [y) \cap Z_t$ is closed increasing in Z_t by (***). Since $Z_{\psi(v)}$ is a Priestley space there exists a clopen decreasing set $V_{\psi(v)}$ with $v \in V_{\psi(v)}$, $U_{\psi(v)} \cap V_{\psi(v)} = \emptyset$ because $v \notin U_{\psi(v)}$. Let $t \in T$ and assume that for every $t' \in T$ with $t' > t$ we constructed a clopen decreasing set $V_{t'} \subseteq Z_{t'}$, with $V_{t'} \cap U_{t'} = \emptyset$ and such that for every $t'' \in T$, $t'' > t'$ we have $V_{t''} \cap Z_{t'} \subseteq V_{t'}$. Set $W_t = \{w \in Z_t;$ there are $t' \in T$, $t' > t$ and $w' \in V_{t'}$ with $w < w'$ in $Z\}$. By (***), W_t is closed decreasing in Z_t and $W_t \cap U_t = \emptyset$. Since Z_t is a Priestley space there exists a clopen decreasing set $V_t \subseteq Z_t$ with $W_t \subseteq V_t$, $U_t \cap V_t = \emptyset$. Set $V = (\bigcup\{V_t;\ t \in T\}) \cup (\bigcup\{\psi^{-1}(x);\ x \in (\psi(v)] \setminus T\})$. Then V is clopen decreasing and $v \in V$, $u \notin V$. Thus Z is a Priestley space. Since for every $x \in Ext(Z)$ we have that $[x) = \psi^{-1}([\psi(x)))$, $(x] = \psi^{-1}((\psi(x)])$ are clopen we conclude by Theorem 1.2 that Z is a dp-space. By Proposition 1.4 we obtain that ψ is a skeletal dp-map from Z onto X, thus $X \cong Sk(Z)$. ∎

We say that Z is *created by means of* X, $\{Z_y; y \in X'\}$ and R.

Lemma 4.2: *Let Z (or Z') be created by means of X, $\{Z_y; y \in X'\}$, (or $\{Z'_y; y \in X'\}$) and R (or R', respectively) where $X' \subseteq Mid(X)$. Assume that for every $y \in X'$, $f_y\colon Z_y \longrightarrow Z'_y$ is a continuous order preserving mapping. Define $f\colon Z \longrightarrow Z'$ such that $f(x) = x$ for $x \in X \setminus X'$, $f(x) = f_y(x)$ for $x \in Z_y$, $y \in X'$.*

Then f is a dp-map if and only if for every $(u, v) \in R$ we have that $(f(u), f(v)) \in R'$.

PROOF : Since f is continuous and $f(Max(x)) = Max(f(x))$, $f(Min(x)) = Min(f(x))$ for every $x \in Z$ it suffices to verify that f is order preserving. Obviously, f is order preserving if and only if $(f(u), f(v)) \in R'$ whenever $(u, v) \in R$ because every f_y is order preserving. ∎

We say that a dp-map f is *created by means of* $\{f_y; y \in Y\}$.

Further we recall two statements proved in [1]. Define the following category T^3: the objects are Priestley spaces $(X, \leqslant, \tau)$ with a decomposition $\{U, V, W, T\}$ of X into non-empty clopen subsets such that U, V, $U \cup W \cup V$, $W \cup T$ are decreasing sets and $U \cup V$, $V \cup W$, T are not decreasing; the morphisms are all order preserving continuous mappings which preserve the decomposition of X. Then it holds:

Theorem 4.3[1]: *The dual category of T^3 is finite-to-finite universal.*

Denote by T_n the category of Priestley spaces with n distinguished points which are open and extremal and morphisms are all order preserving continuous mappings which preserve the distinguished points.

Lemma 4.4[1]:*The category T_5 contains a full subcategory $\underset{\sim}{Q}$ dually isomorphic to a finite-to-finite universal category. The category $\underset{\sim}{Q}$ is formed by Priestley spaces X with constants $a_0, a_1, \ldots, a_4 \in Min(X)$ such that $([A)] = X$, and $|(x] \cap A| \neq 1$ for any $x \in X \setminus A$ where $A = \{a_i; i \in 5\}$. Every morphism g of $\underset{\sim}{Q}$ satisfies $g^{-1}\{g(a_i)\} = \{a_i\}$ for all $i \in 5$.*

To prove that T_3 is finite–to–finite universal we shall define a functor $\Phi: T_5 \longrightarrow T_3$ similarly as in [6]. For any object $Q = (X, \tau, \leqslant, \{a_0, a_1, \dots, a_4\})$ of $\underset{\sim}{Q}$ set $A = \{a_i; i \in 5\}$ and define

$\Phi(Q) = (Y, \sigma, \leqslant, \{c_0, c_1, c_3\})$ as follows:
$F = \bigcup\{E_i; i \in 5\} \cup D$,
$Y = (X \setminus A) \cup F$,

where all unions are disjoint, $D = \{d_i; i < 52\}$ and, for $i \in 5$, $E_i = \{e_{i,k}; 1 \leqslant k \leqslant 14\}$.

The partial order on $(Y, \leqslant)$ is the least order for which

(i) $d_{2i} \leqslant d_{2i+1}$ and $d_{2i+2} \leqslant d_{2i+1}$ for $i \in 26$ with the addition modulo 52;
(ii) for every $i \in 5$, $e_{i,2j} \leqslant e_{i,2j-1}, e_{i,2j+1}$ when $1 \leqslant j \leqslant 6$, and $e_{i,14} \leqslant e_{i,13}$;
(iii) for every $i \in 5$, $d_{8+2i} \leqslant e_{i,1}$ and $e_{i,14} \leqslant d_{43-2i}$;
(iv) for every $i \in 5$, $e_{i,8} \leqslant x \in X \setminus A$ if and only if $a_i \leqslant x$ in $(Q, \leqslant)$;
(v) for every $x, y \in X \setminus A$, $x \leqslant y$ if and only if $x \leqslant y$ in Q.

The topology σ of $\Phi(Q)$ is the union topology given by the discrete topology on the finite set F and by the clopen subspace $X \setminus A$ of Q. It is easily seen that $\Phi(Q)$ is an object of T_3. Set $c_0 = d_0$, $c_1 = d_6$, $c_2 = d_{28}$.

Since every morphism $\varphi: Q \longrightarrow Q'$ of $\underset{\sim}{Q}$ satisfies $\varphi(X \setminus A) \subseteq X' \setminus A$, the extension of $\varphi \restriction (X \setminus A)$ to $\Phi(Q)$ by the identity mapping id_F of F is a continuous order preserving mapping $\Phi(\varphi)$ satisfying $\Phi(\varphi)(c_i) = c_i$ for $i \in \{0,1,2\}$. The functor Φ is obviously an embedding, and moreover, Φ preserves the finite Priestley spaces.

Let $\psi: \Phi(Q) \longrightarrow \Phi(Q')$ be an order preserving continuous mapping such that Q and Q' are objects of $\underset{\sim}{Q}$ and $\{\psi(c_i); i \in \{0,1,2\}\} = \{c_i; i \in \{0,1,2\}\}$. We aim to show that $\psi = \Phi(\varphi)$ for some $\varphi: Q \longrightarrow Q'$.

First we note that the shortest path from c_0 to c_1 is $c_0 = d_0, d_1, \dots, d_6 = c_1$, the shortest path from c_1 to c_2 is $c_1 = d_6, d_7, d_8, \dots, d_{28} = c_2$, and the shortest path from c_0 to c_2 is $c_0 = d_0, d_{51}, d_{50}, \dots, d_{28} = c_2$, and their lengths are distinct. Hence we conclude hat ψ must preserve these paths and therefore $\psi \restriction D$ is the identity.

Since, for each $i \in 5$, the shortest path connecting d_{8+2i} to d_{43-2i} is that consisting entirely of elements of E_i, the restriction of ψ to each E_i must be the identity mapping. Altogether, ψ is the identity on the poset F.

If $x \in X \setminus A \subseteq Y$ satisfies $a_i \leqslant x$ for some $i \in 5$, then $a_j \leqslant x$ for some $j \in 5$ distinct from i. By (iv), $e_{i,8} \leqslant x$ and $e_{j,8} \leqslant x$ in $\Phi(Q)$; since ψ fixes all elements of F and because no elements of F lie above distinct $e_{i,8}$ and $e_{j,8}$, it follows that $\psi(x) \in X' \setminus A$. By the definition, $(X' \setminus A] \subseteq X' \cup \{e_{i,8}; i \in 5\}$, and from $([A)] = X$ it now follows that

$$\psi((X \setminus A) \cup \{e_{i,8}; i \in 5\}) \subseteq (X' \setminus A) \cup \{e_{i,8}; i \in 5\}.$$

Since the latter space is homeomorphic and order isomorphic to Q', the mapping $\psi \restriction Q$ is a morphism in T_5, and $\psi = \Phi(\psi \restriction Q)$ as was to be shown.

Observe that the same result holds if we set $c_0 = d_{51}$, $c_1 = d_5$, $c_2 = d_{27}$. Thus we can summarize

Theorem 4.5 *The category T_3 contains a full subcategory $\underset{\sim}{U}$ dually isomorphic to a finite–to–finite universal category such that*

a) *every order preserving continuous map f between two objects from $\underset{\sim}{U}$ such that $\{f(c_i); i \in \{0,1,2\}\} = \{c_i; i \in \{0,1,2\}\}$ satisfies $f(c_i) = c_i$ for every $i \in \{0,1,2\}$,*
b) *for every object $(X, \leqslant, \tau, \{c_0, c_1, c_2\})$ in $\underset{\sim}{U}$ we have for every $i \in \{0,1,2\}$ that $c_i \in Min(X)$ (or $c_i \in Max(X)$).*

We shall construct a full embedding from T^3 or $\underset{\sim}{U}$ into $P(\underset{\sim}{V})$. Let Y be the dual of a frame algebra satisfying (Y1) and (Y2). Denote by C the unique non–singleton order component of $Mid(Y)$.

First we assume that $C \cong S_2$. We shall construct a full embedding $\Psi\colon T^3 \longrightarrow P(\underset{\sim}{V})$ preserving the finite Priestley spaces. Assume that $C = \{c_0 < c_1 > c_2 < c_3\}$. For $(X, \leqslant, \tau, V_i; i \in 4) \in T^3$, let $\Psi(X, \leqslant, \tau, V_i; i \in 4)$ be created by means of Y, $\{V_i; c_i \in C\}$ and $R = \{(u,v);\ u \leqslant v$ in X and there exist distinct $i,j \in 4$ with $u \in V_i,\ v \in V_j\}$ (we recall that $V_0, V_2, V_0 \cup V_1 \cup V_2, V_2 \cup V_3$ are decreasing sets and $V_0 \cup V_1, V_1 \cup V_2 \cup V_3$ are not decreasing ones). From the properties of decomposition $\{V_i; i \in 4\}$ we get that R has the properties (*), (**), and (***) from Lemma 4.1. For a morphism $f\colon (X, \leqslant, \tau, V_i; i \in 4) \longrightarrow (X', \leqslant, \tau, V_i'; i \in 4)$ of T^3 the morphism Ψf is created by means of $\{f \restriction V_i; c_i \in 4\}$. By Lemmas 4.1 and 4.2 and by Proposition 1.4 we easily obtain that Ψ is an embedding functor from T^3 into $P(\underset{\sim}{V})$. We prove

Proposition 4.6:*Ψ is a full embedding from T^3 into $P(\underset{\sim}{V})$ preserving the finite Priestley spaces.*

PROOF : Let $f\colon (Z, \leqslant, \sigma) \longrightarrow (Z', \leqslant, \sigma)$ be a dp–map where $\Phi(X, \leqslant, \tau, V_i; i \in 4) = (Z, \leqslant, \sigma)$, $\Phi(X', \leqslant, \tau, V_i'; i \in 4) = (Z', \leqslant, \sigma)$ for objects $(X, \leqslant, \tau, V_i; i \in 4)$, $(X', \leqslant, \tau, V_i'; i \in 4)$ of T^3. Since $Sk(Z) = Sk(Z') = Y$ by Proposition 1.4 there exists a dp–map $\bar{f}\colon Y \longrightarrow Y$ with $\varphi_{Z'} \circ f = \bar{f} \circ \varphi_Z$ where $\varphi_Z\colon Z \longrightarrow Y$, $\varphi_{Z'}\colon Z' \longrightarrow Y$ are skeletal dp–maps. Since Y satisfies (Y2) we conclude that $\bar{f}$ is the identity because S_2 is automorphism free. Hence $f \restriction (Y \setminus C)$ is the identity and $f(V_i) \subseteq V_i'$ for every $i = 0,1,2,3$. Since f is a dp–map we conclude that $f \restriction X\colon (X, \leqslant, \tau, V_i; i \in 4) \longrightarrow (X', \leqslant, \tau, V_i'; i \in 4)$ is a morphism of T^3, and moreover, $\Psi f \restriction X = f$, thus Ψ is full. The rest is clear. ∎

Secondly, assume that $C \cong S_3$ where $C = \{c_0 < c_1 < c_2\}$. We shall define a full embedding $\Lambda\colon \underset{\sim}{U} \longrightarrow P(\underset{\sim}{V})$.

Define Priestley spaces D and E: D is a poset on the set $\{d_i; i \in 7\}$ such that $d_{2i+1} < d_{2i}, d_{2i+2}$ for $i \in 3$, E is the poset on the set $\{e_0, e_1\}$ where $e_0 < e_1$ (the topology in both cases is, of course, discrete). For an object $Z = (Z, \leqslant, \sigma, y_i; i \in 3)$ of $\underset{\sim}{U}$ define $\Lambda Z = (W, \leqslant, \eta)$ where W is created by means Y, $\{V_c; c \in C\}$ and R where $V_{c_0} = Z$, $V_{c_1} = E$, $V_{c_2} = D$, $R = \{(y_2, d_0), (e_0, d_2), (y_0, d_6), (y_1, e_1)\}$. For a morphism $f\colon Z \longrightarrow Z'$ of $\underset{\sim}{U}$ a morphism Λf is created by means of $\{f_c; c \in C\}$ where $f_{c_0} = f$, and f_{c_1}, f_{c_2} are the identities. By Lemmas 4.1 and 4.2 and by Proposition

1.4 we easily obtain that Λ is an embedding functor from $\underset{\sim}{U}$ into $P(\underset{\sim}{V})$. We prove that Λ is full. To this end we assume that $f\colon \Lambda Z \longrightarrow \Lambda Z'$ is a dp–map where Z and Z' are objects of $\underset{\sim}{U}$. Since $Sk(\Lambda Z) = Sk(\Lambda Z') = Y$ there exists according to Proposition 1.4 a dp–map $\bar{f}\colon Y \longrightarrow Y$ such that $\varphi_{\Lambda Z'} \circ f = \bar{f} \circ \varphi_{\Lambda Z}$ where $\varphi_{\Lambda Z}\colon \Lambda Z \longrightarrow Y$, $\varphi_{\Lambda Z'}\colon \Lambda Z' \longrightarrow Y$ are skeletal dp–maps. Since S_3 is automorphism free we conclude by (Y2) that $\bar{f}$ is the identity. Hence $f(y) = y$ for every $y \in Y \setminus C$, $f(D) \subseteq D$, $f(E) \subseteq E$ and $f(Z) \subseteq Z'$. Since $(f(u), f(v)) \in R$ for every $(u,v) \in R$ we obtain $f(e_0) = e_0$, $f(e_1) = e_1$, $f(y_1) = y_1$, and $f(d_2) = d_2$. For $X \in \{D, Y\}$ and $x, y \in X$ denote by $D(x,y)$ the length of a shortest path from x to y in X, then we have $D(y_0, y_1) < D(y_0, y_2)$ and $D(d_0, d_2) < D(d_2, d_6)$ and thus $f(d_0) = d_0$, $f(d_6) = d_6$, $f(y_0) = y_0$, $f(y_2) = y_2$ because f preserves the ordering. Therefore $f \restriction D$ and $f \restriction E$ is the identity and $f \restriction Z$ is a morphism of $\underset{\sim}{U}$ from Z into Z'. Then $\Lambda(f \restriction Z) = f$ and Λ is full. Obviously Λ preserves finite Priestley spaces. Thus we proved

Proposition 4.7:$\Lambda\colon \underset{\sim}{U} \longrightarrow P(\underset{\sim}{V})$ *is a full embedding preserving the finite Priestley spaces.*

Finally, assume that $C \cong S_0$ where $C = \{c_0 < c_1, c_2, c_3\}$. We shall construct a full embedding Ω from $\underset{\sim}{U}$ into $P(\underset{\sim}{V})$.

For an object Z of $\underset{\sim}{U}$ a dp–space ΩZ is created by means of Y, $\{Z_c; c \in C\}$ and R where $Z_{c_0} = Z$ and $Z_{c_i} = \{z_i\}$ for $i = 1,2,3$ are singleton dp–spaces, $R = \{(y_i, z_{i+1}); i \in 3\}$. For a morphism $f\colon Z \longrightarrow Z'$ of $\underset{\sim}{U}$ a dp–map Ωf is created by means $\{f_c; c \in C\}$ where $f_{c_0} = f$, $f_{c_i}(z_i) = z_i$ for every $i = 1,2,3$. According to Lemmas 4.1 and 4.2 and Proposition 1.4, Ω is an embedding functor from $\underset{\sim}{U}$ into $P(\underset{\sim}{V})$. We prove that Ω is full. To this end we assume that $f\colon \Omega Z \longrightarrow \Omega Z'$ is a dp–map. Since $Sk(\Omega Z) = Sk(\Omega Z') = Y$ there exists by Proposition 1.4 a dp–map $\bar{f}\colon Y \longrightarrow Y$ such that $\varphi_{\Omega Z'} \circ f = \bar{f} \circ \varphi_{\Omega Z}$ where $\varphi_{\Omega Z}\colon \Omega Z \longrightarrow Y$, $\varphi_{\Omega Z'}\colon \Omega Z' \longrightarrow Y$ are skeletal dp–maps. Since Y fulfils (Y1) we have $\bar{f}(C) = C$ and therefore $\bar{f}(c_0) = (c_0)$. Hence we have $f(Z) \subseteq Z'$ and $f(\{z_i; i = 1,2,3\}) = \{z_i; i = 1,2,3\}$. Thus we conclude that also $f(\{y_i; i \in 3\}) = \{y_i; i \in 3\}$. Theorem 4.5 a) implies that $f(y_i) = y_i$ for every $i \in 3$, because f is continuous and order preserving. Whence $f(z_i) = z_i$ for every $i = 1,2,3$ and $f \restriction Z$ is a morphism of $\underset{\sim}{U}$ from Z into Z'. Further we obtain that $\bar{f}(c_i) = c_i$ for every $i \in 4$ and by (Y2), $\bar{f}$ is the identity. Therefore $f(y) = y$ for every $y \in Y \setminus C$ and $\Omega f \restriction Z = f$. Since Ω preserves finite Priestley spaces we proved

Proposition 4.8: $\Omega\colon \underset{\sim}{U} \longrightarrow P(\underset{\sim}{V})$ *is a full embedding preserving the finite Priestley spaces.*

If $C \cong S_1$ the proof is dual. We summarize these results:

Theorem 4.9: *If $\underset{\sim}{V}$ is a variety of dp–algebras containing a finite frame fulfilling (Y1) and (Y2) then $\underset{\sim}{V}$ is finite-to-finite universal.*

The proof of Theorem 1.5 is complete. ∎

5. Conclusions.

If Y is a dual of a finite frame then for every $y \in Mid(Y)$ denote by $B(y)$ the subposet of Y induced on the set $Ext(Y) \cup \{y\}$. Obviously, $B(y)$ is a dp–space and Davey [4] proved that $D(B(y))$ is a subdirectly irreducible algebra. Moreover, for every variety $\underset{\sim}{V}$ of dp–algebras we have $Y \in P(\underset{\sim}{V})$ if and only if $B(y) \in P(\underset{\sim}{V})$ for every $y \in Mid(Y)$, see [12].Thus d) of Theorem 1.5 implies a) of Corollary 1.7. Moreover, it is easy to see that a variety $\underset{\sim}{V}$ of dp–algebras generated by exactly one subdirectly irreducible algebra is not finite–to–finite universal. The proof of Corollary 1.7 is complete. ■

We show that there exists a finite–to–finite universal variety $\underset{\sim}{V}$ of dp–algebras generated by two subdirectly irreducible algebras. Let A_0 be a dp–algebra such that the poset of its join irreducible elements is isomorphic to $\{a, b\} \cup \{c_i; i \in 5\}$ where a is the biggest element and $b > c_i$ for $i \in 4$ and let A_1 be a dp–algebra such that the poset of its join irreducible elements is isomorphic to $\{a, b\} \cup \{c_i; i \in 5\}$ where a is the biggest element and $b > c_i$ for $i \in 2$. Consider the variety $\underset{\sim}{V}$ generated by A_0 and A_1. Then $\underset{\sim}{V}$ contains a finite frame A such that the poset of its join irreducible element is $\{a, b\} \cup \{c_1; i \in 3\} \cup \{d_i; i \in 5\}$ where a is the biggest element, $b > c_i$ for $i \in 3$ and $c_i > d_i, d_{i+1}$ for $i \in 3$. By a direct inspection we obtain that A satisfies (Y1) and (Y2), whence $\underset{\sim}{V}$ is finite–to–finite universal.

Finally we give an example of a finitely generated universal variety which is not finite–to–finite universal. First consider a dp–algebra A such that the poset X of its join irreducible elements is $\{a_i; i \in 7\}$ where $a_0 < a_1 > a_2 < a_3 > a_4 < a_5$, and $a_3 > a_6$.

Lemma 5.1: *The algebra A is simple and if B is a subalgebra of A then B is either three-element or two–element chain.*

Proof : By Beazer [3], A is simple. Assume that h is the dual dp–map of the inclusion of B in A. Then h is surjective see [13] and by a direct inspection we obtain that the dual of B is either a two element chain or a singleton (see Theorem 1.2 for a characterization of dp–maps). ■

Let Y be a poset such that $Y = X \cup \{b_i; i \in 3\}$ where $a_3 > b_0 > b_1, b_2$ and $b_1 > a_2$, $b_2 > a_4$. Set $A' = D(Y)$, then A' is a dp–algebra and let $\underset{\sim}{V}$ be a variety of dp–algebras generated by A'. By Proposition 1.4, A' is a frame and by Theorem 1.6 we conclude that $\underset{\sim}{V}$ is universal since Y is automorphism free and $\{b_1; i \in 3\}$ is a component of $Mid(Y)$.

Lemma 5.2: *If B is a subdirectly irreducible algebra in $\underset{\sim}{V}$ then either B is a chain of at most four elements, or $B \cong A$ or the poset of its join irreducible elements is isomorphic to the subposet of Y on the set $X \cup \{b_i\}$ for some $i \in 3$.*

Proof : Denote Y_i the subposet of Y on $X \cup \{b_i\}$, for $i \in 3$. By Davey [4], $D(Y_i)$ is a subdirectly irreducible algebra and the dp–algebras $D(Y_i)$, $i \in 3$, generate $\underset{\sim}{V}$, see [12]. Since the congruence lattice of dp–algebras is distributive and $D(Y)$ is finite we obtain by Jónsson Lemma (see [8] or [7]) that every subdirectly irreducible algebra in $\underset{\sim}{V}$ is a quotient of a subalgebra of $D(Y_i)$ for some $i \in 3$. By Lemma 5.1 we obtain that for every $i \in 3$ every proper subalgebra of $D(Y_i)$ is a chain with at most four elements. The rest follows from the result of Davey [4]. ∎

Assume that $\underset{\sim}{V}$ is finite–to–finite universal, then by Theorem 1.5 there exists a frame $D \in \underset{\sim}{V}$ satisfying (Y1). Let Z be the dual of D then for every $z \in Mid(Z)$, the subposet $Z(z)$ of Z on the set $Ext(Z) \cup \{z\}$ is the dual of a subdirectly irreducible algebra in $\underset{\sim}{V}$. From Lemma 5.2 we immediately obtain that $Ext(Z) \cong X$ and thus by Lemma 5.2 we conclude that D is a quotient of the frame $F(A')$. Then every component of $Mid(Z)$ has at most two arcs and this is a contradiction. Thus

Theorem 5.3: *$\underset{\sim}{V}$ is a finitely generated universal variety of dp–algebras which is not finite monoid universal.*

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The Faculty of Mathematics and Physics, Charles Univ., Malostranské nám. 25, 118 00 Prague, Czechoslovakia

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