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On compactness with respect to semi-open sets

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Dedicated to the memory of Zdeněk Frolík

Abstract. This paper shows that semi-compactness of a topological space is equivalent to hereditary compactness of a larger space, namely the associated space of α-open subsets. This indicates the strength of the notion of semi-compactness.

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Classification: 54D30

1. Introduction.

Recently a strong form of compactness has been defined by requiring each cover of the topological space in question by semi-open subsets to have a finite subcover, [1]. A subset of a topological space is semi-open if it is contained in the closure of its interior. In [6] it was shown that semi-compactness of a topological space is equivalent to compactness of a much larger space, namely that having the collection of all semi-open subsets of the given space as a subbase.

Let (X,τ) be a topological space. We denote the closure and the interior of a subset S of X by $cl_X S$ and $int_X S$ respectively. A subset S of (X,τ) is called semi-open [respectively regular closed, α-open] in (X,τ) if $S \subseteq cl_X(int_X S)$ [respectively $S = cl_X(int_X S), S \subseteq int_X(cl_X(int_X S))$]. Clearly every open set is α-open, and α-open sets and regular closed sets are semi-open. A subset S of (X,τ) is called nowhere dense, abbreviated nwd., if $int_X(cl_X S) = \phi$. Njastad [5] has shown that the collection τ^α of all α-open sets in (X,τ) is a topology on X larger than τ. Moreover, $V \in \tau^\alpha$ if and only if $V = U - N$ where $U \in \tau$ and N is nwd. in (X,τ). Hence nwd. subsets of (X,τ) are closed and discrete in (X,τ^α).

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of X. We say that $\mathcal{B}$ refines $\mathcal{A}$ if each member of $\mathcal{B}$ is contained in some member of $\mathcal{A}$. Following Hodel [3], we call a pairwise disjoint collection of non-empty open sets in (X,τ) a cellular family. A simple application of Zorn's lemma yields the following.

Lemma 1.1. *Let $\mathcal{S}$ be a family of semi-open subsets of X,τ. Then there exists a cellular family $\mathcal{G}$ such that $\mathcal{G}$ refines $\mathcal{S}$ and $\cup\{G : G \in \mathcal{G}\}$ is dense in $\cup\{S : S \in \mathcal{S}\}$.*

Finally, the cardinality of a set X is denoted by $|X|$. A cardinal number is the set of all ordinals which precede it. In this paper, κ will always denote an infinite cardinal number.

In this paper we indicate the strength of the semi-compactness property by showing that (X,τ) is semi-compact if and only if (X,τ^α) is hereditarily compact. We prefer to present our results for a general cardinal κ, rather than for the special case $\kappa = \omega$, so that we obtain corresponding results for the Lindelöf case, for example, at the same time. An example is provided to show that the role of the topology τ^α of α-open subsets is crucial.

2. Semi-κ -compact spaces.

Definition 2.1. A space (X,τ) is called κ-compact [respectively semi-κ-compact] if every open cover [respectively semi-open cover] of (X,τ) admits a subcover of cardinality $< \kappa$.

Hence ω-compact = compact, ω_1 - compact = Lindelöf, semi-ω-compact = semi-compact [1] and semi-ω_1-compact = semi-Lindelöf [2]. Let us observe that, for every κ, any set X with the cofinite topology is semi-κ-compact.

The following result is part of our main theorem.

Proposition 2.2. *Let (X,τ) be semi-κ-compact. Then*
i) every nwd. subset of (X,τ) has cardinality $< \kappa$.
ii) every cellular family in (X,τ) has cardinality $< \kappa$.

PROOF : Let N be nwd. in (X,τ). Then N is closed and discrete in (X,τ^α). Since (X,τ^α) is clearly κ-compact we have $|N| < \kappa$.

Now let $\mathcal{G} = \{G_i : i \in I\}$ be a cellular family in (X,τ) and suppose that $|I| = \kappa$. Then we may write $I = \cup\{I_\beta : \beta < \kappa\}$ where $|I_\beta| = \kappa$ for each $\beta < \kappa$ and $I_\beta \cap I_\gamma = \phi$ whenever $\beta \neq \gamma$. Let $G = \cup\{G_i : i \in I\}$ and for each $\beta < \kappa$ let $V_\beta = \cup\{G_i : i \in I_\beta\}$. If $A = cl_X G - G$ then A is nwd. and consequently $|A| < \kappa$. Let $A^* = \{x \in A : \exists \beta < \kappa$ such that $x \in cl_X V_{\beta_X}\}$. For each $x \in A^*$ pick $\beta_x < \kappa$ such that $x \in cl_X V_{\beta_x}$. Since $|A^*| < \kappa$ there exists $\gamma \in \kappa - \{\beta_x : x \in A^*\}$ and thus we have $cl_X G = cl_X(\cup\{G_i : i \notin I_\gamma) \cup V_\gamma$. Now, $\{X - cl_X G\} \cup \{cl_X(\cup\{G_i : i \notin I_\gamma\})\} \cup \{G_i : i \in I_\gamma\}$ is a semi-open cover of (X,τ) having no subcover of cardinality $< \kappa$. Hence ii) is proved. ∎

Theorem 2.3. *For a space (X,τ) the following are equivalent:*

1) *(X,τ) is semi-κ-compact.*
2) *Every nwd. subset of (X,τ) has cardinality $< \kappa$ and every cellular family in (X,τ) has cardinality $< \kappa$.*
3) *(X,τ^α) is hereditarily κ-compact.*

PROOF :

1) $\Rightarrow$ 2): This is Proposition 2.2.
2) $\Rightarrow$ 3): By an analogous result to Theorem 1 of [7], we have to show that each $W \in \tau^\alpha$ is a κ-compact subset of (X,τ^α). So let $\mathcal{W}$ be an α-open cover of $W \in \tau^\alpha$. By Lemma 1.1 there exists a cellular family $\mathcal{G}$ in (X,τ) which refines $\mathcal{W}$ and whose union is dense in W. By assumption, $|\mathcal{G}| < \kappa$. If $A = W - \cup\{G : G \in \mathcal{G}\}$ then A is nwd. in (X,τ) and hence $|A| < \kappa$. For each $G \in \mathcal{G}$ pick $V_G \in \mathcal{W}$ such that $G \subseteq V_G$. Then $W = \cup\{V_G : G \in \mathcal{G}\} \cup A$ proving that W is κ-compact in (X,τ^α).
3) $\Rightarrow$ 1): Since (X,τ^α) is κ-compact, every nwd. subset of (X,τ) has cardinality $< \kappa$. Let $\{S_i : \in I\}$ be a semi-open cover of (X,τ). For each $i \in I$ there exists

$V_i \in \tau$ such that $V_i \subset S_i \subset cl_X(V_i)$. If $V = \cup\{V_i : i \in I\}$ then V is dense and open in (X, τ) and thus $|X - V| < \kappa$. By assumption there exists $I_0 \subseteq I$ whit $|I_0| < \kappa$ and $V = \cup\{V_i : i \in I_0\}$. Hence $X = \cup\{S_i : i \in I_0\} \cup (X - V)$ showing that (X, τ) is semi-κ-compact.

■

Corollary 2.4. *A space $(X\tau)$ is semi-compact if and only if (X, τ^α) is hereditarily compact.*

Corollary 2.5. *(see also [2]). For a space (X, τ) the following are equivalent:*

1) *(X, τ) is semi-Lindelöf.*
2) *(X, τ^α) is hereditarily Lindelöf.*
3) *(X, τ) satisfies the countable chain condition and every nwd. subset of (X, τ) is at most countable.*

Hence, if (X, τ) is an uncountable Hausdorff space then (X, τ) is semi-Lindelöf if and only if (X, τ) is a Luzin space in the sense of Kunen [4].

3. An example.

In this example we point out that in our Theorem 2.3 the condition "(X, τ^α) is hereditarily κ-compact" is essential In particular (X, τ^α) cannot be replaced by (X, τ). We show that for each κ there exists a hereditarily κ-compact T_1 space (X, τ) which is not semi-κ-compact.

Let X be a set with $|X| = \kappa$ and let $X = X_1 \cup X_2$ with $|X_1| = |X_2| = \kappa$ and $X_1 \cap X_2 = \phi$. Define a topology τ on X in the following way: a basic neighbourhood V of $x \in X_1$ is of the form $V = U_1 \cup U_2$ where for $i = 1, 2, U_i \subseteq X_i, |X_i - U_i| < \kappa$ and $x \in U_1$. A basic neighbourhood V of $x \in X_2$ is a subset of X_2 containing x such that $|X_1 - V_2| < \kappa$.

Then (X, τ) is a hereditarily κ-compact T_1 space. Since X_1 is nwd. (X, τ) and $|X_1| = \kappa$, (X, τ) fails to be semi-κ-compact.

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