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The sequentiality is equivalent to the $\mathcal{F}$-Fréchet–Urysohn property

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Dedicated to the memory of Zdeněk Frolík

Abstract. Some relations among sequentiality, $\mathcal{F}$-Fréchet–Urysohn property and $\mathcal{F}$-sequentiality are shown.

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The main results of the paper (necessary definitions will be given below):

Theorem 1. *There exists a filter $\mathcal{F}$ on ω such that every Hausdorff sequential space is $\mathcal{F}$-Fréchet–Urysohn.*

Example 1, [CH]. *For every ultrafilter p on ω which is a P-point in ω^* there exists a compact Franklin's space whose index of p-sequentiality equals* 2.

Example 2, [CH]. *There exists a sequential compact space which is p-Fréchet–Urysohn for no P-point $p \in \omega^*$.*

Theorem 2. *Every compact Franklin's space is p-Fréchet–Urysohn for no P-point $p \in \omega^*$.*

In 1968 M. Katětov [1] introduced the following concept of an $\mathcal{F}$-limit point. Let $\mathcal{F}$ be a filter on ω. A point x of a topological space X is called an $\mathcal{F}$-limit point of $A \subset X$ if there exists a sequence $\{a_n : n \in \omega\}$ such that $\{n \in \omega : a_n \in O_x\} \in \mathcal{F}$ for all neighborhoods O_x of x. It is obvious that if $\mathcal{F}$ is a Fréchet filter, i.e. the filter of cofinite subsets of ω, then an $\mathcal{F}$-limit point is the ordinary limit of some convergent sequence lying in the corresponding subset. So, the concept of an $\mathcal{F}$-limit point is a generalization of that of a convergent sequence.

The property of a space being p-sequential or $\mathcal{F}$-Fréchet–Urysohn is very natural, as well.

A topological space X is said to be *$\mathcal{F}$-Fréchet–Urysohn* if each limit point of any subset $A \subset X$ is an $\mathcal{F}$-limit point.

A topological space X is called *$\mathcal{F}$-sequential* if the nonclosedness of any subset $A \subset X$ is equivalent to the existence of an $\mathcal{F}$-limit point of the subset A laying in $X \setminus A$. This notion is due to A. P. Kombarov [2].

When $\mathcal{F}$ is an ultrafilter p, the notion of p-Fréchet–Urysohn has been studied by I. Savchenko.

PROOF of Theorem 1: First, we recall the notion of the sum of a family of filters $\{\mathcal{F}_a : a \in A\}$ over another filter $\mathcal{F}$ on the set A. Suppose that filters $\mathcal{F}_a$ are defined on disjoint sets K_a. Let K be the union $\bigcup\{K_\alpha : a \in A\}$. The filter on K consisting of all subsets $M \subset K$ with $\{a \in A : M \bigcap K_\alpha \in \mathcal{F}_a\} \in \mathcal{F}$ is called the sum of the family $\{\mathcal{F}_a : a \in A\}$ over the filter $\mathcal{F}$ and is denoted by $\sum\limits_{\mathcal{F}} \mathcal{F}_a$.

Let $\{X_n : n \in \omega\}$ be a disjoint family of countable infinite sets X_n and let X be the sum $\bigcup\{X_n : n \in \omega\}$. For each $n \in \omega$ we fix a bijection $\varphi_n : X \longleftrightarrow X_n$.

Let $\mathcal{F}$ be a Fréchet filter on X and let $\mathcal{F}^1 = \mathcal{F}$.

We describe the first step of the transfinite process of length ω_1.

For each $n \in \omega$ let $\mathcal{F}_n^1 = \varphi_n(\mathcal{F}^1)$ and let $\mathcal{F}^2 = \sum\limits_{\mathcal{F}} \mathcal{F}_n^1$. We remark that $\mathcal{F}^1 \subset \mathcal{F}^2$, hence, for each $n \in \omega$ we have $\mathcal{F}_n^2 = \varphi_n(\mathcal{F}^2) \supset \mathcal{F}_n^1$. Second, we define $\mathcal{F}^3 = \sum\limits_{\mathcal{F}} \mathcal{F}_n^2$ on X thus $\mathcal{F}^2 \subset \mathcal{F}^3$ and, for each $n \in \omega$ we have $\mathcal{F}_n^3 = \varphi_n(\mathcal{F}^3) \supset \mathcal{F}_n^2$. And so on. At the ω-th step we define $\mathcal{F}^\omega = \sum\limits_{\mathcal{F}} \mathcal{F}_n^n$ on X. It is obvious, that $\mathcal{F}^n \subset \mathcal{F}^\omega$ for all $n \in \omega$, therefore, $\mathcal{F}_n^\omega = \varphi_n(\mathcal{F}^\omega) \supset \mathcal{F}_n^k$ for all n, $k \in \omega$. Hence, if we define $\mathcal{F}^{\omega+1} = \sum\limits_{\mathcal{F}} \mathcal{F}_n^\omega$, then all the natural inductive assumptions are fulfilled.

Now, let α be a limit ordinal and $\omega < \alpha < \omega_1$. Let $\{\Theta_n : n \in \omega\}$ be an increasing sequence of ordinals in α with $\sup\{\Theta_n : n \in \omega\} = \alpha$. We define $\mathcal{F}^\alpha = \sum\limits_{\mathcal{F}} \mathcal{F}_n^{\Theta_n}$ on X. It is clear that $\mathcal{F}^\beta \subset \mathcal{F}^\alpha$ for all $\beta < \alpha$, therefore, $\mathcal{F}_n^\alpha = \varphi_n(\mathcal{F}^\alpha) \supset \mathcal{F}_n^\beta$ for all $n \in \omega$ and $\beta < \alpha$. Hence, if we define $\mathcal{F}^{\alpha+1} = \sum\limits_{\mathcal{F}} \mathcal{F}_n^\alpha$ then all the natural inductive assumptions are fulfilled.

So we have the increasing families of filters $\{\mathcal{F}^\alpha : \alpha \in \omega_1\}$ on X, $\{\mathcal{F}_n^\alpha = \varphi_n(\mathcal{F}^\alpha) : \alpha \in \omega_1\}$ on every X_n, where $\mathcal{F}^{\alpha+1} = \sum\limits_{\mathcal{F}} \mathcal{F}_n^\alpha$ for all $\alpha \in \omega_1$ and $\mathcal{F}^\alpha = \sum\limits_{\mathcal{F}} \mathcal{F}_n^{\Theta_n}$ for some increasing sequence of ordinals $\{\Theta_n : n \in \omega\}$ in α such that $\sup\{\Theta_n : n \in \omega\} = \alpha$.

Then $\mathcal{F}^{\omega_1} = \bigcup\{\mathcal{F}^\alpha : \alpha \in \omega_1\}$ is a filter on X and $\mathcal{F}_n^{\omega_1} = \bigcup\limits_{n \in \omega} \{\mathcal{F}_n^\alpha : \alpha \in \omega_1\} = \varphi_n(\mathcal{F}^{\omega_1})$ is a filter on X_n for each $n \in \omega$. It is not hard to verify that $\sum\limits_{\mathcal{F}} \mathcal{F}_n^{\omega_1} = \mathcal{F}^{\omega_1}$.

We shall show now that $\mathcal{F}^{\omega_1}$ is the requested filter. Let Z be any Hausdorff sequential space. We denote by δ its index of sequentiality, hence, if $A \subset Z$ and $z \in \overline{A}$ then $\delta(z, A)$ is a countable ordinal α. Suppose that the corresponding inductive conditions are fulfilled. Let z be the limit of the convergent sequence $\{z_n : n \in \omega\}$ with $\delta(z_n, A) = \beta_n < \alpha$ for all $n \in \omega$. Since Z is a Hausdorff space, there exists a disjoint family $\{V_n\}$ of neighborhoods of the points z_n. By the inductive assumption, each z_n is an $\mathcal{F}^{\omega_1}$-limit of a sequence from $A \cap V_n$, hence an $\mathcal{F}_n^{\omega_1}$-limit of the same sequence. Consequently, z is an $\mathcal{F}^{\omega_1}$-limit of the union of the sequences, which completes the proof. ■

CONSTRUCTION of Example 1: Let us assume that CH is true. Let p be a P-point in ω^*. We shall construct an infinite maximal family Σ_p^* of disjoint non-empty clopen subsets of ω^* such that the set $T(p)$ of all ultrafilters of the same type as p is contained in $\bigcup \Sigma_p^*$.

Under the assumption of CH, $|T(p)| = \aleph_1$. Let $T(p) = \{p_\alpha : \alpha \in \omega_1\}$.

We shall construct the family Σ_p^* by transfinite induction. We assume that a countable part Σ_α^* of Σ_p^* has already been constructed: $\Sigma_\alpha^* = \{V_\beta^* : \beta \in \alpha\}$, where $p_\beta \in \bigcup \Sigma_\alpha^*$ for all $\beta \in \alpha$. Let us describe the α-th step of the transfinite procedure. If $p_\alpha \in \bigcup \Sigma_\alpha^*$, then $\Sigma_{\alpha+1}^* = \Sigma_\alpha^*$, otherwise we put $\Sigma_{\alpha+1}^* = \Sigma_\alpha^* \bigcup \{V_\alpha^*\}$ where V_α^* is a clopen neighborhood of p_α disjoint with $\bigcup \Sigma_\alpha^*$.

Let $\Sigma_p^* = \bigcup\{\Sigma_\alpha^* : \alpha \in \omega_1\}$. It is evident that $T(p) \subset \bigcup \Sigma_p^*$ and Σ_p^* is maximal, i.e. $\left[\bigcup \Sigma_p^*\right] = \omega^*$. The index of p-sequentiality of the corresponding Franklin space $F(\Sigma_p^*)$ is 2. ∎

CONSTRUCTION of Example 2: Again, suppose that CH is true. For any P-point $p \in \omega^*$ denote by F_p a copy of $F(\Sigma_p^*)$ (see the previous example) and by F the disjoint topological sum of all these copies. Thus F is locally compact. If F^* is the one–point compactification of F, then F^* is sequential and its index of sequentiality equals 2. Let p be a P-point in ω^*. Then F^* is not p-Fréchet–Urysohn, because F_p is clopen and F_p is not p-Fréchet–Urysohn. ∎

Theorem 2 easily follows from the following lemma.

Lemma. *If $\mathcal{A}$ is an infinite disjoint family of non–empty clopen subsets in ω^*, then $Fr(\bigcup \mathcal{A})$ contains ultrafilters of all types except of P-points.*

PROOF : Indeed, it is evident, that if $\mathcal{B} \subseteq \mathcal{A}$, so $Fr(\bigcup \mathcal{B}) \subseteq Fr(\bigcup \mathcal{A})$. If ξ is not a P-point, then there exists a countable $\mathcal{C}$ of non–empty clopen subsets in ω^* such that $\xi \in Fr(\bigcup \mathcal{C})$. Let $\mathcal{B}$ be any infinite countable subfamily of $\mathcal{A}$. There exists a permutation $\varphi : \omega \longleftrightarrow \omega$ such that the extension $\varphi^* : \beta\omega \longleftrightarrow \beta\omega$ moves $\mathcal{C}$ onto $\mathcal{B}$, therefore $Fr(\bigcup \mathcal{B})$ contains the same types of ultrafilters as $Fr(\bigcup \mathcal{C})$. This completes the proof. ∎

Question. Is it consistent with ZFC that every Hausdorff sequential (compact) space is ultra-Fréchet–Urysohn?

(A space is said to be ultra-Fréchet–Urysohn if it is ξ-Fréchet–Urysohn for all $\xi \in \omega^*$).

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