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Abstract. We prove the existence of 3-symmetric Riemannian spaces with solvable full isometry group.

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1.Introduction. 3-Symmetric Riemannian spaces constitute an important class of Riemannian manifolds (cf.[5]. In [5] the full classification of 3-symmetric spaces with semisimple groups of holomorphic isometries has been obtained. On the other hand, there exist Riemannian k-symmetric spaces with solvable isometry groups for all even $k > 2$ [2], and for all odd $k > 3$ [12]. Thus, it is still interesting to ask about the existence of 3-symmetric Riemannian spaces with solvable full isometry groups. In the present note we prove the existence of such spaces and obtain a description of them. All the necessary information about generalized symmetric spaces can be found in [9]. For the brevity we call a generalized symmetric Riemannian space of order k "k-symmetric Riemannian space". The symbol $L(G)$ denotes the Lie algebra of a Lie group G. The subgroup of all the fixed points of the automorphism $\sigma : G \to G$ is denoted by G^σ. Respectively, $L(G)^{\sigma_*}$ is the subalgebra of all the fixed vectors of $\sigma_* = (d\sigma)_e$. Everywhere $I_0(M,g)$ denotes the connected component of the full isometry group of a Riemannian manifold (M,g).

Definition 1. A Lie group G is said to be of Frobenius type if it admits an automorphism σ such that G^σ is discrete.

Theorem 1. *There exist 3-symmetric Riemannian spaces (M,g) with solvable isometry group $I_0(M,g)$. In particular, the Riemannian manifold.*

$$(\mathbf{R}_6[x_1,x_2,x_3,y_1,y_2,y_3],g),$$

$$\begin{aligned} g = {} & dx_1^2 + dx_2^2 + (x_1^2+y_1^2+1)dx_3^2 - 2x_1dx_2dx_3 - 2y_1dx_3dy_2 + x_1y_1dx_3dy_3 + \\ & + dy_1^2 + dy_2^2 + dy_3^2 - 2x_1dy_2dy_3 + (x_1^2+1)dy_3^2 \end{aligned} \tag{1}$$

is a 3-symmetric Riemannian space with the above property.

Theorem 2. *Any simply connected 3-symmetric Riemannian space with solvable full isometry group is isometric to the space (U, g_U) where U is a nilpotent Lie group of Frobenius type and of nilpotency class ≤ 4 equipped with a left-invariant Riemannian metric g and a metric-preserving automorphism σ of order 3.*

Remark. Definition 1 is a Lie group analogue of a definition in [7]. The case of algebraic groups has been considered in [7], [11]. Let G be a complex Lie group. The symbol G^R denotes the same group considered as a real Lie group (note that dim $G^R = 2dimG$). Respectively, $L(G)^R$ is the algebra $L(G)$ considered over $\mathbf{R}$. Suppose that G is a connected simply connected Lie group endowed with a left-invariant Riemannian metric g and let K be the connected component of the isotropy subgroup at a fixed point in the isometry group $I_0(G,g)$. Let ∇ be the Levi-Civitta connection corresponding to g. Then ∇ defines for every $X \in L(G)$ a skew-symmetric linear operator $\nabla_X : L(G) \to L(G)$ by the classical formula

$$(2) \qquad 2\langle\nabla_X Y, Z\rangle = \langle[X,Y],Z\rangle + \langle X,[Z,Y]\rangle + \langle Y,[Z,X]\rangle$$

Define the operator $C_X : \wedge^2 L(G) \to \wedge^2 L(G)$ by the formula $C_X(L) = [L, \nabla_X] - \nabla_{L(X)}$. Denote $D^r_s(L(G))$ the subspace of (r,s)-tensors on $L(G)$. We shall need the following

Proposition 1 (Azencott-Wilson [1]). *There exists an isomorphism of Lie algebras*

$$(3) \qquad \sigma : L(K) \to \mathcal{K}$$

where $\mathcal{K}$ is the largest subalgebra in $\wedge^2 L(G)$ which is invariant under C_X for all $X \in L(G)$ and whose elements annihilate the curvature tensor $R \in D^1_3(L(G))$.

2.Proof of theorem 1. Let U be the maximal unipotent subgroup in $SL_3(\mathbf{C})$. It consists of all the matrices of the form

$$U = \{(a_{ij})|a_{ii} = 1, a_{ij} = 0, i > j, a_{ij} \in \mathbf{C}, i < j, i,j = 1,2,3,\}$$

Evidently, $L(U)$ is a nilpotent Lie algebra over $\mathbf{C}$, consisting of the triangular matrices with zeros on the diagonal.

Consider U^R and $L(U^R) = L(U)^R$. Denote by $\tau_{ij}(1)$ the matrix in $L(U)$ with the elements $\alpha_{ks} = 0((k,s) \neq (i,j))$ and $\alpha_{ij} = 1$. Evidently, $\tau_{ij}(1), \sqrt{-1}\tau_{ij}(1)(i < j, i,j = 1,2,3,)$ constitute the $\mathbf{R}$-basis of the Lie algebra $L(U)^R$. Introduce the notations

$$X_1 = \tau_{12}(1), \quad X_2 = \sqrt{-1}\tau_{12}(1), \quad X_3 = \tau_{13}(1), \quad X_4 = \sqrt{-1}\tau_{13}(1),$$
$$X_5 = \tau_{23}(1), X_6 = \sqrt{-1}\tau_{23}(1)$$

Then the direct matrix calculation yields

$$(4) \qquad [X_i, X_j] = 0$$

for all (i,j) except the cases

$$(5) \qquad [X_1,X_5] = X_3, \quad [X_1,X_6] = X_5, \quad [X_2,X_5] = X_4, \quad [X_2,X_6] = -X_3$$

Define the scalar product in $L(U)^R$ by the conditions $\langle X_i, X_j\rangle = \delta_{ij}$ and let g_U be the left-invariant Riemannian metric on U^R corresponding to $<,>$. First, we shall prove the solvability of $I_0(U^R, g_U)$. For this purpose we shall calculate $\mathcal{K}$ in the case $G = U^R$. Using the formula (2) for $\nabla_X : L(U)^R \to L(U)^R$, then (4),(5) and the condition $\langle X_i, X_j\rangle = \delta_{ij}$, one automatically obtains:

$$\tag{6}\begin{aligned}
&\nabla_{X_1} = (b^1_{ij})^6_{i,j=1}, \quad b^1_{35} = \frac{1}{2}, \quad b^1_{56} = \frac{1}{2}, \quad b^1_{ij} = 0 \quad \text{for other } i \le j;\\
&\nabla_{X_2} = (b^2_{ij})^6_{i,j=1}, \quad b^2_{36} = -\frac{1}{2}, \quad b^2_{45} = \frac{1}{2}, \quad b^2_{ij} = 0 \quad \text{for other } i \le j;\\
&\nabla_{X_3} = (b^3_{ij})^6_{i,j=1}, \quad b^3_{15} = \frac{1}{2}, \quad b^3_{26} = -\frac{1}{2}, \quad b^3_{ij} = 0 \quad \text{for other } i \le j;\\
&\nabla_{X_4} = (b^4_{ij})^6_{i,j=1}, \quad b^4_{25} = \frac{1}{2}, \quad b^4_{ij} = 0 \quad \text{for other } i \le j;\\
&\nabla_{X_5} = (b^5_{ij})^6_{i,j=1}, \quad b^5_{13} = b^5_{l6} = \frac{1}{2}, \quad b^5_{24} = \frac{1}{2}, \quad b^5_{ij} = 0 \quad \text{for other } i \le j;\\
&\nabla_{X_6} = (b^6_{ij})^6_{i,j=1}, \quad b^6_{15} = \frac{1}{2}, \quad b^6_{23} = -\frac{1}{2}, \quad b^6_{ij} = 0 \quad \text{for other } i \le j;
\end{aligned}$$

and the equalities $b^s_{ij} = -b^s_{ji}$ hold for all indices.

Now, we obtain $R(X_i, X_j)$ using the formula

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

and the expressions (6), as well as the commutator formulae (4), (5). After the long but direct calculations with matrices one has

$$\tag{7}\begin{aligned}
&R^3_{412} = R^3_{512} = R^4_{612} = -\frac{1}{4}, \quad R^5_{612} = R^1_{313} = \frac{1}{4}, \quad R^1_{613} = R^2_{513} = -\frac{1}{4},\\
&R^2_{314} = R^2_{516} = \frac{1}{4}, \quad R^2_{614} = R^1_{316} = -\frac{1}{4}, \quad R^1_{616} = -\frac{3}{4}, \quad R^2_{416} = -\frac{1}{2},\\
&R^1_{423} = R^2_{323} = R^2_{424} = R^1_{625} = \frac{1}{4}, \quad R^1_{325} = -\frac{1}{4}, \quad R^2_{525} = -\frac{3}{4},\\
&R^1_{426} = R^2_{134} = R^3_{535} = R^3_{636} = R^1_{246} = R^1_{256} = R^5_{656} = \frac{1}{4}, \quad R^1_{526} = \frac{1}{2},\\
&R^2_{626} = -\frac{3}{4}, \quad R^1_{234} = R^1_{235} = R^4_{635} = R^5_{635} = R^3_{546} = R^3_{556} = -\frac{1}{4},\\
&R^l_{ijk} = 0 \quad \text{for the rest of the indices.}
\end{aligned}$$

According to proposition 1, $\mathcal{K}$ consists of the linear operators $A : \wedge^2 L(U^R) \mapsto \wedge^2 L(U^R)$ such that $\widetilde{A}(R) = 0$ where $\widetilde{A} : D^r_s(L(U^R)) \to D^r_s(L(U^R))$ is a derivation of the tensor algebra $D(L(U^R))$ induced by A. The following formula is well-known

$$\tag{8} \widetilde{A}(R(X,Y)) = A \cdot R(X,Y) - R(AX,Y) - R(X,AY) - R(X,Y)\cdot A$$

Substituting $R(X_i, X_j)$ from (7) to(8) and taking into consideration that A is necessarily skew-symmetric, one obtains after the long but direct matrix calculations

$$A = \begin{pmatrix} 0 & \alpha_{12} & & & & \\ -\alpha_{12} & 0 & & & 0 & \\ & & 0 & \alpha_{34} & & \\ & & -\alpha_{34} & 0 & & \\ & 0 & & & 0 & \alpha_{56} \\ & & & & -\alpha_{56} & 0 \end{pmatrix}$$

and thus $\mathcal{K}$ is necessarily abelian (note that the condition $C_X(\mathcal{K}) \subset \mathcal{K}$ need not be verified in that case).

Observe that U^R is a simply connected nilpotent Lie group endowed with the left-invariant Riemannian metric. According to Wilson's theorem [**14**] we have

$$I_0(U^R, g_U) = K * U^R$$

where the star $*$ marks the semidirect product. Because $K * U^R$ is homeomorphic to $K \times U^R, K$ must be connected and hence abelian (we have already proved that $L(K) \cong \mathcal{K}$ is abelian). Thus $I_0(U^R, g_U)$ is solvable as a semidirect product of solvable groups.

Remark. In the latter argument we denoted by K the isotropy subgroup of $I_0(U^R, g_U)$.

Now it is sufficient to find the appropriate s-structure on (U^R, g_U). Define the automorphism $\sigma : U \to U$ by the formula

$$\sigma = Ad(diag(1, \varepsilon, \varepsilon^2)) \tag{8}$$

where $\varepsilon = \sqrt[3]{1}$ is a primitive root, and the symbol $Ad(x)$ denotes as usual the automorphism $Ad(x)(A) = xAx^-1$, $x \in T_3 \subset SL_3(\mathbf{C})$, $A \in L(U^R)$ (T_3 is a subgroup of all the upper triangular matrices in $SL_3(\mathbf{C})$). The direct calculation shows that $U^\sigma = \{e\}$. Further, considering σ_* as an automorphism of $L(U)^R$ one obtains its matrix with respect to the basis $\{X_1, X_2 \ldots, X_6\}$:

$$\sigma_* = \mathrm{diag}(\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}, \begin{pmatrix} \cos\alpha & sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & cos2\alpha \end{pmatrix})$$

where $\varepsilon = \cos\alpha + \sqrt{-1}\sin\alpha$. Thus σ_* is metric-preserving. For any left translation $L_u (u \in U^R)$ one has

$$\sigma \cdot L_u \cdot \sigma^{-1} = L_{\sigma(u)}, u \in U^R$$

and therefore $(\sigma_*)_u \cdot (L_u)_{*e}(\sigma^{-1})_{*e}$ is an isometry. As far as $\sigma_* = (\sigma_*)_e$ is metric-preserving, $(\sigma_*)_u$ is also an isometry and thus $\sigma \in I(U^R, g_U)$. Put

$$s_e = \sigma, \quad s_u = L_u \cdot s_e \cdot L_u^{-1}, \quad \in U^R.$$

It is easy to verify by the definition, that $\{s_u, u \in U^R\}$ is a regular Riemannian s-structure on (U^R, g_U) of order 3. Now it is sufficient to mention that (U^R, g_U) is 3-symmetric because it admits no Riemannian s-structure of order 2 (otherwise (U^R, g_U) would be symmetric, which is impossible for the Riemannian manifold with solvable full isometry group [6]).

To finish the proof, it is sufficient to notice that Riemannian metric (1) coincides with g_U. To show the latter, consider the group U^R in the following matrix representation:

$$U^R = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad A + \sqrt{-1}B \in U \right\}.$$

Then $L(U^R)$ consists of the matrices

$$(10)\qquad L(U^R)=\left\{\begin{pmatrix} L & M\\ -M & L\end{pmatrix},\quad L+\sqrt{-1}M\in L(U)\right\}$$

Introduce the coordinates $x_1=\alpha_{12}, x_2=\alpha_{13}, x_3=\alpha_{23}, y_1=\beta_{12}, y_2=\beta_{13}, y_3=\beta_{23}$, where $\alpha_{ij}+\sqrt{-1}\beta_{ij}$ are the elements of the matrix (10). Identify $L(U^R)$ with U^R by means of the exponential mapping (observe that U^R is a simply connected unipotent group and therefore exp: $L(U^R)\to U^R$ is a diffeomorphism).Consider $x_1,\dots,y_3$ as local coordinates in U^R. Then the left-invariant vector fields $\widetilde{X}_1,\dots,\widetilde{X}_6$ on U^R generated by the vectors $X_1,\dots,X_6\in L(U^R)$ can be expressed as follows:

$$(11)\qquad (\widetilde{X}_i)_u=\frac{d}{dt}\Big|_0(u\cdot\exp tX_i)\quad ,u\in U^R$$

Note that exp : $L(U^R)\to U^R$ is expressed in our particular case by

$$(12)\qquad \exp T=E+T+T^2$$

where E is a unit matrix, T is of the form (10). Using (12), one easily obtains that any element $u\in U^R$ having the local coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$ is of the form $u=\begin{pmatrix} A & B\\ -B & A\end{pmatrix}$, where

$$A=\begin{pmatrix}1 & x_1 & x_2+x_1x_3-y_1y_3\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{pmatrix}$$
$$B=\begin{pmatrix}0 & y_1 & y_2+x_1y_3+y_1x_3\\ 0 & 0 & y_3\\ 0 & 0 & 0\end{pmatrix}$$

Using formula (11) one easily finds the expressions for the vector fields

$$\frac{\partial}{\partial x_i},\ \frac{\partial}{\partial y_j}\quad :$$

$$\frac{\partial}{\partial x_1}=\widetilde{X}_1,\quad \frac{\partial}{\partial x_2}=\widetilde{X}_2,\quad \frac{\partial}{\partial x_3}=-x_1\widetilde{X}_2-\widetilde{X}_3-y_1\widetilde{X}_5$$
$$(13)\qquad \frac{\partial}{\partial y_1}=\widetilde{X}_4,\quad \frac{\partial}{\partial y_2}=\widetilde{X}_5,\quad \frac{\partial}{\partial y_3}=-x_1\widetilde{X}_5-\widetilde{X}_6$$

The scalar products $\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\rangle$, or $\langle\frac{\partial}{\partial x_k},\frac{\partial}{\partial y_l}\rangle$, or $\langle\frac{\partial}{\partial y_s},\frac{\partial}{\partial y_t}\rangle$ are calculated taking into consideration the equalities $\langle\widetilde{X}_i,\widetilde{X}_j\rangle=\delta_{ij}$. The result is given by (1). ∎

3.Proof of theorem 2. Before the proof we shall introduce Definition 2 [4]. Let G be connected Lie group and $L(G) = L_1 + L_2$ be a decomposition of its Lie algebra in the sense of Oniščik ($L(G) = L_1 + L_2$, where L_1 and L_2 are the Lie subalgebras). A decomposition $L(G) = L_1 + L_2$ is said to be global if $G = G' \cdot G''$ for the Lie subgroups G' and G'' such that $L(G') = L_1, L(G'') = L_2$.

We shall use the notion "torus" in a usual sense and the notion "algebraic torus" for an abelian algebraic group isomorphic to $(\mathbf{C}^*)^n$ (here $\mathbf{C}^*$ is a multiplicative group of the field $\mathbf{C}$).

Now start the proof. Let $\sigma : I_0(M,g) \to I_0(M,g)$ be the automorphism defined by the formula

$$\sigma(a) = s_o a s_o^{-1}$$

where s_o is a symmetry at a fixed point o of the 3-symmetric Riemannian space. Suppose that $I_0(M,g)$ is solvable. Let $G = Tr(M, \{s_x\})$ be the transvection group (see [9]). Recall that G is σ-invariant Lie subgroup in $I_0(M,g)$ acting transitively on M. Then its isotropy subgroup S is compact and hence S_0 is abelian. Thus S_0 is a torus, or $S_0 = \{e\}$. Then there are two possibilities: 1) $G_0^\sigma = S_0$ is a torus (see [9]), 2) $G_0^\sigma = \{e\}$. Consider the former case. Let $\mathcal{D}L(G) = [L(G), L(G)]$ be the subalgebra generated by all the commutators in $L(G)$. As far as $L(G)$ is solvable, $\mathcal{D}L(G)$ is its nil-radical and $\mathcal{D}L(G) \subset L_1$, where L_1 is a maximal nilpotent ideal in $L(G)$ [3] . According to [3] one has

$$L_1 = \{X \in L(G) | ad_{L(G)}X \text{ is nilpotent}\}.$$

Let $L(S) = L(S_0)$ be the Lie algebra of S. Suppose that $L(S) \cap L_1 \neq \{0\}$. Let $N_1 \subset G$ be the connected Lie subgroup in G corresponding to L_1. Consider $S_0 \cap N_1$. Let $\exp : L(G) \to G$ be the exponential mapping. For $X \in L(S) \cap L_1$ one has $n = \exp X \in S_0 \cap N_1$. Then $Ad(n) = Ad(\exp X) = e^{adX}$ and as far as $adX = ad_{L(G)}X$ is nilpotent one easily obtains $Ad(n)$ to be a unipotent element in $GL(L(G))$. On the other hand, $n \in S_0$. Consider the complexification $S_0(\mathbf{C})$ of the torus S_0. Then $S_0(\mathbf{C})$ is an algebraic torus [13] and therefore any complex representation of it is semisimple. In particular, $AD_{L(G)(\mathbf{C})}(n)$ is simultaneously a semisimple and unipotent linear transformation of a complex Lie algebra $L(G)(\mathbf{C}) = L(G) \otimes \mathbf{C}$. Therefore $Ad_{L(G)}(n) = id$ and hence $Int(n) = id$ as far as G is connected. Thus $n \in Z(G)$. But $G \subset I(M,g)$ and by the definition G acts effectively on $M = G \setminus S$. Hence $n = e$, because $n \in S_0 \subset S$. Thus for any $X \in L_1 \cap L(S)$ one has $\exp X = e$ and hence

$$L_1 \cap L(S) = \{0\} \Rightarrow [L(G), L(G)] \cap L(S) = \{0\}.$$

Define any bilinear positively-definite σ_* - invariant symmetric form on $L(G)$ (it is possible because the group $\langle\sigma_*\rangle$ is finite).Consider the decomposition relatively to that form

$$(14) \qquad L(G) = (L(S) + \mathcal{D}L(G)) \oplus (L(S) + \mathcal{D}L(G))^\perp = (L(S) + \mathcal{D}L(G)) \oplus \widetilde{M}$$

The vector space $L_2 = \mathcal{D}L(G) \oplus \widetilde{M}$ is in fact a subalgebra, as far as $L_2 \supset [L(G), L(G)]$ by the definition. Further,

$$\sigma_*(L(S)) \subset L(S), \quad \sigma_*(\mathcal{D}L(G)) \subset \mathcal{D}L(G)$$

and (14) implies $\sigma_*(\widetilde{M}) \subset \widetilde{M}$. Therefore $\sigma_*(L_2) \subset L_2$. Consider the connected Lie subgroup $U \subset G$ corresponding to L_2. Observe that the triple

$$(L(G), L_2, L(S)) \tag{15}$$

is a decomposition in the sense of Oniščik. Further,

$$L_2 \supset [L(G), L(G)] \Rightarrow [L(S), L_2] \subset [L(G), L(G)] \subset L_2$$

and hence L_2 is an ideal in $L(G)$. Then any inner automorphism $\alpha \in Int(L(G))$ has the property $\alpha(L_2) \subset L_2$ (recall that G is always generated by the set exp $(L(G))$. Then the criterion of the decomposition being global [4] shows that (15) is global. Hence $G = G^\sigma \cdot U'$, where U' is a certain Lie subgroup in G acting transitively on M and such that $L(U') = L_2$. Then $U = U'_0$ also acts transitively on M and $U \subset I_0(M,g)$. Further, $\sigma_*(L_2) \subset L_2 \Rightarrow \sigma(U) \subset U$. Thus (M,g) is isometric to $U \setminus U \cap S$.Evidently, U^σ is discrete as far as its Lie algebra is $L_2 \cap L(S) = \{0\}$ according to (14).Thus U is a Frobenius Lie group. Now consider the simply connected case. Then $U \cap S = \{e\}$ (otherwise M would not be simply connected). Moreover, as far as $\sigma_3 = \mathrm{id}$ and $L(U)^\sigma_* = \{0\}, L(U)$ and consequently U are nilpotent according to the well-known Jacobson's theorem [8]. The nilpotency class is $\leqslant 4$ according to Kreknin's theorem [10]. Recall now that $G = Tr(M, \{s_x\})$ is the minimal σ - invariant Lie subgroup in $I_0(M,g)$ acting transitively on M. It means, that necessarily $G = U$ and in fact only the case $G^\sigma = \{e\}$ may occur. Thus, $G = U$ satisfies all the conditions theorem 2. ∎

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