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Abstract. We construct a family of compact 4-dimensional symplectic manifolds which admit no Kähler structures, and hence no Kähler polarizations. Moreover, we prove that these spaces have no polarizations with non-zero real index.

Keywords: Kähler manifolds, symplectic manifolds, polarizations, geometric quantization

Classification: 53C55, 53C15, 81D07

1. Introduction.

As it is well known, the most important in the geometric quantization of a symplectic manifold is the choice of polarization ([1], [2], [3]). The existence of symplectic manifolds which do not admit polarizations has significant implications for geometric quantization theory. Few examples of such manifolds are known. For, instance, the symplectic manifold $S^2 \times S^2$ has no polarizations with non-zero real index. On the other hand, a symplectic manifold carries totally complex (respectively Kähler) polarizations iff it admits compatible complex (respectively Kähler) structures [2]. So the bundles E^4 of [4] (which are circle bundles over circle bundles over a torus T^2) with $b_1(E^4) = 2$ or 3 have no Kähler polarizations. Moreover, if $b_1(E^4) = 2$, then E^4 has no complex polarizations. Gotay [5] has obtained a class of symplectic 4-manifolds E^4_λ which do not admit polarizations of any type whatever. These E^4_λ are constructed by repeatedly blowing up E^4 with $b_1(E^4) = 2$. Recently, Cordero, Fernández, de León and Saralegui [6] have obtained, by a similar way, a new class of compact symplectic four solvmanifolds without polarization.

In this paper, we extend the results of Gotay in the following sense. We consider circle bundles E^4_g over the product manifold $M_g \times S^1$, where M_g is a Riemann surface of geometric genus $g > 1$. Then E^4_g possesses a symplectic structure but carries no Kähler structures. Therefore, E^4_g has no Kähler polarizations. Moreover, following the construction of Gotay, we obtain a family of compact 4-dimensional symplectic manifolds $E^4_g(\lambda)$ blowing up E^4_g at λ distinct points. We prove:

(1) $E^4_g(\lambda)$ has no Kähler structures, and then $E^4_g(\lambda)$ has no Kähler polarizations;
(2) $E^4_g(\lambda)$ has no polarizations with non-zero real index.

2. The manifolds E^4_g.

Let M_g be a compact Riemann surface of geometric genus $g > 1$. Then there exist $2g$ harmonic differential 1-forms $\xi_1, \dots, \xi_{2g}$ on M_g such that $H^1(M_g, Z) = \{[\xi_1], \dots, [\xi_{2g}]\}$. Let F_g be the Kähler form corresponding to the canonical Kählerian structure on M_g. Therefore we have $H^2(M_g, Z) = \{[F_g]\}$. We denote by $\star$ the Hodge star operator on M_g. Then we can suppose that $\star\xi_i = \xi_{i+g}$, $\star\xi_{i+g} = -\xi_i$, $1 \leq i \leq g$. Consequently we have

$$[F_g] = \sum_{i=1}^{g}[\xi_i \wedge \ast\xi_i].$$

The classification of circle bundles over a manifold is well known (see [7], [8]). We shall use the following result:

Theorem 2.1. *Let M be a manifold. Then there is a one to one correspondence between equivalence classes of circle bundles over M and the cohomology group $H^2(M, Z)$. Furthermore, given a harmonic 2-form Φ on M there is a circle bundle $\pi : E \to M$ with connection form η such that Φ is the curvature of η, that is, $\pi^*\Phi = d\eta$.*

Then, since $H^2(M_g, Z) = \{[F_g]\}$, for any integer n there is a circle bundle $E_g^3(n) \to M_g$ corresponding to $n[F_g]$. Obviously, when $n = 0$, $E_g^3(0)$ is a product $M_g \times S^1$.

The Gysin sequence can be used to compute the integral cohomology groups of $E_g^3(n)$. For $n \neq 0$ they are given by

$$\left.\begin{array}{l} H^0(E_g^3(n), Z) = Z, H^1(E_g^3(n), Z) = Z^{2g}, \\ H^2(E_g^3(n), Z) = Z^{2g} \oplus Z_{|n|}, H^3(E_g^3(n), Z) = Z. \end{array}\right\} \tag{1}$$

Moreover, the real cohomology of $E_g^3(n)$ can be written out explicitly in terms of differential forms. In fact, Theorem 2.1 implies that the connection for form γ of $E_g^3(n) \to M_g$ can be chosen so that the curvature of γ is nF_g (we remark that the same notation for differential forms on M_g and their pullbacks to $E_g^3(n)$ is used). Then (1) can be rewritten as

$$\begin{gathered} H^0(E_g^3(n), Z) = \{[1]\}, H^1(E_g^3(n), Z) = \{[\xi_1], \dots [\xi_{2g}]\}, \\ H^2(E_g^3(n), Z) = \{[\xi_1 \wedge \gamma], \dots, [\xi_{2g} \wedge \gamma]\}, \\ H^3(E_g^3(n), Z) = \{[F_g \wedge \gamma]\}. \end{gathered}$$

Let us now recall the following result due to Bouyakoub [9].

Theorem 2.2. *Let E be a circle bundle over a compact, orientable, connected 3-dimensional manifold M. If M is fibred over S^1 and $b_1(M) \geq 2$, then there is a symplectic structure on E.*

From Theorem 2.2, we have been interested in the circle bundles E_g^4 over $E_g^3(n)$ such that $E_g^3(n)$ is fibred over S^1. $E_g^3(n)$ is a Seifert manifold with associated surface M_g. Since a Seifert manifold E (with associated surface M_g) which is fibred over S^1 must have first Betti number $2g + 1$, we conclude that the only circle bundle $E_g^3(n)$ which is fibred over S^1 is, precisely, the trivial bundle $M_g \times S^1$.

Next, let us consider a circle bundle $E_g^4 \to E_g^3(0)$. These bundles are classified by $H^2(E_g^3(0), Z)$, which is Z^{2g+1}. In particular, for each $(2g + 1)$–tupla

$(p_1,\ldots,p_g,q_1,,q_g,r) \in Z^{2g+1}$, there is a circle bundle corresponding to the cohomology class

$$\sum_{i=1}^{g}(p_i[\xi_i \wedge \gamma] + q_i[*\xi_i \wedge \gamma]) + rF_g.$$

Again we can use Kobayashi's theorem and conclude that the connection form η on $E_g^4 \to E_g^3(n)$ can be chosen so that its curvature form $d\eta$ is

$$\sum_{i=1}^{g}(p_i(\xi_i \wedge \gamma) + q_i(*\xi_i \wedge \gamma)) + rF_g.$$

(As above, we use the same notation for differential forms on $E_g^3(n)$ and their pullbacks to E_g^4.) In the sequel, we denote by E_g^4 the circle bundle over $E_g^3(0)$ corresponding to $(p_1,\ldots,p_g,q_1,\ldots,q_g,0) \in Z^{2g+1}$, where one of p_i, q_i is different from zero.

Theorem 2.3. *E_g^4 has a symplectic structure but no Kähler structures.*

PROOF : In fact, $\Omega_g = \gamma \wedge \eta + F_g$ is closed and has maximal rank 4. Hence Ω_g is a symplectic form on E_g^4. On the other hand, the first Betti number of E_g^4 is odd; in fact, $b_1(E_g^4) = 2g+1$. Consequently, E_g^4 can have no Kähler structure. ∎

3. The manifolds $E_g^4(\lambda)$.

First, we recall some facts about the manifolds E_g^4 considered in Theorem 2.3. They are compact symplectic manifolds. Moreover, they have Euler characteristic and signature zero. In fact, their Betti numbers are $b_0(E_g^4) = b_4(E_g^4) = 1$, $b_1(E_g^4) = b_3(E_g^4) = 2g+1$ and $b_2(E_g^4) = 4g$.

Now, blow up these E_g^4 at λ distinct points using the technique of Gromov and McDuff (see [10]). The resulting manifolds $E_g^4(\lambda)$ are compact 4–manifolds diffeomorphic to

$$E_g^4 \# \lambda \overline{CP}^2,$$

where $\overline{CP}^2$ denotes CP^2 with the reversed orientation. Then $E_g^4(\lambda)$ has signature $\sigma(E_g^4(\lambda)) = -\lambda$ and Betti numbers

$$b_0(E_g^4(\lambda)) = b_4(E_g^4(\lambda)) = 1,$$
$$b_1(E_g^4(\lambda)) = b_3(E_g^4(\lambda)) = 2g+1,$$
$$b_2(E_g^4(\lambda)) = 4g+\lambda.$$

Therefore the Euler characteristic of $E_g^4(\lambda)$ is $\chi(E_g^4(\lambda)) = \lambda$.

Proposition 3.1. *The manifolds $E_g^4(\lambda)$ have a symplectic structure but no Kähler structures.*

PROOF : That $E_g^4(\lambda)$ are symplectic is a direct consequence of [**10**, Proposition 3.7]. Now, since $b_1(E_g^4(\lambda)) = 2g+1$, then $E_g^4(\lambda)$ cannot be Kählerian. ∎

To end this section, we shall prove our main result. First, let us recall some well known facts about polarizations of symplectic manifolds (see [1], [2], [5]).

Let (X,ω) be a $2n$-dimensional symplectic manifold. A *polarization* of (X,ω) is an integrable complex subbundle P of the complexified tangent bundle $T^C X$ which is Lagrangian with respect to the complexification ω^C of ω, that is,

(1) P is of rank n,
(2) $\omega^C/P \times P = 0$,
(3) the involutive real distribution L defined by $L^C = P \cap \overline{P}$ has constant dimension, and
(4) the real distribution K defined by $K^C = P + \overline{P}$ is involutive.

The dimension l of L is called the *real index* of P. When $l = n, P = \overline{P}$ and P is said to be the real polarization. Then $L = K = P \cap TX$. Now, let J be an almost complex structure on X determined by ω (see [2]). We have a Lagrangian splitting $TX = L \oplus JL$ so that (TX, J) may be identifies with L^C. It follows that the odd real Chern classes of (TX, J) vanish.

On the other hand, when $l = 0$, P is called a *totally complex*-polarization. Then $P \cap \overline{P} = 0$, $K = TX$ and P determines an almost complex structure J on X, which is actually a complex structure because P is integrable (see [2]). Moreover, since $\omega(Ju, Jv) = \omega(u,v)$ for all $u, v \in TX$, we can define an Hermitian metric $\langle,\rangle$ on X by $\langle u,v\rangle = \omega(u, Jv)$. If $\langle,\rangle$ is positive definite, then $(X, J, \langle,\rangle)$ is a Kähler manifold and P is said to be *Kähler*.

Remark. The symplectic manifold E_g^4 cannot admit Kähler polarizations since Theorem 2.3.

Theorem 3.2.

(1) *The symplectic manifolds $E_g^4(\lambda)$ have no polarizations of the real index $l \neq 0$.*
(2) *Moreover, $E_g^4(\lambda)$ have no Kähler polarizations.*

PROOF : (2) follows directly from Proposition 3.1. To prove (1), we shall consider two cases, depending upon the value of the real index l, $1 \leq l \leq 2$.
$l = 1$: In this case L would define a field of line elements on $E_g^4(\lambda)$. But this is impossible since $\chi(E_g^4(\lambda)) = \lambda \neq 0$.
$l = 2$: In this case the first real Chern class of $(TE_g^4(\lambda), J)$ must vanish. But we have $c_1^2(TE_g^4(\lambda), J) = 3\sigma(E_g^4(\lambda)) + 2\chi(E_g^4(\lambda)) = -\lambda \neq 0$. ■

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