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On the adjunction of any semiset to the system of Sd_V–classes in AST

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Abstract. It is proved that the closure of the system of Sd_V–classes and of any proper semiset σ on Gödel's operations can be described as the system of classes of the form $R''\rho_n$, where $R \in Sd_V$ and $\rho_n (n \in FN)$ are defined by recursion as follows: ρ_0 is the disjoint union of σ and the complement of σ to any superset and ρ_{n+1} is the powerclass of the complement of ρ_n to any superset. If σ is no set, then no finite number of ρ_n suffices for the description of the whole closure as it is in the case of support. Some other information concerning possible generalizations and importance of Morse's schema is given.

Keywords: Alternative set theory, dependence, Gödel's operations, semiset, endomorphic universe with standard extension

Classification: 03E70, 03H15

Introduction.

In this paper we describe the minimal system of classes closed on definitions by normal formulas containing Sd_V–classes and a chosen semiset $\sigma \subset a$. An obvious, but almost nothing saying description of the system is: "Do the closure on Gödel's operations iterated for finitely times." This procedure is possible in AST due to Morse's schema for definitions of classes. In the paper we prove that this system of classes can be described as the system of classes of the form $R''\rho_n$ where $R \in Sd_V$, $n \in FN$ and ρ_n are defined by recursion as follows:

$$\rho_0 = \{0\} \times \sigma \cup \{1\} \times (a - \sigma), \quad b_0 = \{0,1\} \times a, \quad b_{i+1} = \mathcal{P}(b_i) \text{ and } \rho_{i+1} = \mathcal{P}(b_i - \rho_i).$$

We prove also that if σ is no set, then finitely many ρ_i do not suffice (as it may happen in the case of Boolean models in classical set theory). To the end we mention possible generalizations for some other systems of classes (not only Sd_V–classes) and we point out the substantial role of Morse's schema in the proof of minimality of the described system.

A description of "the minimal envelope" is given (in the framework of the theory of semisets) in [C]. The fact that there is no nonstandard support (a semiset for which the zeroth step in the description of "the minimal envelope" suffices) was proved by B.Balcar [B]. We give here a somewhat different, easier, proof which suffices in the framework of AST.

The reader is supposed to be acquainted with some basic concepts of AST (see [V], [SV]).

Definition. (cf.[TSS])

1) $\mathrm{Dep}(X,Y) \equiv (\exists R \in Sd_V)(X = R''Y)$.
2) $\mathrm{Dep}_d(X,Y) \equiv (\exists F \in Sd_V)(X = (F^{-1})''Y)$.

The defined notions have the following obvious properties which we give without any proof.

$$\mathrm{Dep}_d(X,Y) \Rightarrow \mathrm{Dep}(X,Y), \mathrm{Dep}(X,X), \mathrm{Dep}(X,Y)\ \&\ \mathrm{Dep}(Y,Z) \Rightarrow \mathrm{Dep}(X,Z),$$
$$\mathrm{Dep}_d(X,Y)\ \&\ \mathrm{Dep}_d(Y,Z) \Rightarrow \mathrm{Dep}_d(X,Z), \mathrm{Dep}(X,X_1)\ \&\ \mathrm{Dep}(Y,Y_1) \Rightarrow$$
$$\Rightarrow \mathrm{Dep}(Y \times X, X_1 \times Y_1).$$
If X is a filter or ideal, then $\mathrm{Dep}(X \times X, X)$.

The following two lemmas describe other useful properties of dependence and disjoint dependence.

Lemma 1. *If $\sigma \subset a$, then*

1) $\mathrm{Dep}(X,\sigma) \Rightarrow \mathrm{Dep}_d(V - X, \mathcal{P}(a-\sigma))\ \&\ \mathrm{Dep}_d(X, \mathcal{P}(a) - \mathcal{P}(a-\sigma))$.
2) $\mathrm{Dep}(\mathcal{P}(a) - \mathcal{P}(a-\sigma), \sigma)$.

PROOF :

1) Let us put $F(t) = (R^{-1})''\{t\} \cap a$ and note that $\mathcal{P}(a) - \mathcal{P}(a-\sigma) = \{x \subseteq a; (\exists t \in \sigma)(t \in x)\}$.
2) Let us put $r''\{t\} = \{x \subseteq a; x \ni t\}$.

■

Lemma 2. *Let $\sigma \subset a$. Let us denote $\rho = \mathcal{P}(a-\sigma)$ and $b = \mathcal{P}(a)$.*

1) $\mathrm{Dep}(a-\sigma,\sigma) \Rightarrow \mathrm{Dep}(b-\rho,\rho)\ \&\ \mathrm{Dep}_d(\sigma,\rho)$.
2) $\mathrm{Dep}(\mathcal{P}(a-\sigma),\sigma) \Rightarrow \mathrm{Dep}_d(\mathcal{P}(b-\rho),\rho)$.

PROOF :

1) We have $\mathrm{Dep}_d(a-\sigma, b-\rho)$ by L1, 1), hence $\mathrm{Dep}_d(\sigma,\rho)$, hence $\mathrm{Dep}(b-\rho,\rho)$ by L1, 2) and the transitivity of Dep.
2) $\mathrm{Dep}(\mathcal{P}(b) - \mathcal{P}(b-\rho),\rho) \Rightarrow \mathrm{Dep}(\mathcal{P}(b) - \mathcal{P}(b-\rho),\sigma)$ (by transitivity and the assumption), hence $\mathrm{Dep}_d(\mathcal{P}(b) - \mathcal{P}(b-\rho), (\mathcal{P}(a) - \mathcal{P}(a-\sigma))$ (by L1, 1)), hence $\mathrm{Dep_d}(\mathcal{P}(b-\rho), \mathcal{P}(a-\sigma))$.

■

Theorem 1. *Let $\rho_0 \subset b_0$ be such a semiset that* $\mathrm{Dep}(b_0 - \rho_0, \rho_0)$. *For $n \in FN$ let us define (by recursion) $b_{n+1} = \mathcal{P}(b_n)$ and $\rho_{n+1} = \mathcal{P}(b_n - \rho_n)$. The system of classes of the form $R''\rho_n$, where $R \in Sd_V$ is the least system of classes containing Sd_V classes and the class ρ_0.*

PROOF : As we have the Morse's schema for definitions of classes at our disposal, the definition of ρ_n is correct. Moreover, we can prove by the induction that for k, $n \in FN, k \leq n \Rightarrow \mathrm{Dep}(\rho_k, \rho_n)$ (we use L2, 1)). We prove that the considered system of classes contains all Sd_V–classes and it is closed on the operations $V - X, X \cup Y, X \times Y, \mathrm{dom}(X), X^{-1}$ and $\mathrm{Cnv}(X) = \{\langle t,u,v\rangle; \langle u,v,t\rangle \in X\}$. It follows, from this fact, that the system is closed on Gödel's operations, as the class $E = \{\langle x,y\rangle; x \in y\}$ is set–theoretically definable and except for the operation $\{\cdot,\cdot\}$, Gödel's operations are definable from the given ones. To prove that the system is closed on $\times$, it suffices to mention that for $n > 0$ we have $\rho_n = \mathcal{P}(b_{n-1} - \rho_{n-1})$, hence ρ_n is an ideal and hence $\mathrm{Dep}(\rho_n \times \rho_n, \rho_n)$. For the other operations, the assertion is an immediate consequence of the given properties or it is obtained by an easy definition of the required Sd_V-relation.

The minimality of the system follows from Morse's schema. Formally: Let us put $Y = \{\alpha \in N; (\exists X)(\mathrm{dom}(X) = \alpha \ \& \ X''\{0\} = \rho_0 \ \& \ (\forall \beta \in \alpha - 1)(X''\{\beta + 1\} = \mathcal{P}(b_\beta - X''\{\beta\})))\}$ (where b_β has the obvious sense). We have $Y \supseteq FN$. ∎

Remark. Note that for $\sigma \leq a$ we have $\mathrm{Dep}_d(\{0,1\} \times a - (\{0\} \times \sigma \cup \{1\} \times (a - \sigma)), \{0\} \times \sigma \cup \{1\} \times (a - \sigma))$.

Theorem 2 (B.Balcar). *If* $\sigma \subset a$, *then* $\mathrm{Dep}(\mathcal{P}(a - \sigma), \sigma) \Rightarrow \sigma \in V$.

PROOF : Without loss of generality we can suppose that σ is an ideal (by L2, 2) $\mathcal{P}(a - \sigma)$ has the same property). Let $\mathcal{P}(a - \sigma) = r''\sigma$. Let us define the relation s as follows: $s''\{x\} = \bigcup r''\{t; t \subseteq x\}$. We have $x \subseteq y \Rightarrow s''\{x\} \subseteq s''\{y\}$, $(\forall x \in \sigma)(s''\{x\} \subseteq b - \sigma)$, $(\forall u \subseteq a - \sigma)(\exists t \in \sigma)(u \subseteq s''\{t\})$, $(\forall t \in \sigma)(t \notin s''\{t\})$. Let $b = \{t \in a; t \notin s''\{t\}\}$, hence $b \supseteq \sigma$. Let $c = \{t \in b; s''\{t\}$ be maximal (in inclusion) w.r. to the elements of $b\}$. There are two possibilities.

1) $c \subset a - \sigma$. There is $t \in \sigma$ such that $c \subset s''\{t\}$ in this case. As $s''\{t\}$ is not maximal, there is $u \in b$ such that $s''\{t\} \subset s''\{u\}$ and $s''\{u\}$ is maximal, hence $u \in c$. But we have $s''\{u\} \supset c$, which is in contradiction with $u \notin s''\{u\}$.
2) There is $t \in \sigma \cap c$. We have $(\forall u \in \sigma)(s''\{u\} \subseteq s''\{t\})$ in this case, as $s''(u \cup t) \supseteq s''\{u\} \cup s''\{t\}$ and $u \cup t \in \sigma$ (σ is an ideal). Hence $a - \sigma = s''\{t\}$.

∎

The last theorem proves that if σ is no set, then in no case the system $R''\rho_k$ creates the minimal envelope for any $k \in FN$, as $\mathrm{Dep}(\mathcal{P}(b_k - \rho_k), \rho_k)$ would be a consequence.

Note that the usage of recursively defined sets $b_n (b_{n+1} = \mathcal{P}(b_n))$ is not substantial. We require only $b_n \supseteq \rho_n$.

The given method may serve also in other cases for an adjunction of a semiset to a system of classes closed on Gödel's operations. Almost immediate is the usage of Sd_V^* classes instead of Sd_V ones. In fact, the only requirement which we have, is that for a suitable superset a of the semiset σ, we have that $\underbrace{\mathcal{P}(\ldots \mathcal{P}(a) \ldots)}_{n\text{-times}} \cap X \in V$ for every $n \in FN$ and every X of the considered system of classes. A nontrivial example of such usage is the following one. Let $\sigma \subset \alpha \in \mathrm{Ex}(FN)$ (see [SV]) and let the considered system of classes be the system of classes of the form $\mathrm{Ex}(X)''\{x\}$, where X are taken from any system of subclasses (e.g.from the system of all subclasses) of an endomorphic universe A with a standard extension, having the property that the system $X''\{a\}$ $(a \in A)$ is closed on Gödel's operations.

In the case that we do not have Morse's schema at our disposal, we can adopt the given method in the following way. We consider the system of classes of the form $R''X$, where $\mathrm{dom}(X) \in N \ \& \ X''\{0\} = \rho_0 \ \& \ (\forall \alpha \in \mathrm{dom}(X) - 1)(X''\{\alpha + 1\} = \mathcal{P}(b_\alpha - X''\{\alpha\}))$. The just described system of classes is closed on Gödel's operations, too, but the minimality cannot be proved as the following example shows. Let us put $\sigma = \mathrm{Ex}(FN)$ and $a \in N - \mathrm{Ex}(FN)$. The iteration in the definition of ρ_α goes over the whole $\mathrm{Ex}(FN)$ in this case, but for $\alpha \in \mathrm{Ex}(FN) - FN$, the semiset ρ_α is not in the minimal envelope.

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