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Remarks on bounded solutions of a semilinear dissipative hyperbolic equation

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Abstract. Global estimates for solutions of a hyperbolic equation with a nonlinear dissipative term of polynomial or arbitrary growth are proved. Moreover, a global estimate of Hölder constant for a solution is derived.

Keywords: A priori estimate, semilinear hyperbolic equation, bounded solution.

Classification: 35B45, 35B30, 35L20

Introduction. A.Haraux in [1] proved the estimate $E(t) \leqq c(\|f\|_\infty^4 + 1)$ of the energy for the equation

$$u_{tt} + Lu + g(u_t) = f \tag{1}$$

for arbitrary growth of the function g and confronted it with better estimate $E(t) \leqq c(\|f\|_\infty^2 + 1)$ in the case of polynomial growth of g.

The first remark is devoted to a connection between these two estimates, whose slightly different proofs are given here. The second remark gives the global estimate of the difference of two solutions,that implies continuous dependence of a solution u on f and initial data and a global estimate of Hölder constant of u.

Assumptions and results. Let us recall some assumptions given in [1] and [2].

Assumptions on L. Let Ω be a bounded domain in R^n, a Hilbert space V be densely and continuously imbedded in $L^2(\Omega)$ and let $L : V \to L^2(\Omega)$ be a linear symmetric positive operator.Denote $\langle u,v\rangle$ the duality between V and V', $\|u\| = \langle Lu,u\rangle^{\frac{1}{2}}$ the norm in V and (u,v) the inner product in $L^2(\Omega)$. Let V be continuously imbedded into $L^s(\Omega)$:

$$|u|_s \leqq c_0\|u\|, u \in V, |u|_2 \leqq c_0|u|_s, u \in L^s(\Omega) \tag{2}$$

where $|u|_s$ is the norm of u in $L^s(\Omega), s \leqq \gamma + 1$ (γ is defined in the embedding theorem).

Assumptions on g. Let $g(y)$ be a continuous non-decreasing function on R satisfying the following

(3) $g(0) = 0,$ there exist $\eta > 0, C_1 : g(y)y \geqq \eta|y|^{p+1} - c_1|y|, \quad y \in R,$
for some $p \in [1,\gamma]$

(4) there exist $C_3, C_4 : \|g(v)\|_{V'} \leqq c_3 + c_4\langle g(v), v\rangle \quad v \in V$

(6) there exist $c_2, \eta > 0, p \in [1,\gamma]$ such that

$$\eta|y_1 - y_2|^{p-1} \leqq \frac{g(y_1) - g(y_2)}{y_1 - y_2} \leqq c_2(1 + [g(y_1)y_1 + g(y_2)y_2]^{\frac{\gamma-1}{\gamma+1}})$$

hold for every $y_1, y_2 \in R, y_1 \neq y_2$.

Remark 1. The polynomial growth of g.i.e.

(4') $$|g(y)| \leqq c_5(|y|^q + 1), y \in R, \text{ for some } q \leqq \gamma,$$

$$\text{gives } |g(y)|^{\frac{q+1}{q}} = |g(y)|.|g(y)|^{\frac{1}{q}} \leqq c_5^{\frac{1}{q}}(g(y)y + |g(y)|)$$

$$\text{which implies } |g(y)|^{\frac{q+1}{q}} \leqq 2c_5 g(y)y + c_6,$$

where c_6 depends on c_5 and q only. Hence

$$|\langle g(v), u\rangle| \leqq |g(v)|_{\frac{q+1}{q}}|u|_{q+1} \leqq (2C_5\langle g(v), v\rangle + C_6\mu(\Omega))^{\frac{q}{q+1}}.\|u\|,$$
$$u \in V, v \in L^{q+1}(\Omega)$$

and (4) holds for arbitrary c_4 and $c_3 = \frac{(2qc_5)^q}{(q+1)^{q+1}}c_4^{-q} + c_6\mu(\Omega)$.

Assumptions on $f(t,x)$.

(5) $$f \in L^1_{loc}(R^+, L^2(\Omega)), \quad \sup_{t \geqq 0} \int_t^{t+1} |f(s,.)|^{\frac{p+1}{p}}_{\frac{p+1}{p}} ds \equiv H_p^{\frac{p+1}{p}} < +\infty.$$

The existence, uniqueness and regularity of the initial value problem associated to the equation (1) is known (see e.g. [3], [4]). Let us remind some properties of the solution $u(t,x)$:

(7) $$u \in W^{2,\infty}_{loc}(R^+, L^2(\Omega)) \cap W^{1,\infty}_{loc}(R^+, V)$$

(8) $$g(u_t)u_t \in L^1_{loc}(R^+, L^1(\Omega))$$

(9) $$\sup_{t \geqq 0} \int_t^{t+1} \langle g(u_t), u_t\rangle ds \leqq \sup_{t \geqq 0} E_0(t) + \sqrt{2}\sup_{t \geqq 0}\sqrt{E_0(t)}H_1,$$

$$\text{where } E_0(t) = \frac{1}{2}(\|u(t,.)\|^2 + |u_t(t,.)|_2^2)$$

for the initial data $u^0 \in D(L)$, $v^0 \in V$, $g(v^0) \in L^2(\Omega)$ and for $f \in W^{1,1}_{loc}(R^+, L^2(\Omega))$.

We prove the following estimates.

Theorem 1. *Under the assumptions (2) - (5) there exists a constant c such that*

(10) $$E_0(t) \leqq c(E_0(0) + H_p^2 + c_4^2 H_p^{2\frac{p+1}{p}} + c_3^2 + c_4^2 + 1), \quad t \geqq 0,$$

where c does not depend on u, f, c_3, c_4 and the initial data.

Theorem 2. *Let the assumptions (2) - (6) be satisfied and let u_i be a solution of (1) with the right hand side f_i and the initial data u_i^0, v_i^0, $i = 1,2$. Denote $f = f_1 - f_2, u = u_1 - u_2, u^0 = u_1^0 - u_2^0, v^0 = v_1^0 - v_2^0$ and*

$$E_\varepsilon(t) \equiv E_0(t) + \varepsilon(u_t, u) \tag{11}$$

the modified energy functional. Then there exist constants $c, \varepsilon_1 > 0$ such that the following inequality

$$E_\varepsilon(t) \leqq E_\varepsilon(0) + c(H_p^2 + \frac{1}{\varepsilon} H_p^{\frac{p+1}{p}} + \varepsilon^{\frac{2}{p-1}}) \equiv E_\varepsilon(0) + cM(\varepsilon) \tag{12}$$

holds for every $\varepsilon \in (0, \varepsilon_1]$ and every $t \geqq 0$. Especially, choosing $\varepsilon = \varepsilon_0$ to minimize $M(\varepsilon)$, we get

$$\begin{aligned} E_0(t) &\leqq 2E_{\varepsilon_0}(t) \leqq 2E_0(0) + 2c(H_p^2 + H_p^{\frac{2}{p}}) && \text{for} \quad H_p \leqq a \\ E_0(t) &\leqq 2E_{\varepsilon_1}(t) \leqq 2E_0(0) + 2cM(\varepsilon_1) && \text{for} \quad H_p \geqq a, \end{aligned} \tag{13}$$

where $a = (\frac{p-1}{2})^{-\frac{p}{p+1}} \varepsilon_1^{\frac{p}{p-1}}$, $\varepsilon_0 = (\frac{p-1}{2})^{\frac{p-1}{p+1}} H_p^{\frac{p-1}{p}}$, $p > 1$.
In the case $p = 1$ (12) holds without the last term.

Remark 2. The estimate (13) gives continuous dependence of a solution of (1) on the right hand side f and the initial data u^0, v^0 globally in $t \in R^+$. Moreover, putting $u_1(t,x) = u(t+h,x), u_2(t,x) = u(t,x), f_1(t,x) = f(t+h,x), f_2(t,x) = f(t,x)$ the inequality (13) gives the upper bound of the Hölder constant of u and u_t (in t) for $t \geqq 0$ with the exponent p^{-1}, if $u^0 \in D(L), v^0 \in V$ and f_t satisfies (5).
The estimates (10), (12) and (13) will be proved under stronger assumptions on smoothness of u, but the general case may be obtained approximating the functions f, u^0 and v^0.

Proof of Theorem 1. First, we formulate some estimates which will be used in the proof. Having in mind (3), (5) and (8), we may estimate the scalar product

$$|\langle f, u_t + \varepsilon u\rangle| \leqq |f|_{\frac{p+1}{p}}(|u_t|_{p+1} + \varepsilon|u|_{p+1}) \leqq |f|_{\frac{p+1}{p}}(|u_t|_{p+1} + \varepsilon c_0\|u\|), \tag{14}$$

the duality pairing

$$\begin{aligned} |\langle g(u_t), u\rangle| &\leqq \|g(u_t)\|_{V'}\|u\| \leqq (c_3 + c_4\langle g(u_t), u_t\rangle)\|u\| \leqq \\ &\leqq [\text{illegible}]\,\|u\| + c_4\sqrt{2E_0(t)}\langle g(u_t), u_t\rangle \end{aligned} \tag{15}$$

and the polynomial

$$-\frac{\eta}{2}x^{p+1} + \frac{\varepsilon c_0^2}{2}(3+\varepsilon)x^2 + (b + c_1)x \leqq c(b^{\frac{p+1}{p}} + \varepsilon^{\frac{p+1}{p-1}} + c_1^{\frac{b+1}{p}}), \quad x \in R^+. \tag{16}$$

In the whole paper, a constant c is independent of ε, f, c_3 and c_4. The second term on the right hand side of (16) is absent for $p = 1$.

Multiplying the equation (1) by the sum $u_t + \varepsilon u$(ε being a small positive number) and integrating it over Ω, we get

(17)
$$E'_\varepsilon(t) + \varepsilon E_\varepsilon(t) = -\frac{\varepsilon}{2}\|u\|^2 + \frac{3}{2}\varepsilon|u_t|_2^2 + \varepsilon^2(u_t, u) - \langle g(u_t), u_t + \varepsilon u\rangle + (f, u_t + \varepsilon u).$$

Using (14) and (15) to the last two terms of (17), we may write

$$\begin{aligned} E'_\varepsilon(t) + \varepsilon E_\varepsilon(t) \leqq & -\frac{\varepsilon}{2}\|u\|^2 + \frac{3}{2}\varepsilon|u_t|_2^2 + \frac{\varepsilon^2}{2}(|u_t|_2^2 + c_0^2\|u\|^2) - \frac{1}{2}\langle g(u_t), u_t\rangle + \\ & + (-\frac{1}{2} + \varepsilon c_4\sqrt{2E_0(t)}) \quad \langle g(u_t), u_t\rangle + \varepsilon c_3\|u\| + \\ & + |f|_{\frac{p+1}{p}}(|u_t|_{p+1} + \varepsilon c_0\|u\|). \end{aligned}$$

Using (3), (16) for $x = |u_t|_{p+1}, b = |f|_{\frac{p+1}{p}}$ and the inequality

$$-\frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon^2}{2}c_0^2\|u\|^2 + \varepsilon c_3\|u\| \leqq \varepsilon c_3^2$$

which holds for $0 \leqq \varepsilon \leqq \frac{1}{2c_0^2}$, we get

$$\begin{aligned} E'_\varepsilon(t) + \varepsilon E_\varepsilon(t) \leqq & \varepsilon(c_3^2 + c_0|f|_{\frac{p+1}{p}}\|u\|) + c(|f|_{\frac{p+1}{p}}^{\frac{p+1}{p}} + \varepsilon^{\frac{p+1}{p-1}} + c_1^{\frac{p+1}{p}} + \\ & + (\varepsilon c_4\sqrt{2E_0(t)} - \frac{1}{2})\langle g(u_t), u_t\rangle. \end{aligned}$$

Now, let us multiply this inequality by $e^{\varepsilon t}$, integrate it over the interval $(0, t)$, $t \in [0, T]$ (T being a fixed but arbitrary positive number), denote $\overline{E}_\varepsilon = \max_{0 \leqq t \leqq T} E_\varepsilon(t)$ and take ε so small that the last term is not positive, i.e.$0 \leqq \varepsilon \leqq \varepsilon_T \equiv \frac{1}{2C_4\sqrt{2\overline{E}_0}}$. Since

$$c_0\varepsilon \int_0^t |f(s,\cdot)|_{\frac{p+1}{p}} \cdot \|u(s,\cdot)\| e^{\varepsilon s} ds \leqq c\sqrt{2\overline{E}_0} H_p e^{\varepsilon t} \leqq (\frac{1}{2}\overline{E}_0 + cH_p^2)e^{\varepsilon t},$$

we get

$$E_\varepsilon(t)e^{\varepsilon t} - E_\varepsilon(0) \leqq c(c_3^2 + H_p^2)e^{\varepsilon t} + \frac{1}{4}\overline{E}_0 e^{\varepsilon t} + \frac{c}{\varepsilon}(H_p^{\frac{p+1}{p}} + c_1^{\frac{p+1}{p}} + \varepsilon^{\frac{p+1}{p-1}})(e^{\varepsilon t} - 1)$$

(18) $\qquad$ for $t \in [0, T]$ and $0 < \varepsilon \leqq \min(\frac{1}{2c_0^2}, \varepsilon_T)$.

As the modified energy functional $E_\varepsilon(t)$ may be estimated for $0 \leqq \varepsilon \leqq \frac{1}{2c_0}$ by the energy functional $E_0(t)$:

$$E_0(t) \leqq 2E_\varepsilon(t) \leqq 3E_0(t), \quad t \geqq 0, \tag{19}$$

we obtain from (18)

$$\overline{E}_0 \leqq 6E_0(0) + c(c_3^2 + H_p^2 + \varepsilon^{\frac{2}{p-1}}) + \frac{c}{\varepsilon}(H_p^{\frac{p+1}{p}} + c_1^{\frac{p+1}{p}})$$

for $\varepsilon \in (0, \min(\varepsilon_1, \varepsilon_T))$, where $\varepsilon_1 = \frac{1}{2}\min(c_0^{-1}, c_0^{-2})$, which implies (using the definition of ε_T)

$$\overline{E}_0 \leqq 6E_0(0) + c(c_3^2 + H_p^2 + 1) + \frac{1}{2}\overline{E}_0 + cc_4^2(H_p^{2\cdot\frac{p+1}{p}} + 1).$$

It says that E_0 does not exceed the right hand side of (10), which is independent of T and then (10) holds for every $t \geqq 0$.

Remark 3. Let (3), (4') and (5) be satisfied. Having in mind Remark 1, choosing $c_4 = H_p^{-\frac{p+1}{p(q+1)}}$ and c_3 from Remark 1, we may write the inequality (10) in the following from

$$E_0(t) \leqq c(E_0(0) + H_p^2 + H_p^{2\cdot\frac{q}{q+1}\cdot\frac{p+1}{p}} + 1). \tag{20}$$

If $q = p$ (e.g. $g(y) = |y|^{p-1}y$), (20) gives the known results (see e.g. [5])

$$E_0(t) \leqq c(E_0(0) + H_p^2 + 1). \tag{21}$$

The estimate (20) may be deduced from (17) in another way: Using the last inequality of Remark 1 in (15) and (17), we get

$$E_\varepsilon'(t) + \varepsilon E_\varepsilon(t) \leqq \frac{1}{2}\langle g(u_t), u_t\rangle^{\frac{q}{q+1}}(4c_5\varepsilon\sqrt{E_\varepsilon(t)} - \langle g(u_t), u_t\rangle^{\frac{1}{q+1}}) +$$
$$+ \varepsilon c_0^2|f|^2_{\frac{p+1}{p}} + c(|f|_{\frac{p+1}{p}} + c_1 + \varepsilon)^{\frac{p+1}{p}}.$$

Similarly to the proof of Theorem 1 we obtain (multiplying the above inequality by $e^{\varepsilon t}$ and integrating it over $(0,t)$)

$$\overline{E}_\varepsilon \leqq E_\varepsilon(0) + c\varepsilon^q(4c_5)^{q+1}\overline{E}_0^{\frac{q+1}{2}} + cH_p^2 + \frac{c}{\varepsilon}(|f|_{\frac{p+1}{p}} + c_1)^{\frac{p+1}{p}} + c\varepsilon^{\frac{1}{p}}, \tag{22}$$

where $\overline{E}_\varepsilon equiv \max_{0\leqq t\leqq T} E_\varepsilon(t)$. Now, ε may be chosen such that the sum of the second and forth terms of the right side of (22) might be minimal, i.e. $\varepsilon = \varepsilon_0 = c(H_p^{\frac{p+1}{p}} + c_1)^{\frac{1}{q+1}}\overline{E}_0^{-\frac{1}{2}}$, which gives the inequality (20) (putting into (22)).

Proof of Theorem 2. Since the difference $u = u_1 - u_2$ satisfies the equation

$$u_{tt} + Lu + g(u_{1,t}) - g(u_{2,t}) = f,$$

we can proceed similarly to the proof of Theorem 1,i.e. multiply this equation by $u_t+\varepsilon u$ and integrate it over Ω. Instead of (15) we must estimate $\langle g(u_{1,t})-g(u_{2,t}),u\rangle$. Denoting $\varphi(t) \equiv (|g(u_{1,t})u_{1,t}|_1 + |g(u_{2,t})u_{2,t}|_1)^{\frac{\gamma-1}{\gamma+1}}$ and using (6) and (8) we have

$$\begin{aligned}|\langle g(u_{1,t}) - g(u_{2,t}), u\rangle| &\leqq \frac{1}{2}|\langle \frac{g(u_{1,t}) - g(u_{2,t})}{u_t}, \frac{1}{\delta}u_t^2 + \delta u^2\rangle| \leqq \\ &\leqq \frac{1}{2\delta}\langle g(u_{1,t}) - g(u_{2,t}), u_t\rangle + \\ + \frac{\delta}{2}c_2 &\left\{\int_\Omega [1 + (g(u_{1,t})u_{1,t} + g(u_{2,t})u_{2,t})^{\frac{\gamma-1}{\gamma+1}}]^{\frac{\gamma+1}{\gamma-1}}\, dx\right\}^{\frac{\gamma-1}{\gamma+1}} |u|_{\gamma+1}^2 \leqq \\ &\leqq \frac{1}{2\delta}\langle g(u_{1,t}) - g(u_{2,t}), u_t\rangle + \frac{\delta c_2 c_0^2}{2}[1+\varphi(t)]\|u\|^2\end{aligned}$$

The modified energy functional $E_\varepsilon(t)$ for the difference $u = u_1 - u_2$ must satisfy the following (due to (16))

$$\begin{aligned}(23)\quad E'_\varepsilon(t) + \varepsilon E_\varepsilon(t) &\leqq -\frac{\varepsilon}{2}\|u\|^2 + \frac{3\varepsilon}{2}|u_t|_2^2 + |f|_{\frac{p+1}{p}}|u_t|_{p+1} + \varepsilon|f|_{\frac{p+1}{p}}c_0\|u\| - \\ &-\langle g(u_{1,t}) - g(u_{2,t}), (1-\frac{\varepsilon}{2\delta})u_t\rangle + \frac{\varepsilon c_2 c_0^2\delta}{2}\|u\|[1+\varphi(t)] \leqq \\ &\leqq \frac{\varepsilon c_0^2}{4}|f|_{\frac{p+1}{p}}^2 + c(|f|_{\frac{p+1}{p}}^{\frac{p+1}{p}} + \varepsilon^{\frac{p+1}{p-1}}) + \frac{\varepsilon c_2 c_0^2\delta}{2}[(1+\varphi(t)]\|u\|^2.\end{aligned}$$

Choosing δ, ε_1 so small, to satisfy $1 - \frac{\varepsilon_1}{2\delta} \geqq \frac{1}{2}$, $\frac{1}{2} - \delta c_2 c_0^2 \geqq 0$ and $c_2 c_0^2 \delta \sup_{t\geqq 0} \int_t^{t+1} \varphi(s)ds \leqq \frac{1}{2}$, multiplying (23) by $e^{\varepsilon t}$ and integrating it over $(0,t)$, $t \in [0,T]$, we get

$$E_\varepsilon(t) \leqq E_\varepsilon(0) + c(\varepsilon^{\frac{2}{p-1}} + H_p^2) + \frac{c}{\varepsilon}H_p^{\frac{p+1}{p}} + \frac{1}{4}\overline{E}_0.$$

Using (19), we obtain

$$\overline{E}_\varepsilon \leqq 4E_\varepsilon(0) + c(\varepsilon^{\frac{2}{p-1}} + H_p^2) + \frac{c}{\varepsilon}H_p^{\frac{p+1}{p}}, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_1].$$

Since T was chosen arbitrary, the last inequality implies (12).

REFERENCES

[1] A.Haraux, *Two remarks on hyperbolic dissipative problems.*, (Nonlinear part. diff. equations and their appl., College de France seminar, vol.7, Editors: H.Brezis, J.L.Lions.

[2] A.Haraux, E.Zuazua, *Decay estimates for some semilinear damped hyperbolic problems*, Archive Rat.Mech.Anal. **150** (1988), 191–206.

[3] A.Haraux, *Nonlinear Evolution Equations - Global Behavior of Solutions*, (Lecture Notes in Math. 841, Springer-Verlag Berlin, Heidelberg,New York 1981.

[4] A.Haraux, *Semilinear hyperbolic problems in bounded domain.*, (Math.Reports., J.Dieudon ne' Editor, Harwook Acad.Publ., Gordon & Breach).

[5] G.Prouse, *Soluzioni quasi periodiche delle equazione delle onde con termine dissipativo non lineare*, I-IV, (Rend.Acad.Naz.Lincei 38,39 (1965)).

[6] J.L.Lions, W.A.Strauss, *Some non-linear evolution equations*, Bull.Soc.Math.France **93** (1965), 43–96.

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