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Cobalanced exact sequences

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Abstract. A sequence of abelian groups $\mathcal{E} : 0 \to A \to B \to C \to 0$ is said to be balanced exact if for every generalized height vector h, the induced sequence $0 \to A(h) \to B(h) \to C(h) \to 0$ is exact. If C is torsion-free, then $\mathcal{E}$ is balanced if and only if every rank one torsion-free group is projective with respect to $\mathcal{E}$. Dually, we consider sequences $\mathcal{E}$ with A torsion-free, and say that $\mathcal{E}$ is cobalanced if the torsion-free rank ones are injuctive with respect to it. It is a well known result of Bican and Salce that for torsion-free finite rank groups C, the group of balanced exact sequences $\mathrm{Bext}(C, T) = 0$ for all torsion groups T if and only if C is a Butler group. We will show that in the dual case that a countable torsion-free group A satisfies that the group of cobalanced exact sequences $\mathrm{Cobext}(T, A) = 0$ for all torsion groups T is and only if A is locally completely decomposable.

Keywords: Cobalanced sequences, vector groups, locally completely decomposable groups

Classification: Primary: 20K27, 20K40, Secondary: 20K15, 20K20

I. Cobalanced Sequences.

If $f : A \to B$ is an epimorphism with kernel D, then the pull back of

$$\begin{array}{ccccccccc} &&&&&& C &&\\ &&&&&& \downarrow &&\\ \mathcal{E}:0 & \longrightarrow & D & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0\end{array}$$

is readily seen to be cobalanced if $\mathcal{E}$ is cobalanced. Also, the pushout of a cobalanced monomorphism is cobalanced. One can then define $\mathrm{Cobext}(C, A)$ in the standard fashion and for any cobalanced $\mathcal{E} : 0 \to A \to B \to C \to 0$ we have the derived long exact sequence: $0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Cobext}(G, A) \to \mathrm{Cobext}(G, B) \to \mathrm{Cobext}(G, C)$ by applying $\mathrm{Hom}(G, \ )$ to $\mathcal{E}$ together with the analogous sequence when one applies $\mathrm{Hom}(\ , G)$ to the sequence (cf, [**9**]).

Let π be the set of primes. Given a type τ, we let $A[\tau] = \cap\{\mathrm{Ker}(f) : f : A \to X_\tau\}$ where X_τ is a torsion-free group of rank-1 and type τ, and let $\pi(\tau) = \{p \in \pi : pX_\tau \neq X_\tau\}$. For $S \subset \pi$, a sequence $0 \to A \to B \to C \to 0$ is said to be S-pure if $nA = nB \cap A$ for all n in the multiplicative closure of S in $\mathbb{Z}$.

Proposition I.1. *Let $A, B,$ and X be countable torsion-free groups with A of finite rank and X a rank-1 of type τ. Then X is injective with respect to $0 \to A \to B \to C \to 0$ if and only if $A[\tau] = B[\tau] \cap A$ and $0 \to A/A[\tau] \to B/B[\tau]$ is $\pi(\tau)$-pure.*

PROOF : $\Rightarrow$) Let $x \in A$. Then $x \neq A[\tau]$ implies there is an $f : A \to X$ with $f(x) \neq 0$. This map can be extended to B. Thus $x \notin B[\tau] \cap A$. Since generally $A[\tau] \subset B[\tau] \cap A, A[\tau] = B[\tau] \cap A$.

Let $\overline{A} = A/A[\tau]$, $\overline{B} = B/B[\tau]$, and $p \in \pi(\tau)$. If $a \in \overline{A}$ has p–height 0, then Proposition 2.1 in [5] implies that there is a map $f : \overline{A} \to X$ such that $f(a)$ has p–height zero in X. This map is easily shown to lift to $\overline{B}$. Thus a has p–height zero in $\overline{B}$.

$\Leftarrow$) We must show that $0 \to \overline{A} \to \overline{B}$ is cobalanced with respect to X. The localizing at $S = \pi(\tau)$ produces a pure exact sequence $\mathcal{E} : 0 \to (\overline{A})_S \to (\overline{B})_S$, with $(\overline{A})_S$ finite rank torsion–free and $(\overline{B})_S$ countable torsion–free. Since $(\overline{B})_S[\tau] = 0$, X is injective with respect to $\mathcal{E}$ by Proposition 2.1 of [5]. ■

This proposition sheds light on Example 1.10 in [8] and generalizes Proposition 4.2 in [1]. If F is a countably infinite ranked free group with $F/K \cong \mathbf{Q}$, then $F[\tau] = K[\tau] = 0$ for all types τ, but no reduced rank–1 torsion–free group is injective with respect to $0 \to K \to F$. Also, if B is not separable with $B[\text{type}(\mathbf{Z})] = 0$, then there is a finite rank free pure subgroup A of B which is not a summand of B. Thus we see that the hypotheses are necessary.

Corollary I.2. *If B is a finite rank torsion–free τ–homogeneous group, then $B[\tau] = 0$ if and only if B is completely decomposable.*

PROOF : ($\Leftarrow$) Clear.

($\Rightarrow$) Let B have rank–n and A be a pure rank–1 subgroup of B. Then the proposition implies that $B \cong A \oplus B/A$. Thus B/A satisfies the hypothesis of the corollary and so induction on rank will prove it. ■

Corollary I.3. *If B is a countable torsion–free τ–homogeneous group, then $B[\tau] = 0$ if and only if B is completely decomposable.*

PROOF : ($\Leftarrow$) Clear.

($\Rightarrow$) Let A be a pure finite rank subgroup of B. Then A satisfies the hypothesis of the previous corollary and thus is completely decomposable τ–homogeneous. Hence Proposition I.1 implies that A is a summand of B. Therefore, by Proposition 87.2 in [3], B is separable and hence completely decomposable (Theorem 87.1, [3]). ■

The torsion–free vector groups are cobalanced injective. Given any torsion–free group G, the canonical embedding

$$0 \to G \to \prod\{G/K : G/K \text{ is rank–1 torsion–free}\}$$

is cobalanced. It is not hard to see that the reduced cobalanced injectives are summands of (reduced) vector groups and are thus vector groups ([7]).

Proposition I.4. *Let G be torsion–free. The following are equivalent:*

(a) *G is a subgroup of a vector group V with V/G torsion–free (cotorsion–free).*
(b) Cobext$(T, G) = 0$ *for all torsion (cotorsion) groups T.*
(c) *If $0 \to G \to H \to H/G \to 0$ is cobalanced and H is torsion–free (cotorsion–free), then H/G is torsion–free (cotorsion–free).*

PROOF : We will only prove the characterization of G when Cobext$(T, G) = 0$ for all cotorsion T, since the argument in the other case is similar.

(a)⇒(b) Let $\iota : G \to V$ be the inclusion map and $q : V \to V/G$ be the quotient map. If T is cotorsion and $\mathcal{E} : 0 \to G \xrightarrow{f} H \xrightarrow{g} T \to 0$ is cobalanced, then there is a map $\iota' : H \to V$ s.t. $\iota' f = \iota$. Thus $q\iota' f = 0$ and hence there is a unique $q' T \to V/G$ s.t. $q'g = q\iota'$. Since V/G is cotorsion–free, $q' = 0$. Thus there is a unique $f' : H \to G$ s.t. $\iota f' = \iota'$. Thus $\iota f' f = \iota' f = \iota$. Hence $f' f = 1_G$, i.e., $\mathcal{E}$ splits.
(b)⇒(c) Let $\mathcal{E} : 0 \to G \to H \to H/G \to 0$ be cobalanced with H cotorsion–free. Then applying Hom(T,) to $\mathcal{E}$ for a cotorsion group T, we get the exact sequence Hom$(T, H) \to$ Hom$(T, H/G) \to$ Cobext(T, G). Since the two ends are zero, the middle is zero and thus H/G is cotorsion–free.
(c)⇒(a) Clear. ■

II. Locally Completely Decomposable Groups.

The next result is the cobalanced analog to Theorem 1.4 on balanced extensions in [2]. Let τ_p = type($\mathbb{Z}_p$), where $\mathbb{Z}_p$ is the group of integers localized at the prime p.

Theorem II.1. *Let G be a countable torsion–free group. The following are equivalent:*

(a) *G is locally completely decomposable.*
(b) $(G/p^\omega G)[\tau_p] = 0$ *for all p.*
(c) *G is a pure subgroup of a vector group.*
(d) Cobext$(T, G) = 0$ *for all torsion groups T.*

PROOF : (a)⇒(c) Let $\iota : G \to \Pi\{G/K : G/K$ is rank–1$\} = V$ be the canonical cobalanced embedding of G into a vector group. Let $x \in G$ such that the p–height of x, denoted $ht_p^G(x)$, is k. Since G is locally completely decomposable, there is an $f : G \to \mathbb{Z}_p$ with $f(x) = p^k$. Thus, because ι is a cobalanced embedding, there is a map $f' : V \to \mathbb{Z}_p$ such that $f'\iota = f$. Hence $ht_p^V(\iota(x)) \le ht_p^{\mathbb{Z}_p}(f'\iota(x)) = k$. Therefore, G is pure in V.
(c)⇒(b) Let ι be as in the previous part, and $0 \ne x + p^\omega G$. Thus the p–height of x in G is finite. Proposition I.4 and (c) imply that G is pure in $\Pi\{G/K : G/K$ is rank–1$\}$ and hence there is a corank–1 subgroup $K \subset G$ with $ht_p^{G/K}(x + K)$ finite. This implies that the type$(G/K) \le \tau_p$ and so there is a map $f : G/K \to \mathbb{Z}_p$ with $f(x + K) \ne 0$. Let $g : G \to G/K$ and $q : G \to G/p^\omega G$ be the appropriate quotient maps. Then $p^\omega G$ must be contained in K and thus there is a map $f' : G/p^\omega G \to \mathbb{Z}_p$ such that $f_g = f'q$. Hence $f'(x + p^\omega G) \ne 0$. Then (b) follows immediately.
(b)⇒(a) Consider the split exact sequence $0 \to \mathbb{Z}_p \otimes p^\omega G \to \mathbb{Z}_p \otimes G \to \mathbb{Z}_p \otimes G/p^\omega G \to 0$. By Corollary I.2, $\mathbb{Z}_p \otimes G/p^\omega G$ is a free $\mathbb{Z}_p$–module. Hence $\mathbb{Z}_p \otimes G$ is completely decomposable.
(c)⇔(d) This follows from Proposition I.4. ■

The implications (a)⇒(b), (c), or (d) do not require a cardinality restriction on G. The class of torsion–free locally completely decomposable groups is a strictly larger class than that of Butler groups (even in the finite rank case). Hence if G is a Butler group (possibly of infinite rank) Cobext$(T, G) = 0$ for all torsion groups T.

If G is a Butler group, then Bext$(G, T) = 0$ for all torsion and cotorsion groups T. A natural question to ask is: must Cobext$(T, G) = 0$ for all cotorsion T?

Proposition II.2. *Let G be a reduced finite rank Butler group. Then* $\text{Cobext}(T,G) = 0$ *for all cotorsion T if and only if there is a cobalanced exact sequence of Butler groups* $0 \to G \to C \to A \to 0$ *where C is completely decomposable and A is reduced.*

PROOF : By Theorem 1.4 in [1], there is a cobalanced exact sequence $\mathcal{E} : 0 \to G \to C \to A \to 0$ with $C = \oplus_{i=1}^{n} C_i$ and each C_i isomorphic to a rank-1 quotient of G, and with A a Butler group. Thus G is pure in C and so without loss of generality we can assume that C is reduced.
($\Rightarrow$) If A is reduced, then it is cotorsion free. Thus by Proposition I.4, $\text{Cobext}(T,G) = 0$ for all cotorsion T.
($\Leftarrow$) Applying $\text{Hom}(\mathbf{Q},\ \ )$ to $\mathcal{E}$, we have that $\text{Hom}(\mathbf{Q},A) \cong \text{Cobext}(\mathbf{Q},G)$. Thus, if $\text{Cobext}(T,G) = 0$ for all cotorsion T, $\text{Hom}(\mathbf{Q},A) = 0$ and hence A is reduced. ■

This proposition clearly holds if $OT(G) < \text{type}(\mathbf{Q})$.

We conclude with an example of a large class of Butler groups for which the cotorsion groups are not injective with respect to cobalanced sequences.

Example II.3. For each $n \geq 3$, there is a rank-n completely decomposable group C and a cobalanced exact sequence $0 \to G \to C \to \mathbf{Q} \to 0$ that does not split. Consequently, $\text{Cobext}(T,G) \neq 0$ for all cotorsion groups T which are not torsion.

PROOF : For $n \geq 2$, choose a set $S = \{p_1, \dots, p_n\}$ of distinct primes. For each $1 \leq i < n$, we let $Z_i = \mathbf{Z}_{p_i} \cap \mathbf{Z}_{p_{i+1}}$ and $Z_n = \mathbf{Z}_{p_n} \cap \mathbf{Z}_{p_1}$. Take $G = \dfrac{\oplus_{i=1}^{n} Z_i}{\langle(1,\dots,1)\rangle_*}$. Then using the construction due to Lee [6] (since G has a corepresenting graph that is a cycle with edges labelled by $\mathbf{Z}_{p_i}$ for $1 \leq i \leq n$) or that found in Theorem 1.4 in [1], we can construct a cobalanced exact sequence $\mathcal{E} : 0 \to G \to \oplus_{i=1}^{n} \mathbf{Z}_{p_i} \to \mathbf{Q} \to 0$. This sequence is readily seen not to split.

Since $\oplus_{i=1}^{n} \mathbf{Z}_{p_i}$ is cotorsion-free, when one applies $\text{Hom}(T,\ \ )$ to $\mathcal{E}$ for cotorsion T, the result is that $\text{Cobext}(T,G) \cong \text{Hom}(T,\mathbf{Q})$ which is not zero when T is not torsion. ■

It is interesting to note that for the groups G so constructed the sequence $\mathcal{E}$ in the proof is an injective resolution for these groups in the category of finite rank Butler groups with regular homomorphisms (cf., [4]). Hence this class of Butler groups has the property that if $0 \to G \to B \to C \to 0$ is exact in the category of Butler groups of finite rank, then the sequence is cobalanced.

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