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How to make many-sorted algebras one-sorted

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Abstract. An equivalence is established between any category of algebras based on finite collections of sets (as exemplified by the category of modules over variable rings) and a category of single-set algebras.

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Classification: 08A99

Introduction.

There are many algebraic contexts in which the natural objects of study are built up not on individual sets but on collections of sets, e.g. graded rings and algebras, modules over variable rings [**2**], [**11**], group representations [**9**], [**10**] [**12**]; for other examples, see [**3**]. Higgins ([**4**], p.115) and Plotkin ([**8**], pp. 53-54) make the point that such structures are more general than (universal) algebras. Nevertheless the two theories run pretty well parallel. In [**4**], for instance, notions of identity and variety are introduced for multi-set structures, with the same relationship between these concepts as one finds in orthodox algebra. Barr [**1**] makes oblique reference to a widespread supposition that multi-set algebras are in some sense special cases of ordinary algebras. We shall provide a rather strong justification for such a supposition by proving that categories of multi-set algebras of a given type where the number of underlying sets is finite are equivalent to varieties of ordinary algebras, showing, moreover, how to obtain a set of equations defining these latter varieties. The special case of sets acted on by variable monoids was treated in the M.Sc. thesis of Richard Wood [**13**]. The phrase "of a given type" needs some explanation; for this we give a rudimentary account of the language and approach of Higgins [**4**].

For a set I equipped with a set Ω of partial operations, Higgins calls by the name $\langle I,\Omega\rangle$*-algebra* a collection $\{S_i : i \in I\}$ of sets such that for every $\omega \in \Omega$, whenever $\omega(i_1,\dots i_n) = j$ there is an associated function $f : S_{i_1} \times S_{i_2} \times \cdots \times S_{i_n} \to S_j$. Clearly an $\langle I,\Omega\rangle$-algebra is a kind of graded structure, the grading being supplied by I, though the sets of elements of different "degrees" may be quite dissimilar.

More precisely, $\langle I,\Omega\rangle$-algebras are *externally* graded. Now as is well known, graded rings, modules etc. come in two versions, external and internal. In showing that these two kinds of graded structure are effectively the same one makes essential use of the zero element. Nothing analogous is available in the general case. Nonetheless, as we shall shortly demonstrate, it is possible to replace $\langle I,\Omega\rangle$-algebras

I am very grateful to Richard Wood for drawing my attention to the possibilities explored in this paper while I was visiting Dalhousie University and for helpful correspondence subsequently, and also for telling me about reference [**6**].

by "internally graded" structures defined on cartesian product indexed by I. Being possible for arbitrary I, this is of some independent interest. In the case of finite I, however, we can do better, passing from structures of cartesian products to structures on sets. We do this by making use of the *diagonal algebras* of Plonka [7]. These algebras have a single n-ary fundamental operation and any such is isomorphic to an algebra defined on a cartesian product, with the product structure and the algebra structure defining each other. (Diagonal algebras whose operation is binary have a somewhat longer history as *rectangular bands.*)

We thus end up with a category equivalence between the category of $\langle I, \Omega\rangle$-algebras for finite I and a variety of universal algebras whose fundamental operations are obtained from the partial operations in Ω and the diagonal algebra operation. As a further refinement varieties (in the sense of [4]) of $\langle I, \Omega\rangle$-algebras are equivalent to varieties of algebras.

Kelly and Pultr [6] investigated the problem of "algebraically recognizing products" in categories and in the course of their study they independently introduced diagonal algebras. Thus there are connections between [6] and the present paper. However, we are concerned with the synthesis of an algebra from a collection of possibly quite disparate algebras rather than with the decomposition of an algebra with subalgebras of the same type. As well, our discussion is presented in quite elementary terms and should be of use to a fairly wide audience of working algebraists.

Results.

Let I be a non-empty set, $\{i_i, \ldots i_n, j\}$ a finite subset. It is not assumed that $i, i_2, \ldots j$ are distinct. We introduce two categories which we shall shortly prove to be isomorphic, thereby demonstrating the equivalence of "internally" and "externally" graded algebras.

Our first category, which we shall call SUM_I^f is defined as follows. Objects are sets $\{S_i : i \in I\}$ of non-empty sets equipped in each case with a function $f : S_{i_1} \times \cdots \times S_{i_n} \to S_j$. A morphism from one object $\{S_i : i \in I\}$ to another $\{T_i : i \in T\}$, is a set $\{g_i : i \in I\}$ where for each i,$g_i : S_i \to T_i$ is a function, such that

$$g_j(f(s_{i_1}, \ldots, s_{i_n})) = f(g_{i_1}(s_{i_1}), \ldots, g_{i_n}(s_{i_n})).$$

The objects of the category PROD_I^ω are cartesian products $\prod_I S_i$ of non-empty sets equipped with an n-ary operation ω such that the i'th component of $\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I)$ is $s_i^{(n)}$ i.e. the i'th component of $(s_i^{(n)})_I$, for all $i \neq j$, and such that $\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I)$ and $\omega((t_i^{(1)})_I, (t_i^{(2)})_I, \ldots, (t_i^{(n)})_I)$ have the same j'th component whenever $s_{i_1}^{(1)} = t_{i_1}^{(1)}$, $s_{i_2}^{(2)} = t_{i_2}^{(2)} \ldots$, $s_{i_n}^{(2)} = t_{i_n}^{(n)}$. Morphisms from $\prod_I S_i$ to $\prod_I T_i$ are families $(g_i)_I$, the $g_i : S_i \to T_i$ being functions, such that

$$\begin{aligned}&\omega((g_i(s_i^{(1)}))_I, (g_i(s_i^{(2)}))_I, \ldots\ (g_i(s_i^{(n)}))_I)\\ &= (g_i(v_i))_I, \quad \text{where} \quad (v_i)_I = \omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I).\end{aligned}$$

Note that giving $\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I)$ and $(s_i^{(n)})_I$ the same "irrelevant" components is a bit arbitrary: there are other equally good ways of getting the effect we want, which is an operation defined on all of $\prod_I S_i$ but with its activities essentially confined to $S_{i_1}, \ldots, S_{i_n}$ and S_j.

Proposition. *The categories* SUM_I^f *and* PROD_I^ω *are isomorphic.*

PROOF : Define $\Phi : \mathrm{SUM}_I^f \to \mathrm{PROD}_I^\omega$ by setting

$$\Phi(\{S_i : i \in I\}, f) = (\prod_I S_i, \omega), \qquad \text{with}$$
$$\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I) = (v_i)_I,$$

where $v_i = s_i^{(n)}$ for $i \neq j$ and $v_j = f\left(s_{i_1}^{(1)}, s_{i_2}^{(2)}, \ldots, s_{i_n}^{(n)}\right)$; $\Phi(\{g_i : i \in I\}) = (g_i)_I$.

Define Ψ: $\mathrm{PROD}_I^\omega \to \mathrm{SUM}_I^f$ by setting $\Psi(\prod_I S_i, \omega) = (\{S_i : i \in I\}, f)$ with $f : S_{i_1} \times \cdots \times S_{i_n} \to S_j$ given by the condition

$f(s_{i_1}, si_2, \ldots s_{i_n})$ is the j'th component of any

$$\omega((t_i^{(1)})_I, (t_i^{(2)})_I, \ldots, (t_i^{(n)})_I) \quad \text{in which}$$
$$t_{i_1}^{(1)} = s_{i_1}, t_{i_2}^{(2)} = s_{i_2}, \ldots\ t_{i_n}^{(n)} = s_{i_n};$$

$\Psi(\{g_i : i \in I\}) = (g_i)_I$.

It is clear that Φ and Ψ take objects to objects. Let $\{g_i : i \in I\}$ be a morphism in SUM_I^f from $\{S_i : i \in I\}$ to $\{T_i : i \in I\}$. Then in PROD_I^ω we have

$$\omega((g_i(s_i^{(1)}))_I, (g_i(s_i^{(2)}))_I, \ldots, (g_i(s_i^{(n)}))_I) = (v_i)_I$$

where $v_i = g_i(s_i^{(n)})$ for $i \neq j$ and

$$v_j = f(g_{i_1}(s_{i_1}^{(1)}), g_{i_2}(s_{i_2}^{(2)}), \ldots, g_{i_n}(s_{i_n}^{(n)})).$$

If we let $\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I) = (u_i)_I$, then we have

$$\begin{aligned} v_i &= g_i(s_i^{(n)}) = g_i(u_i) \quad \text{for} \quad i \neq j \text{ and} \\ v_j &= f(g_{i_1}(s_{i_1}^{(1)}), g_{i_2}(s_{i_2}^{(2)}), \ldots, g_{i_n}(s_{i_n}^{(n)})) \\ &= g_j(f(s_{i_1}^{(1)}, s_{i_2}^{(2)}, \ldots\ s_{i_n}^{(n)})) = g_j(u_j). \end{aligned}$$

Thus $\Phi(\{g_1 : i \in I\})$ is a PROD_I^ω - morphism. Conversely, if $(g_i)_I$ is a PROD_I^ω-morphism, then in SUM_I^f we have

$$\begin{aligned} g_j(f(s_{i_1}, s_{i_2}, \ldots,\ s_{i_n})) &= g_j(j\text{'th component of any } \omega((t_i^{(1)})_I, (t_i^{(2)})_I, \ldots, (t_i^{(n)})_I)) \\ &\quad \text{with } t_{i_1}^{(1)} = s_{i_1}, \quad t_{i_2}^{(2)} = s_{i_2}, \quad \ldots, \quad t_{i_n}^{(n)} = s_{i_n} \\ &= [\text{ j'th component of } \omega((g_i(t_i^{(1)}))_I, (g_i(t_i^{(2)}))_I, \ldots (g_i(t_i^{(n)}))_I)] \\ &= f(g_{i_1}(s_{i_1}), g_{i_2}(s_{i_2}), \ldots, g_{i_n}(s_{i_n})), \end{aligned}$$

so that $\Phi((g_i)_I)$ is a SUM_I^f-morphism.

In $\Psi\Phi(\{S_i : i \in I\})$ we have

$$\begin{aligned} f(s_{i_1}, s_{i_2}, \ldots s_{i_n}) &= [j\text{'th component of any } \omega((t_i^{(1)})_I, (t_i^{(2)})_I, \ldots, (t_i^{(n)})_I) \\ &\quad \text{in which } t_{i_1}^{(1)} = s_{i_1}, t_{i_2}^{(2)} = s_{i_2}, \ldots t_{i_n}^{(n)} = s_{i_n}] \\ &= f(t_{i_1}^{(1)}, t_{i_2}^{(2)}, \ldots, t_{i_n}^{(n)}) \text{ as defined in } \{S_i : i \in I\} \\ &= f(s_{i_1}, s_{i_2}, \ldots, s_{i_n}) \text{ as defined in } \{S_i : i \in I\}; \end{aligned}$$

hence $\Psi\Phi(\{S_i : i \in I\}) = \{S_i : i \in I\}$.

In $\Phi\Psi(\prod_I S_i)$, we have

$$\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I) = (v_i)_I, \text{where } v_i = \begin{cases} s_i^{(n)} \text{ if } i \neq j \\ f(s_{i_1}^{(1)}, \ldots, s_{i_n}^{(n)}) \text{ if } i = j \end{cases}$$

Thus the j'th component v_j is the j'th component in $(\prod_I S_i)$ of any $\omega((t_i^{(1)})_I, (t_i^{(2)})_I, \ldots, (t_i^{(n)})_I)$ in which $t_{i_1}^{(1)} = s_{i_1}^{(1)}, t_{i_2}^{(2)} = s_{i_2}^{(2)}, \ldots, t_{i_n}^{(n)} = s_{i_n}^{(n)}$. In particular it is the j'th component of $\omega((s_i^{(1)})_I, (s_i^{(2)})_I, \ldots, (s_i^{(n)})_I)$ in $\prod_I S_i$. Since i'th components for $i \neq j$ are the same when computed in both $\Phi\Psi(\prod_I S_i)$ and $\prod_I S_i$, we conclude that $\Phi\Psi(\prod_I S_i) = \prod_I S_i$.

The functors Φ and Ψ are therefore mutually inverse on objects. It is clear that they are on morphism too. ∎

If instead of a single f we have collection of operations of arity ≥ 1 defined by functions from finite cartesian subproducts to individual sets S_i, it is clear from the proof just given that we can define operations on the corresponding $\prod_I S_i$ to get a more general equivalence. Nullary operations are not automatically covered, but we can take account of them quite straightforwardly.

If in the SUM-type category there is a nullary with value e_j in S_j we define an associated unary operation $\#$ in the PROD-type category by setting $(s_i)_I^{\#} = (t_i)_I$ where $t_j = e_j$ and $t_i = s_i$ for $i \neq j$. Thus we require in the PROD-type category a unary operation which always has the same effect on the j'th component.

The following theorem covers these observations.

Theorem 1. *For every $\langle I, \Omega \rangle$, the category of $\langle I, \Omega \rangle$-algebras is isomorphic to a category of structures with a forgetful functor to cartesian products indexed by I.*

We wish to obtain a category equivalence between $\langle I, \Omega \rangle$-algebras and a variety of genuinely one-set algebras. This can be done if the category of cartesian products can be so represented. At least in the case of finite I this is possible: the diagonal algebras of Plonka [7] are a sort of one-set version of cartesian products. Fuller details can be found in [7], but in the interest of a more-or-less self-contained account, we shall present a few of the salient features of these algebras.

A ***diagonal algebra*** is an algebra with a single fundamental operation δ of arity $m \geq 2$ satisfying the conditions

$$\delta(x,x,\dots,x)=x \quad \text{and}$$
$$\delta(\delta(x_1^{(1)},x_2^{(1)},\dots,x_m^{(1)}),\delta(x_1^{(2)},x_2^{(2)},\dots,x_m^{(2)}),\dots,\delta(x_1^{(m)},x_2^{(m)},\dots,x_m^{(m)})) = \delta(x_1^{(1)},x_2^{(2)},\dots,x_m^{(m)}).$$

As an example , we have an arbitrary cartesian product $S_1 \times S_2 \times \cdots \times S_m$ of non-empty sets with

$$\delta((x_1^{(1)},x_2^{(1)},\dots,x_m^{(1)}),(x_1^{(2)},x_2^{(2)},\dots,x_m^{(2)}),\dots,(x_1^{(m)},x_2^{(m)},\dots;x_m^{(m)})) = = (x_1^{(1)},x_2^{(2)},\dots,x_m^{(m)}).$$

In this case, for two elements $a=(a_1,a_2,\dots,a_m)$ and $b=(b_1,b_1,\dots,b_m)$ we have $a_{i_1}=b_{i_1}$, $a_{i_2}=b_{i_2},\dots$ and $a_{i_k}=b_{i_k}$ if and only if

$$a=\delta(a,a,\dots,a)=\delta(c^{(1)},c^{(2)},\dots,c^{(m)}),$$
$$\text{where} \quad c^{(i)}=\begin{cases} b & \text{if } i=i_1,i_2,\dots \text{ or } i_k \\ a & \text{otherwise.}\end{cases}$$

Now it turns out [7] that every diagonal algebra is isomorphic to one of the kind just described: for elements a and b of a general m-ary diagonal algebra D we write $a \equiv_i b$ $(i=1,2,\dots,m)$ if $a=\delta(\underbrace{a,\dots,a}_{i-1},b,a,\dots,a)$. Then each $\equiv_i$ is a congruence and the correspondence

$$a \mapsto (a\equiv_1, a\equiv_2,\dots,a\equiv_m)$$

(the $a\equiv_i$ being congruence classes) defines an isomorphism from D to the "cartesian product diagonal algebra"

$$(D/\equiv_1)\times(D/\equiv_2)\times\cdots\times(D/\equiv_m).$$

Now let us re-consider the category PROD_I^ω with $I=\{1,2,\dots,m\}$. The condition

"the i'th component of $\omega((s_i^{(1)})_I,(s_i^{(2)})_I,\dots,(s_i^{(n)})_I)$ is $s_i^{(n)}$ for $i\neq j$"

translates as the equation

$$\omega((s_i^{(1)})_I,(s_i^{(2)})_I,\dots,(s_i^{(n)})_I) =$$
$$\delta(\underbrace{(s_i^{(n)})_I,(s_i^{(n)})_I,\dots,(s_i^{(n)})_I}_{j-1},\omega((s_i^{(1)})_I,(s_i^{(2)})_I,\dots,(s_i^{(n)})_I),(s_i^{(n)})_I,\dots,(s_i^{(n)})_I).$$

For two elements $(s_i)_I$, $(t_i)_I$, we have $s_{i_1}=t_{i_1}$ if and only if

$$(t_i)_I=\delta(\underbrace{(t_i)_I,\dots,(t_i)_I}_{i_1-1},(s_i)_I,(t_i)_I,\dots,(t_i)_I).$$

Thus the condition

$$\text{“}\omega((s_i^{(1)})_I,(s_i^{(2)})_I,\ldots,(s_i^{(n)})_I) \text{ and } \omega((t_i^{(1)})_I,(t_i^{(2)})_I,\ldots,(t_i^{(n)})_I)$$
$$\text{have the same j'th component whenever}$$
$$s_{i_1}^{(1)}=t_{i_1}^{(1)},s_{i_2}^{(2)}=t_{i_2}^{(2)},\ldots,s_{i_n}^{(n)}=t_{i_n}^{(n)}\text{”}$$

translates as the equation

$$\sigma=\delta(\underbrace{\sigma,\sigma,\ldots,\sigma}_{\longleftarrow j-1\longrightarrow},\tau,\sigma,\ldots,\sigma),$$

where $\sigma=\omega((s_i^{(1)})_I,(s_i^{(2)})_I,\ldots,(s_i^{(n)})_I)$ and

$$\begin{aligned}\tau=\omega\big[&\delta(\underbrace{(t_i^{(1)})_I,\ldots,(t_i^{(1)})_I}_{\longleftarrow i_1-1\longrightarrow},(s_i^{(1)})_I,(t_i^{(1)})_I,\ldots,(t_i^{(1)})_I),\\ &\delta(\underbrace{(t_i^{(2)})_I,\ldots,(t_i^{(2)})_I}_{\longleftarrow i_2-1\longrightarrow},(s_i^{(2)})_I,(t_i^{(2)})_I,\ldots,(t_i^{(2)})_I),\ldots,\\ &\delta(\underbrace{(t_i^{(n)})_I,\ldots,(t_i^{(n)})_I}_{\longleftarrow i_n-1\longrightarrow},(s_i^{(n)})_I,(t_i^{(n)})_I,\ldots,(t_i^{(n)})_I)\big].\end{aligned}$$

As before, there is no essential difference when the single operation f is replaced by a set of operations. Thus (for finite I) the category of all $\langle I,\Omega\rangle$-algebras is equivalent to a category of algebras. In both these categories there is a notion of variety - as defined by Higgins [4] in the former and in the classical sense in the latter. These types of variety correspond to each other in a quite transparent manner; moreover, an identity $g \doteq h$ of $\langle I,\Omega\rangle$-algebras corresponds to a condition in diagonal algebras that (terms corresponding to) g and h agree in certain components. Use of the operation δ then enables us to turn this condition into an identity.

We summarize all this as

Theorem 2. *For finite I, every variety of $\langle I,\Omega\rangle$-algebras is equivalent to a variety of algebras.*

A diagonal algebra with a binary operation is a type of semigroup known as a *rectangular band*. Rectangular bands are also characterized by the identity $xyx=x$. For further details, see, e.g., [5] pp. 96 *et seqq.*

Let a and b be elements of a rectangular band which have the “same first component”. Then $b=\delta(a,b)$. On the other hand, for any c we have

$$\delta(\delta(a,c),a)=\delta(\delta(a,c),\delta(a,a))=\delta(a,a)=a,$$

i.e. $\delta(a,c)$ and a have the “same first component”. Analogously, a and b have the “same second component” if and only if $b=\delta(c,a)$ for some c. There are similar results in diagonal algebras of greater arity, including the cases where elements are required to “agree in several components”. See [7] §I and Lemma 1.

We now consider some examples.

Example 1. The category of (left) M-sets for (variable) monoids M is equivalent to the variety of all algebras $(A, ., *, ', \circ)$ of type $(2,2,1,2)$ satisfying the identities

(i) $(x \circ y) \circ z = x \circ (y \circ z)$,
(ii) $x \circ y \circ x = z$,
(iii) $(xy) \circ y = xy$,
(iv) $(x \circ z)(y \circ w) = xy$,
(v) $(xy)z = [x(yz)] \circ [(xy)z]$,
(vi) $y \circ (x * y) = x * y$,
(vii) $x * y = (x * y) \circ [(x \circ w) * (z \circ y)]$,
(viii) $(xy) * z = [(xy) * z] \circ [x * (y * z)]$,
(ix) $y' \circ x' = x'$,
(x) $x' * y = (x' * y) \circ y$.

Here we are using a standard binary symbol $\circ$ for the rectangular band operation. Equations (i) and (ii) say that A is a regular band (binary diagonal algebra). Equation (iii) says that xy has the same "second component" as y, i.e. . is active in the first component only, and (iv) says that if x, y are replaced by elements with the "same first components", $x \circ z$, $y \circ w$ respectively, the product is unchanged. The operation . is thus effectively a binary operation on the first component of A, which will be the monoid of scalars. By (v) this product is associative. The operation $*$ is going to be scalar multiplication (of elements of the "second component" by those of the "first") so (vi) says that $x * y$ has the same "first component" as y, i.e. only the "second component" is really relevant. By (vii), scalar product is unchanged (i.e. has the same "second component") if the first (second) factor is replaced by something with the same "first (second) component". Equation (viii) takes care of multiplication by a product of scalars. Equations (ix) and (x) refer to the identity element of the monoid: by (ix) all elements x' have the same "first component" and by (x) $x' * y$ always has the same "second component" as y.

It should be noted that the equational description of A just given is not the simplest possible, but rather the one that our general discussion has established. Various shortcuts and simplifications will suggest themselves in specific examples.

We illustrate this with our second example, modules over arbitrary rings.

Let M be a left unital module over a ring R with identity. Let $+_1$, $+_2$ denote the addition on R, M respectively. Going back to our original procedure, we extend these operations (keeping the same names) to $R \times M$ as follows

$$(r,m) +_1 (r',m') = (r +_1 r', m');$$
$$(r,m) +_2 (r',m') = (r', m +_2 m').$$

Let us replace $+_2$ by a new operation $\hat{+}_2$ defined by

$$(r,m) \hat{+}_2 (r',m') = (r, m +_2 m').$$

Then $\hat{+}_2$ is as good as $+_2$ for representing in $R \times M$ the internal addition of M. Next we define $+$ on $R \times M$ by

$$(r,m) + (r',m') = (r +_1 r', m +_2 m').$$

Introducing the standard rectangular band (binary diagonal algebra) operation $\circ$ on $R \times M$, we get

$$\begin{aligned}(r,m)+(r',m') &= (r +_1 r', m +_2 m')\\ &= (r +_1 r', m') \circ (r, m +_2 m')\\ &= [(r,m) +_1 (r',m')] \circ [(r,m) \hat{+}_2 (r',m')]\,;\\ [(r,m)+(r',m')] \circ (r',m') &= (r +_1 r', m +_2 m') \circ (r',m');\\ &= (r +_1 r', m') = (r,m) +_1 (r',m');\\ (r,m) \circ [(r,m)+(r',m')] &= (r,m) \circ (r +_1 r', m +_2 m')\\ &= (r, m +_2 m') = (r,m) \hat{+}_2 (r',m').\end{aligned}$$

Passing to "abstract" diagonal algebras, we therefore have the equations

$$x+y = (x +_1 y) \circ (x \hat{+}_2 y); (x+y) \circ y = x +_1 y; x \circ (x+y) = x \hat{+}_2 y,$$

so the two additions can be replaced by a single one. The same goes for the two unary operations of taking additive inverses and the two unary operations developing from the nullaries (zeros) of R and M. We now make use of this.

Example 2. The category of (left unital) modules over variable rings with identity is equivalent to the variety of algebras $(A, +, -, 0, ., *, ', \circ)$ of type $(2,1,0,2,2,1,2)$ satisfying the following conditions

(a) $(A, +, -, 0)$is an abelian group,
(b) $x(y+z) = (xy+yz) \circ [x(y+z)]$,
(c) $(x+y)z = (xz+yz) \circ [(x+y)z]$,
(d) $x * (y+z) = [x * (y+z)] \circ ((x*y) + (x*z))$,
(e) $(x+y) * z = [(x+y) * z] \circ ((x*z) + (y*z))$,
(f) (i) - (x) of Example 1.

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