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Two-point boundary value problems for nonlinear perturbations of some singular linear differential equations at resonance

JEAN MAWHIN, WALO OMANA

Dedicated to the memory of Svatopluk Fučík

Abstract. We extend to some second order nonlinear two-point boundary value problems with a singular resonant linear part some existence results known for the regular case. The proof uses degree arguments and sharp estimates for an associated Green's function.

Keywords: Boundary value problems, singular equations, coincidence degree.

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1. Introduction.

This paper is devoted to the existence of solutions for some nonlinear boundary value problems at resonance of the form

$$-\frac{1}{p(t)}(p(t)u'(t))' - \lambda_1 u(t) = f(t, u(t)), \qquad t \in]0,1[\tag{1}$$

and either

$$u(0) = u(1) = 0 \tag{2_1}$$

or

$$\lim_{t \to 0_+} p(t)u'(t) = u(1) = 0, \tag{2_2}$$

where $f : I \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function, $I = [0,1], p \in C(I) \cap C^1(]0,1])$, $p(0) = 0$, $p(t) > 0$ on $]0,1]$, and $1/p \in L^1(I)$, λ_1 is the first eigenvalue of $-p^{-1}(\frac{d}{dt})(p\frac{d}{dt})$ with boundary conditions (2_1) or (2_2).

Our results extend to some class of functions p a technique introduced in [$\mathbf{Maw}_1$] when the linear part does not contain singularities. For example, our Theorem 2 will apply to the nonlinear perturbation of a Lommel equation (whose regular counterpart corresponding to $\alpha = 0$ was considered in [$\mathbf{Maw}_1$])

$$\begin{gathered} -t^{-\alpha}(t^{\alpha}u'(t))' - \lambda_1 u(t) = A \exp u(t) - h(t) \\ u(0) = u(1) = 0 \end{gathered} \tag{3}$$

provided $0 \leq \alpha < 1$, $t^\alpha h \in L^1(I)$, and will imply the existence of a solution if and only if

$$A \int_I h(t) t^{(1+\alpha)/2} J_{(1-\alpha)/2}(\sqrt{\lambda_1} t)\, dt > 0 \tag{4}$$

where $J_{(1-\alpha)/2}$ is the Bessel function of the first kind of order $(1-\alpha)/2$ and λ_1 is the square of its first positive zero. Condition (4) is a Landesman-Lazer condition which is thus extended here to equations with a singular linear part and some nonlinearities having not necessarily a linear growth. A similar result holds for the other boundary condition.

Another example, which, in the regular case $\alpha = 0$, corresponds to a question by Fučík [**Fuc**] (see [**Maw$_1$**]), is

$$-t^{-\alpha}(t^\alpha u'(t))' - \lambda_1 u(t) = g(u(t)) - h(t),$$
$$u(0) = u(1) = 0,$$

where $0 \leq \alpha < 1$, $g(u) = 0$ for $u \geq 0$ and $u^{-1} g(u) \to \beta > 0$ when $u \to -\infty$. In this case, Theorem 1 will imply that a solution exists if

$$\int_I h(t) t^{(1+\alpha)/2} J_{(1-\alpha)/2}(\sqrt{\lambda_1} t)\, dt < 0$$

Moreover, if g is nondecreasing, the condition above with non strict inequality will be a necessary and sufficient condition for solvability, as shown by Theorem 2. Again, a similar result holds for the other boundary condition.

The method uses coincidence degree arguments (see [**Maw$_2$**] or [**Maw$_3$**]) and the required a priori bounds rely on sharp estimates for the Green function associated to the singular differential linear operator. Those estimates rely very much upon the fact that 1/p is integrable over I and hence the interesting question of extending the results in the examples to the case where $\alpha \geq 1$ remains open. In another paper, we shall show that extension is possible for a restricted class of nonlinearities.

2.Some results on the linear problem.

Let $I = [0,1]$, $p \in C(I) \cap C^1(]0,1])$ such that $p(0) = 0$ and $p(t) > 0$ on $]0,1]$. We denote by $L^1_p(I)$ the space of measurable functions u on I such that $|u|p \in L^1(I)$, with the norm

$$\|u\|_{1;p} = \int_I |u(t)| p(t)\, dt$$

We shall impose to p the condition used in [**DuK**]

$$1/p \in L^1(I), \tag{P}$$

which will be assumed throughout the paper.

We define the operators $L_i : D(L_i) \subset C(I) \to L^1_p(I)$, $(i = 1, 2)$ by
$D(L_1) = \{u \in C(I) : u(0) = u(1) = 0,\ u$ and pu' are absolutely continuous on I and $(pu')' \in L^1_p(I)\}$
$D(L_2) = \{u \in C(I)$: u and pu' are absolutely continuous on I, $(pu')' \in L^1p(I)$ and $\lim_{t\to 0+} p(t)u'(t) = u(1) =: 0\}$
and $L_i u = -(1/p)(pu')'$, $(i = 1, 2)$.

It is easy to check that $\ker L_i = \{0\}$ and that, for each $h \in L^1_p(I)$, the problem

$$L_i u = h$$

has a unique solution u given by

$$u(t) = \int_0^1 G_i(t, s)h(s)p(s)\, ds,$$

where G_i is the Green function defined by

$$G_1(t,s) = \begin{cases} P^{-1} \int_t^1 (1/p(r))\, dr \int_0^s (1/p(r))\, dr & \text{if} \quad 0 \le s \le t \le 1 \\ P^{-1} \int_s^1 (1/p(r))\, dr \int_0^t (1/p(r))\, dr & \text{if} \quad 0 \le s \le t \le 1 \end{cases}$$

and

$$G_2(t,s) = \begin{cases} \int_t^1 (1/p(r))\, dr, & \text{if} \quad 0 \le s \le t \le 1 \\ \int_s^1 (1/p(r))\, dr, & \text{if} \quad 0 \le t \le s \le 1 \end{cases}$$

with

$$P = \int_I (1/p(r))\, dr$$

(see e. g. [**DuK**]). Notice that each G_i is continuous on $I \times I$ and, with $|\cdot|_0$ denoting the uniform norm in $C(I)$,

$$L_i^{-1} : L^1_p(I) \to D(L_i) \subset C(I)$$

exists and, as

$$|L_i^{-1} u|_0 \le \max_{|x|} |G_i| \|h\|_{1,p}$$

$L_i^{-1} : L^1p(I) \to C(I)$ is continuous. Now, by the compactness of G_i, we can find, for each $\varepsilon > 0$, some $\delta > 0$ such that

$$|(L_i^{-1} h)(t_1) - (L_i^{-1} h)(t_2)| \le \varepsilon \|h\|_{1,p}$$

whenever $|t_1 - t_2| \leq \delta$, which shows, by Ascoli-Arzela theorem, that L_i^{-1} is compact $(i = 1,2)$. Using the results of [CoL] on Sturm-Liouville problems (see Ch.8 and the remarks ending Section 1 of Ch. 9), we know that L_i has an infinite number of real eigenvalues λ_j^i $(j = 1,2,\dots)$ forming an increasing sequence with $\lambda_j^i \to +\infty$ as $j \to \infty$. Moreover, the eigenfunction v_n^i associated to λ_n^i has exactly $(n-1)$ zeros on $]0,1[$. Now, to simplify the notation, let us denote by λ_i the smallest eigenvalue and by v_i a corresponding eigenfunction positive on $]0,1[$ $(i = 1,2)$. Notice that $v_2(t) > 0$ for all $t \in [0,1[$. From the equality

$$v_i(L_i v_i)p = \lambda_i v_i^2 p$$

we get, by integration by parts and use of the boundary conditions, for each $0 < c < 1$,

$$\lambda_i \int_c^1 v_i^2(t)p(t)\,dt =$$

$$p(c)v_i'(c)v_i(c) + \int_c^1 p(t)(v_i')^2(t)\,dt\,, \qquad (i = 1,2)$$

But, for each $u \in D(L_i)$, we have

$$p(t)u'^2(t) \leq (1/p(t))(p(t)u'(t))^2\,,$$

and hence $pu'^2 \in L^1(I)$. Hence, letting $c \to 0$, we get

$$\lambda_i = (\int_I p(t)(v_i'(t))^2\,dt)/(\int_I p(t)v_i^2(t)\,dt) \geq 0$$

and therefore $\lambda_i > 0$ as L_i is invertible.

Lemma 1. *If condition (P) holds, the function Γ_i defined by*

(6) $$\Gamma_i :]0,1[\times]0,1[\to \mathbf{R}, \quad (t,s) \to G_i(t,s)/v_i(s)$$

belongs to $L^\infty(]0,1[\times]0,1[)$.

PROOF : We have, for $(t,s) \in [0,1] \times]0,1[$

$$0 \leq \frac{G_1(t,s)}{v_1(s)} \leq \frac{G_1(s,s)}{v_1(s)}$$

and hence

$$0 \leq \frac{G_1(t,s)}{v_1(s)} \leq \frac{\int_0^s (1/p(r))\,dr}{v_1(s)}$$

By L'Hospital's rule,

$$\lim_{s\to 0} \frac{\int_0^s (1/p(r))\,dr}{v_1(s)} = \lim_{s\to 0} \frac{1}{p(s)v_1(s)}\,.$$

Now, from the identity

$$v_1(s) = \lambda_1 \int_I G_1(s,r)v_1(r)p(r)\,dr\,,$$

we deduce, for $s \in]0\,,1[$,

$$\begin{aligned}
p(s)v_1'(s) &= \lambda_1 \int_I p(s)D_sG_1(s,x)v_1(x)p(x)\,dx = \\
&= \lambda_1\Big\{[\int_0^s P^{-1}p(s)(-1/p(s)) \int_0^x (1/p(r))\,dr]v_1(x)p(x)\,dx+ \\
&+ \int_s^1 [P^{-1}p(s)(1/p(s)) \int_x^1 (1/p(s))\,dr]v_1(x)p(x)\,dx\Big\} = \\
&= \lambda_1 P^{-1}\Big\{-\int_0^s (\int_0^x 1/p(s)\,dr)v_1(x)p(x)\,dx+ \\
&+ \int_s^1 (\int_x^1 (1/p(r))\,dr)v_1(x)p(x)\,dx\Big\}
\end{aligned}$$

so that

$$\lim_{s\to 0} p(s)v_1'(s) = \lambda_1 P^{-1} \int_0^1 (\int_x^1 (1/p(r))\,dr)v_1(x)p(x)\,dx > 0$$

is finite. Thus

$$\lim_{s\to 0} \frac{\int_0^s (1/p(r))\,dr}{v_1(s)}$$

exists and $G_1(t,s)/v(s)$ is bounded for s close to 0. Similarly

$$0 \le \frac{G_1(t,s)}{v_1(s)} \le \frac{\int_s^1 (1/p(r))\,dr}{v_1(s)}$$

and one shows that

$$\lim_{s\to 1} \frac{\int_s^1 (1/p(r))\,dr}{v_1(s)} = \lim_{s\to 1} -\frac{1}{p(s)v_1(s)}$$

exists, so that $G_1(t,s)/v_1(s)$ is bounded for s close to 1. As $G_1(t,s)/v_1(s)$ is clearly bounded for $s \in [\delta, 1-\delta]$ for any $\delta > 0$, and G_1, continuous on $]0,1[\times]0,1[$ is measurable on this set, the result is proved for G_1. For $i = 2$, we first notice that for each $0 < \delta < 1$ there exists M such that

$$0 \le G_2(t,s)/v_2(s) \le M$$

on $[0,1]\times[0,1-\delta]$. By definition of G_2, we have

$$G_2(t,s) \le G_2(s,s)$$

for all $(t,s) \in I \times I$, and hence

$$0 \le \Gamma_2(t,s) \le G_2(s,s)/v_2(s)$$

for all $(t,s) \in I \times [0,1[$. Now, from the equation

$$-(p(s)v_2'(s))' = \lambda_2 p(s)v_2(s)$$

and the boundary condition, we deduce

$$p(s)v_2'(s) = -\lambda_2 \int_0^s p(r)v_2(r)\,dr$$

and hence

$$\lim_{s\to 1-} p(s)v_2'(s) = -\lambda_2 \int_0^1 p(r)v_2(r)\,dr < 0.$$

Then, using L'Hospital's rule, we obtain

$$\lim_{s\to 1-} G_2(s,s)/v_2(s) = \lim_{s\to 1-} (-1/p(s)v_2'(s))$$

where the right-hand member exists, and hence $G_2(s,s)/v_2(s)$ is bounded above on I. This completes the proof as Γ_2, continuous on $]0,1[\times]0,1[$, is measurable on this set. ∎

Corollary 1. *If condition (P) holds then for each $h \to L^1_p(I)$, one has*

$$|L_i^{-1}h|_0 \le |\Gamma_i|_\infty \|hv_1\|_{1;p}\,, \quad (i=1,2)\,,$$

where $|\,.\,|_\infty$ denotes the norm in $L_\infty(]0,1[\times]0,1[)$ and Γ_i is defined in (6).

PROOF : For each $t \in I$, one has, using Lemma 1,

$$|(L_i^{-1}h)(t)| = \left|\int_0^1 G_i(t,s)h(s)p(s)\,ds\right| \le \int_0^1 \frac{G_i(t,s)}{v_i(s)}|h(s)|v_i(s)p(s)\,ds \le |\Gamma_i|\|hv_i\|_{1;p}\,.$$

■

Let us now define the linear operator A_i by $D(A_i) = D(L)$ and $A_i = L - \lambda_i I$, so that $A_i : D(A_i) \subset C(I) \to L^1_p(I)$ and, by the discussion above $ker A_i = span\, v_i$, $(i = 1,2)$. Hence the space $C(I)$ can be splitted as the (topological) direct sum $C(I) = span\, v_i \oplus \widetilde{C}_i(I)$ and each $u \in C(I)$ can be written accordingly $u = \bar{u}_i + \widetilde{u}_i \quad (i = 1,2)$.

Lemma 2. *If condition (P) holds, there exists $\Lambda_i \ge 0$ such that for each $u = \bar{u}_i + \widetilde{u}_i \in D(A_i)$, with $\bar{u}_i \in ker A_i$ and $\widetilde{u}_i \in \widetilde{C}_i$ one has*

$$|\widetilde{u}_i| \le \Lambda_i\|(A_i\widetilde{u}_i v_i\|_{1;p} = \Lambda_i\|(A_i u)v_i\|_{1;p}\,, \quad (i=1,2).$$

PROOF : If it is not the case, we can find a sequence $(\widetilde{u}_n)$ in $\widetilde{C}_i(I)$ with $|\widetilde{u}_n|_0 = 1$ such that

$$1 > n\|(A_i\widetilde{u}_n)v_i\|_{1;p}$$

and hence, by Corollary 1,

$$n^{-1}|\Gamma_i|_\infty > |L^{-1}(A_i\widetilde{u}_n)|_0 = |\widetilde{u}_n - \lambda_i L^{-1}\widetilde{u}_n|_0$$

for all $n \in \mathbb{N}^*$. Now, L^{-1} being compact, there is a subsequence $(\widetilde{u}_{n_k})$ such that $L^{-1}\widetilde{u}_{n_k} \to y \in \widetilde{C}_i(I)$ in $C(I)$ and hence $\widetilde{u}_{n_k} \to \lambda_i y$ in $C(I)$, which implies

$$y = \lambda_i L^{-1} y$$

i.e. $y \to$ Ker A_i, and hence $y = 0$, a contradiction with $|\widetilde{u}_{n_k}| = 1$ for all $k \in \mathbb{N}^*$ and $\lambda_i > 0$. ■

Remark 1. A_i is obviously a densely defined operator and if $A_i x_n \to y$ in $L^1_p(I)$ and $x_n \to x$ in $C(I)$, then $x_n - \lambda_i L^{-1}x_n \to L^{-1}y$ in $C(I)$ so that $x = \lambda_i L^{-1}x + L^{-1}y \in D(L)$ and $A_i x = y$; thus A_i is closed $(i = 1,2)$. Finally,

$$L^{-1}A_i = I - \lambda_i L^{-1} \text{ on } D(A_i)$$
$$A_i L^{-1} = I - \lambda_i L^{-1} \text{ on } L^1_p(I)$$

which implies by a known result [Sch] that A_i is a Fredholm operator $(i = 1,2)$. Moreover, as L is Fredholm of index zero as well as $I - \lambda_i L^{-1}$, we have, for the Fredholm indices Ind,

$$0 = Ind(I - \lambda_i L^{-1}) = Ind(L^{-1}) + Ind(A_i) = Ind(A_i),$$

and A_i is Fredholm of index zero.

Remark 2. If $h \in Im\, A_i$, then we have

$$-(pu')' - \lambda_i pu = ph$$

for some $u \in D(L)$, and hence

$$\int_I (p(t)u'(t))'v_i(t)\,dt - \lambda_i \int_I u(t)p(t)v_i(t)\,dt = \int_I h(t)p(t)v_i(t)\,dt \qquad (i = 1,2).$$

Integrating by parts and using the boundary conditions, we get

$$-\int_I \left[(p(t)v_i{}'(t))' + \lambda_i v_i(t)p(t)\right] u(t)\,dt = \int_I h(t)p(t)v_i(t)\,dt,$$

i.e.

$$\int_I h(t)p(t)v_i(t)\,dt = 0, \qquad (i = 1,2).$$

3. The solvability of the nonlinear problem.

Let $f : I \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function, i.e. $f(t,.)$ is continuous on $\mathbf{R}$ for a.e. $t \in I$, $f(.,x)$ is measurable on I for each $x \in I$ and for each $r > 0$, there exist $a_r \in L^1_p(I)$ such that

$$|f(t,u)| \leq a_r(t)$$

for a.e. $t \in I$ and each u with $|u| \leq r$. The Nemitsky operator F defined by

$$(Fu)(t) = f(t,u(t))$$

maps $C(I)$ into $L^1_p(I)$.

Lemma 4. *F is A_i-completely continuous on $C(I)$* $(i = 1,2)$.

PROOF : By definition (see e.g. [Maw$_2$]) we have to prove that if $B : C(I) \to L^1_p(I)$ is continuous, of finite-rank and such that $A_i + B : D(A_i) \to L^1_p(I)$ is bijective, then $(A_i + B)^{-1}F : C(I) \to C(I)$ is completely continuous. For such a B we have

$$(A_i + B)^{-1}F = (L - \lambda_i I + B)^{-1}F = \left[I + L^{-1}(B - \lambda_i I)\right]^{-1} L^{-1}F$$

Now $L_{-1}(B - \lambda_i I)$ is compact on $C(I)$ and hence $\left[I + L^{-1}(B - \lambda_i I)\right]^{-1}$ is continuous on $C(I)$ and, L^{-1} being compact, it is standard to check that $L^{-1}F : C(I) \to C(I)$ is completely continuous, and the proof is complete. ∎

Theorem 1. *Assume that p satisfies condition (P) and that f satisfies the following conditions $(f_1) - (f_2)$ or $(f_1) - (f_2')$.*

(f_1) *There exists $\gamma \in L^1_p(I)$, such that*

$$|f(t,u)| \leq \epsilon f(t,u) + \gamma(t)$$

for a.e. $t \in I$*, all* $u \in \mathbf{R}$ *and* $\varepsilon = +1$ *or* -1.

(f_2) *There exist* $\delta_+ \in L_p^1(I)$, $\delta_- \in L_p^1(I)$ *such that*

$$f(t,u) \leq \delta_+(t) \quad \textit{for } u \geq 0$$
$$f(t,u) \geq \delta_-(t) \quad \textit{for } u \leq 0$$

and

$$\int_I f^-(t)v_i(t)p(t)\,dt < 0 < \int_I f_+(t)v_i(t)p(t)\,dt \tag{7}$$

where

$$f^-(t) = \limsup_{u \to -\infty} f(t,u), \qquad f_+(t) = \liminf_{u \to +\infty} f(t,u)$$

(f_2') *There exist* $\delta_+ \in L_p^1(I)$, $\delta_- \in L_p^1(I)$ *such that*

$$f(t,u) \leq \delta_+(t) \textit{ for } u \geq 0$$
$$f(t,u) \geq \delta_-(t) \textit{ for } u \leq 0$$

and

$$\int_I f^+(t)v_i(t)p(t)\,dt < 0 < \int_I f_-(t)v_i(t)p(t)\,dt \tag{8}$$

where

$$f_-(t) = \liminf_{u \to -\infty} f(t,u), \quad f^+(t) = \limsup_{u \to +\infty} f(t,u).$$

Then equation

$$-(1/p(t))(p(t)u'(t))' - \lambda_i u(t) = f(t,u(t)), \quad t \in]0,1[, \tag{9}$$

has at least one solution satisfying (2_i) $(i = 1,2)$.

PROOF : We fix $i = 1$ or 2 and, to apply Theorem IV.13 of [Maw$_3$] to the abstract equivalent version

$$A_i u = Fu$$

of (9), we first find an a priori bound for the possible solutions of

$$A_i u = \lambda F u, \quad \lambda \in]0,1[. \tag{10}$$

If u is a solution of (10) for some $\lambda \in]0,1[$, then

$$0 = \int_I p(t)(A_i u)(t)v_i(t)\,dt = \lambda \int_I f(t,u(t))v_i(t)p(t)\,dt$$

and hence

$$\int_I f(t,u(t))v_i(t)p(t)\,dt = 0 \tag{11}$$

Moreover, using (f_1) and (11),

$$\int_I |(A_iu)(t)|v_i(t)p(t)\,dt = \lambda \int_I |f(t,u(t))|v_i(t)p(t)\,dt \le$$

$$\le \lambda\varepsilon \int_I f(t,u(t))v_i(t)p(t)\,dt + \lambda \int_I \gamma(t)v_i(t)p(t)\,dt \le \|\gamma v_i\|_{1;p} = C_1.$$

Hence, by Lemma 3,

$$|\tilde{u}|_0 \le \Lambda_i C_1 = C_2.$$

Therefore, if the set of solutions of (10) is not a priori bounded, we can find sequences (u_n) in $D(A_i)$ and (λ_n) in $]0,1[$ such that u_n is a solution of (10_{λ_n}), $|\tilde{u}_n|_0 \le C_2$, $\bar{u}_n(t) = c_n v_i(t)$ with $c_n \to +\infty$ or $c_n \to -\infty$ as $n \to \infty$. Supposing, say that $c_n \to +\infty$ and condition (f_2) holds (the three other cases are treated similarly), we have

$$0 = \int_I f(t,u_n(t))v_i(t)p(t)\,dt \tag{12}$$

and $u_n(t) \ge c_n v_i(t) - |\widetilde{u_n}|_0 \ge c_n v_i(t) - C_2$, so that $u_n(t) \to +\infty$ for a.e. $t \in I$. Consequently, using (12) and Fatou's lemma, we get

$$0 = \liminf_{n\to\infty} \int_I f(t,u_n(t))v_i(t)p(t)\,dt \ge$$

$$\ge \int_I [\liminf_{n\to\infty} f(t,u_n(t))]v_i(t)p(t)\,dt \ge \int_I f_+(t)v_i(t)p(t)\,dt,$$

a contradiction with (7). It then remains to find an a priori bound for the set of solutions of the real equation

$$\bar{f}(c) \equiv \int_I f(t,cv_i(t))v_i(t)\,dt = 0$$

which again follows by contradiction from (f_1) or (f_2) using Fatou's lemma, and condition (7) or (8) then easily implies that

$$|d_B(\bar{f},]-r,r[,0)| = 1$$

for r sufficiently large, as $\bar{f}(-r)\bar{f}(r) < 0$. Thus all conditions of Theorem IV.13 in [Maw$_3$] are satisfied and the proof is complete. ∎

Remark 3. The class of nonlinearities verifying (f), which contains of course the bounded nonlinearities but also various classes of unbounded ones, was introduced by Ward in [**War**] for periodic problems.

Remark 4. If we take $p(t) = t^\alpha$ where $0 \le \alpha < 1$ in order that condition (P_2) holds, then the eigenvalue problem associated to L is

$$-t^{-\alpha}(t^\alpha u'(t))' - \lambda u(t) = 0, \quad t \in]0,1[\tag{13}$$

$$u(0) = u(1) = 0 \quad \text{or} \quad \lim_{t\to 0+} t^\alpha u'(t) = u(1) = 0 \tag{14}$$

$$tu''(t) + \alpha u'(t) + \lambda t u(t) = 0, \quad t \in]0,1[\tag{i.e.}$$

$$u(0) = u(1) = 0 \quad \text{or} \quad \lim_{t\to 0+} t^\alpha u'(t) = u(1) = 0.$$

(13) is a special case of the Lommel equation [**Nik**] and the general solution of the equation is

$$u(t) = t^{(1-\alpha)/2}\left[C_1 J_{(1-\alpha)/2}(\sqrt{\lambda}\,t) + C_2 J_{(\alpha-1)/2}(\sqrt{\lambda}\,t)\right],$$

where C_1 and C_2 are arbitrary constants and J_ν denotes the Bessel function of first kind of order ν. Hence

$$t^\alpha u'(t) = t^{(1+\alpha)/2}\left[C_1 J_{-(1+\alpha)/2}(\sqrt{\lambda}\,t) - C_2 J_{(1+\alpha)/2}(\sqrt{\lambda}\,t)\right].$$

As $J_\nu(z) \sim \dfrac{(z/2)^\nu}{\Gamma(\nu+1)}$ for $z \to 0$, the boundary condition $u(0) = 0$ in (14) implies that $C_2 = 0$ and the second $u(1) = 0$ will be satisfied for a nontrivial u if and only if

$$J_{(1-\alpha)/2}(\sqrt{\lambda}) = 0$$

i.e. if only if λ is the square of a zero of the Bessel function $J_{(1-\alpha)/2}$. In particular, λ_1 is in this case the square of the smallest positive zero of $J_{(\alpha-1)/2}$. Similarly, the condition $\lim_{t\to 0+} t^\alpha u'(t) = 0$ implies that $C_1 = 0$ and the condition $u(1) = 0$ is then satisfied for a nontrivial u if and only if

$$J_{(\alpha-1)/2}(\sqrt{\lambda}) = 0$$

i.e. if and only if λ is the square of a zero of $J_{(\alpha-1)/2}$. In particular, λ_2 is the square of the smallest positive zero $J_{(\alpha-1)/2}$. Notice that for $\alpha = 0$, $J_{1/2}(t) = \sqrt{2/\pi}t^{-1/2}\sin t$ and we recover the classical results. The results mentioned in the introduction are easy consequences of this Remark 2, Theorem 1 and the following Theorem 2.

When $f(t,.)$ is monotone for a.e. $t \in I$ and satisfies condition (f_1), one can give a necessary and sufficient condition for the solvability of (9).

Theorem 2. *Assume that p satisfies condition (P) and F satisfies condition (f_1). Assume moreover that $f(t,.)$ is monotone for a.e. $t \in I$. Then equation (9) has a solution verifying (2_i) if and only if there exists $c \in \mathbf{R}$ such that*

$$\int_I f(t, cv_i(t))v_i(t)p(t)\,dt = 0, \quad (i = 1, 2). \tag{15}$$

PROOF : Necessity. If (9) has a solution u verifying $(2_i)(i = 1$ or $2)$, then

$$\int_I f(t, u(t))v_i(t)p(t)\,dt = 0.$$

Now we have also

$$u(t) = \int_I G_i(t, s)\,[\lambda_i u(s) + f(s, u(s))]\,p(s)\,ds$$

and hence, using Lemma 1 and the symmetry of G_i, we have, on $]0, 1[$

$$\left|\frac{u(t)}{v_i(t)}\right| \leq \int_I \left|\frac{G_i(s,t)}{v_i(t)}\,[\lambda_i u(s) + f(s, u(s))]\right| p(s)\,ds \leq$$
$$\leq |\Gamma_i|_{L^\infty} \|\lambda_1 u + f(., u(.)\|_{p;1} \leq C.$$

Consequently, if we assume, say, that $f(t,.)$ is nondecreasing, we have

$$\int_I f(t, -Cv_i(t))v_i(t)p(t)\,dt \leq 0 \leq \int_I f(t, Cv_i(t))v_i(t)p(t)\,dt$$

and the result follows from the intermediate value theorem.

Sufficiency. Let c be a solution of (15) and let us consider the case where $f(t,.)$ is non decreasing, the other one being similar. If

$$\int_I f(t, bv_i(t))v_i(t)p(t)\,dt = 0 \tag{16}$$

for all $b \geq c$, then necessarily

$$f(t, bv_i(t)) = f(t, cv_i(t))$$

for all $b \geq c$ and a.e. $t \in I$. Let w_i be a solution of the problem

$$-(1/p(t))(p(t)w'(t))' = \lambda_i w(t) + f(t, cv_i(t)),\ t \in]0, 1[,$$

satisfying (2_i) (which exists because of (15)). By an argument similar to that used in the necessity proof, we shall have

$$|\frac{w_i(t)}{v_i(t)}| \leq C$$

for some $C > 0$ and all $t \in]0,1[$, and if we choose $c_1 \geq c$ so large that

$$c_1 + \frac{w_i(t)}{v_i(t)} \geq c$$

for $t \in]0,1[$, which will be the case if $c_1 \geq c+C$, then the function $u = c_1 v_i(t) + w_i(t)$ will be such that

$$\begin{aligned} -(1/p(t))(p(t)u'(t)) - \lambda_i u(t) &= -(1/p(t))(p(t)w_i'(t)) - \lambda_1 w_i(t) = \\ &= f(t, cv_1(t)) = f(t, u(t)), \quad t \in]0,1[\end{aligned}$$

and will satisfy (2_i), i.e. u will be a solution of (9). We construct similarly a solution if (16) holds for all $b \leq c$. Thus it remains to consider the case where there exist $b_1 < c < b_2$ such that

$$\int_I f(t, b_1 v_i(t)) v_i(t) p(t)\, dt < 0 < \int_I f(t, b_2 v_i(t)) v_i(t) p(t)\, dt$$

But, in this situation, one has of course

$$\begin{aligned} f(t,u) &\geq f(t,0) \quad \text{for } u \geq 0 \\ f(t,u) &\leq f(t,0) \quad \text{for } u \leq 0. \end{aligned}$$

and

$$\int_I f^-(t) v_i(t) p(t)\, dt < 0 < \int_I f_+(t) v_i(t) p(t)\, dt$$

where

$$f^-(t) = \lim_{u \to -\infty} f(t,u),\ f_+(t) = \lim_{u \to +\infty} f(t,u)$$

so that the conclusion follows from Theorem 1. ∎

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