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Hysteresis operators - a new approach to evolution differential inequalities

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Dedicated to the memory of Svatopluk Fučík

Abstract. The hysteresis effects appear in a natural way in mathematical problems leading to evolution differential inequalities. This observation is formulated by means of hysteresis or relaxation – hysteresis operators. Examples (phase transitions, elasto – plastic vibrations) illustrate the typical situations where these operators can be used.

Keywords: relaxation – hysteresis operators, evolution differential inequalities, free boundary problem, elasto – plastic vibrations

Classification: 35R45, 47H15, 35R35

The theory of evolution differential inequalities and the mathematical theory of hysteresis phenomena are known not to be independent (cf. e.g. [12]). We continue in this direction and our aim is to show that a large class of evolution inequalities can be not only considered as partial differential equations with hysteresis operators, but also this approach can have nontrivial applications, since hysteresis operators exhibit particular memory effects and the structure of the memory has been recently extensively studied. In many cases the knowledge of the memory structure provides some more information about the solution.

In Sections 1 and 2 we introduce the notion of a relaxation-hysteresis operator and investigate its properties. In Section 3 we show briefly an example of transforming the one-dimensional one-phase Stefan problem into an equation with hysteresis and we outline the proof of existence and uniqueness of the solution. In Section 4 we give a survey of results concerning linear and nonlinear elasto-plastic vibrations. Let us note that the nonlinear-elasto-plastic constitutive law generates the same hysteresis effects as the Preisach model of ferromagnetism.

1. Relaxation-hysteresis operators

In the theory of elasticity the mechanical properties of a material are characterized by the stress-strain relation (constitutive law or Hooke's law). For sake of simplicity we treat here only one-dimensional models, i.e. the stress σ and the strain ε are supposed to be scalar quantities. Indeed, generalizations concerning the vector (tensor) case are possible.

Following [12] the constitutive law for a serial visco-elasto-plastic model can be written in the form

$$(\sigma' - \varepsilon')(\sigma - x) + \Phi(\sigma) - \Phi(x) \leq 0 \qquad \forall x \in D_\Phi, \tag{1.1}$$

where the prime denotes differentiation (with respect to the time) and the function $\Phi : R^1 \to [0, +\infty]$ with a nonvoid domain $D_\Phi = \{x \in R^1; \Phi(x) < +\infty\}$ is assumed to satisfy the requirements

(1.2).

(i) Φ is convex lower semicontinuous,
(ii) $0 \in D_\Phi$, $\Phi(0) = 0$.

Let us mention typical special cases:

(1.3).

(i) **Pure hysteresis.** D_Φ is closed, $\Phi = I_{D_\Phi}$ (indicator function), i.e. $\Phi(x) = 0$ for $x \in D_\Phi$ (pure elasto-plasticity),
(ii) **Pure relaxation.** D_Φ is open, Φ continuously differentiable in D_Φ. Then (1.1) is equivalent to the equation $\sigma' + \Phi'(\sigma) = \varepsilon'$ (pure visco-elasticity).

In general, the domain D_Φ is an interval (we exclude the trivial case $D_\Phi = \{0\}$) with boundary points $-\infty \le a < b \le +\infty$. We denote by $\tilde{D}_\Phi$ the interval $(\tilde{a}, \tilde{b})$, where $\tilde{b} = b$ if $b \notin D_\Phi$ and $\tilde{b} = +\infty$ if $b \in D_\Phi$ and analogously for $\tilde{a}$. Notice that in the case $+\infty > b \notin D_\Phi$ we have $\Phi(x) \nearrow +\infty$ as $x \nearrow b$.

1.4. Lemma. *There exists a sequence $\{\Phi_n\}$ of convex continuously differentiable functions on D_Φ such that $\varphi_n = \Phi_n'$ are absolutely continuous, $|\varphi_n(x)| \le |\Phi'(x)|$ a.e., $\Phi_n(0) = \varphi_n(0) = 0$, $\Phi_n(x) \to \infty$ for $x \in \tilde{D}_\Phi \backslash D_\Phi$ and $\Phi_n \to \Phi$ in D_Φ locally uniformly.*

PROOF : The derivative $\varphi = \Phi'$ is a nondecreasing function in D_Φ, $\varphi(x)x \ge 0$ a.e. For $x \in D_\phi$ we define φ_n as the solution of the equation

$$\frac{1}{n}\varphi_n'(x) + \varphi_n(x) = \varphi(x), \quad \varphi_n(0) = 0, \qquad \text{if} \qquad x > 0,$$

$$-\frac{1}{n}\varphi_n'(x) + \varphi_n(x) = \varphi(x), \quad \varphi_n(0) = 0, \qquad \text{if} \qquad x < 0, \quad \text{i.e.}$$

$$\varphi_n(x) = n \int_0^x e^{n(y-x)} \varphi(y)\, dy = \int_0^{nx} e^{-y} \varphi(x - y/n)\, dy \quad \text{for } x > 0$$

and similarly for $x < 0$. If $b \in D_\Phi$, then we put $\varphi_n(x) = \varphi_n(b) + n(x - b)$ for $x > b$ (penalization) and analogously for $a \in D_\Phi$. The functions φ_n are absolutely continuous, nondecreasing and converge to φ in $L^1_{loc}(D_\Phi)$. We put $\Phi_n(x) = \int_0^x \varphi_n(y)\, dy$ and the proof follows easily. ∎

1.5 Existence Theorem. *Let $\sigma_0 \in D_\Phi$ and $\varepsilon \in W^{1,2}(0,T)$ be given such that $\partial\Phi(\sigma_0) \neq \emptyset$. Then there exists $\sigma \in W^{1,\infty}(0,T)$, $\sigma(t) \in D_\Phi$ for every $t \in [0,T]$, $\sigma(0) = \sigma_0$, satisfying (1.1) almost everywhere.*

PROOF : Let Φ_n be the sequence from Lemma (1.4). We denote by σ_n the solution of the equation.

$$\sigma_n' + \varphi_n(\sigma_n) = \varepsilon', \qquad \sigma_n(0) = \sigma_0. \tag{1.6}$$

In its domain of definition the solution of (1.6) belongs to $W^{2,1}$ and satisfies the relations

$$\begin{aligned} &(i) && \sigma_n' \operatorname{sign}\sigma_n + |\varphi_n(\sigma_n)| = \varepsilon' \operatorname{sign}\sigma_n, \\ &(ii) && \sigma_n'' \operatorname{sign}\sigma_n' + |\varphi_n'(\sigma_n)\sigma_n'| = \varepsilon'' \operatorname{sign}\sigma_n' \\ &(iii) && \sigma_n'' \sigma_n' + \varphi'(\sigma_n)\sigma_n'^2 = \varepsilon'' \sigma_n'. \end{aligned} \tag{1.7}$$

Integrating (1.7)(i),(ii) from 0 to t we see that the following estimates hold:

(1.8)

$$\begin{aligned} &(i) && |\sigma_n(t)| + \int_0^t |\varphi_n(\sigma_n)|\, d\tau \le \int_0^t |\varepsilon'|\, d\tau + \sigma_0 \\ &(ii) && |\sigma_n'(t)| + \int_0^t |\varphi_n'(\sigma_n)\,\sigma_n'|\, d\tau \le \int_0^t |\varepsilon''|\, d\tau + |\varphi_-(\sigma_0)| + |\varepsilon'(0)| \end{aligned}$$

where $\varphi_-(\sigma_0) = \lim_{|x|\to|\sigma_0|-} \varphi(x)$ for $\sigma_0 \neq 0, \varphi_-(0) = 0$. The hypothesis $\partial\Phi(\sigma_0) \neq \emptyset$ guarantees that $|\varphi_-(\sigma_0)| < +\infty$. Consequently, there exists a constant $c > 0$ independent of n such that

$$\begin{aligned} &(i) && |\varphi_n(\sigma_n(t))| \le c, \qquad \Phi_n(\sigma_n(t)) \le \sigma_n(t)\,\varphi_n(\sigma_n(t)) \le c, \\ &(ii) && |\sigma_n(t)| + |\sigma_n'(t)| + \int_0^t |\sigma_n''(\tau)|\, d\tau \le c. \end{aligned} \tag{1.9}$$

These estimates imply the global existence of the solution of (1.6) in $[0,T]$ for every n. Moreover, we can choose a subsequence (we denote it again by $\{\sigma_n\}$) and an element $\sigma \in W^{1,\infty}(0,T)$ such that

(1.10)

$\sigma_n \to \sigma$	uniformly in	$C([0,T])$,	
$\sigma_n' \to \sigma'$	in $L^p(0,T)$	strong for $1 \le p < \infty$	and almost everywhere,
$\sigma_n' \to \sigma'$	in $L^\infty(0,T)$	weak-star.	

Using (1.6) and the convexity of Φ_n we obtain

$$(\sigma_n' - \varepsilon')(\sigma_n - x) + \Phi_n(\sigma_n) - \Phi_n(x) \le 0 \qquad \forall x \in D_\Phi \tag{1.11}$$

We check that $\sigma(t) \in D_\Phi$ for every $t \in [0,T]$. Indeed, let us suppose $\sigma(t) \notin D_\Phi$ for some t. We introduce the convex closed set $M = \{x \in D_\Phi; \Phi(x) \le c+1\}$, where c is the constant from (1.9)(i). We have $\operatorname{dist}(\sigma(t), M) > 0$, hence there exists $\delta > 0$ such that $(1-\delta)\,\sigma(t) \notin M$. We find n_0 such that $|\sigma_n(t) - \sigma(t)| < \delta|\sigma(t)|$ and $\Phi_n((1-\delta)\,\sigma(t)) > c$ for $n \ge n_0$. Consequently, $\Phi_n(\sigma_n(t)) \ge \Phi_n((1-\delta)\,\sigma(t)) > c$, but this contradicts (1.9)(i).

Therefore, $\sigma(t) \in D_\Phi$ and it remains to verify that σ is a solution of (1.1). We have either $\sigma_n(t) \in D_\Phi$ for every n sufficiently large, hence $\Phi_n(\sigma_n(t)) \to \Phi(\sigma(t))$, or $\sigma_n(t) \notin D_\Phi$ for infinitely many n, and for such n we have $\Phi_n(\sigma_n(t)) \ge \Phi_n(\sigma(t))$. This implies for every $t \in [0,T]$ $\liminf_{n\to\infty} \Phi_n(\sigma_n(t)) \ge \Phi(\sigma(t))$. We take the limit in (1.11) and the proof is complete. ■

1.12. Uniqueness and Continuity Theorem. *Let $\varepsilon_1, \varepsilon_2 \in W^{2,1}(0,T)$ be given and let σ_1, σ_2 be the corresponding solutions of (1.1) with initial conditions $\sigma_1^0, \sigma_2^0 \in D_\Phi$, respectively. Let us denote $\xi_i = \varepsilon_i - \sigma_i$, $i = 1,2$. Then we have*

(i) $$(\xi_1' - \xi_2')(\sigma_1 - \sigma_2) \geq 0 \qquad \textit{almost everywhere,}$$

(ii) $$|\xi_1(t) - \xi_2(t)| \leq \max\{\|\varepsilon_1 - \varepsilon_2\|_{[0,t]}, |\xi_1(0) - \xi_2(0)|\}$$

for every $t \in [0,T]$, *where* $\|w\|_{[0,t]}$ *denotes* $\max\{|w(s)|, s \in [0,t]\}$,

(iii) $$\int_0^T |\xi_1' - \xi_2'|\, dt \leq \int_0^T |\varepsilon_1' - \varepsilon_2'|\, dt + |\sigma_1^0 - \sigma_2^0|.$$

PROOF :

(i) We put simply $x = \sigma_2$ in (1.1) for σ_1 and vice versa and sum up.

(ii) Let $t \in [0,T]$ be fixed. If $\sigma_1(t) = \sigma_2(t)$, then the assertion is trivial. Thus, let for example $\sigma_1(t)$ be greater than $\sigma_2(t)$. The continuity of σ_i implies that only two cases are possible:

a) $\sigma_1(s) > \sigma_2(s)$ for every $s \in [0,t]$, hence $\xi_1(s) - \xi_2(s) < \varepsilon_1(s) - \varepsilon_2(s)$ for every $s \in [0,t]$. By (i) we have $\xi_1'(s) - \xi_2'(s) \geq 0$ a.e. in $[0,t]$, i.e. $\xi_1(t) - \xi_2(t) \geq \xi_1(0) - \xi_2(0)$ and (ii) follows immediately,

b) $\exists t_0 \in [0,t)$ such that $\sigma_1(t_0) = \sigma_2(t_0)$. We take the maximal t_0 with this property. Using the argument from a) in $[t_0,t]$ we obtain an analogous conclusion.

(iii) By (1.10) it suffices to consider the case $\Phi = \Phi_n$. In other words, we may assume that D_Φ is open and Φ is continuously differentiable in D_Φ, and σ_1, σ_2 are solutions of (1.3)(ii) with right-hand sides $\varepsilon_1, \varepsilon_2$, respectively.

The set $M = \{t \in (0,T), \sigma_1(t) \neq \sigma_2(t)\}$ is open and can be described as the union of open disjoint intervals (a_k, b_k). For $t \notin M$ we have $\sigma_1'(t) - \sigma_2'(t) = \varepsilon_1'(t) - \varepsilon_2'(t)$, hence $\xi_1'(t) = \xi_2'(t)$. For $t \in (a_k, b_k)$ we have e.g. $\sigma_1(t) > \sigma_2(t)$, hence $\xi_1' \geq \xi_2'$ in (a_k, b_k), and

$$\int_{a_k}^{b_k} |\xi_1' - \xi_2'|\, dt = \int_{a_k}^{b_k} (\varepsilon_1' - \varepsilon_2')\, dt - (\sigma_1(b_k) - \sigma_2(b_k)) + (\sigma_1(a_k) - \sigma_2(a_k))$$
$$\leq \int_{a_k}^{b_k} |\varepsilon_1' - \varepsilon_2'|\, dt + |\sigma_1(a_k) - \sigma_2(a_k)|,$$

but $\sigma_1(a_k) \neq \sigma_2(a_k)$ only if $a_k = 0$. Therefore,

$$\int_0^T |\xi_1' - \xi_2'|\, dt = \int_M |\xi_1' - \xi_2'|\, dt \leq \int_M |\varepsilon_1' - \varepsilon_2'|\, dt + |\sigma_1^0 - \sigma_2^0|$$

and the Theorem is proved.

(1.13) Definition. Let $a : R^1 \to D_\Phi \backslash \{x; \partial\Phi(x) = \emptyset\}$ be a given nondecreasing α-Lipschitz continuous function with $\alpha \leq 1$, $a(0) = 0$. The operator $f_\Phi : W^{2,1}(0,T) \to W^{1,\infty}(0,T)$ given by the formula

$$f_\Phi(\varepsilon)(t) = \sigma(t),\ t \in [0,T],$$

where σ is the solution of (1.1), $\sigma(0) = a(\varepsilon(0))$, is called *relaxation-hysteresis operator* (or RH-operator) corresponding to the convex lower semicontinuous function Φ.

As an immediate consequence of (1.12) we have

(1.14) Corollary. *Every RH-operator can be extended to a Lipschitz continuous operator $C[0,T]) \to C([0,T])$ and $W^{1,1}(0,T) \to W^{1,1}(0,T)$, with the Lipschitz constant 2 in both cases, if the norm in $W^{1,1}(0,T)$ is given by the formula $\|\varepsilon\|_{1,1} = \int_0^T |\varepsilon'|\,dt + |\varepsilon(0)|$.*

(1.15) Remark. Let $\varphi = \Phi'$ be bounded in the interior of D_Φ (say, $|\varphi(x)| \le M$ a.e.), and let $t \in (0,T)$ be a Lebesgue point of σ'. Putting $x = \sigma(t-h)$ in (1.1) and dividing by h we obtain for $h \to 0+$

$$(\sigma' - \varepsilon')\sigma' \le M|\sigma'|, \quad \text{hence} \quad |\sigma'(t)| \le |\varepsilon'(t)| + M \quad \text{a.e.}.$$

In particular, f_Φ maps $W^{1,p}(0,T)$ into itself for $p \ge 1$ and by Lemma 3 of [10] this mapping is continuous for $1 \le p < \infty$.

(1.16) Example. In the situation of (1.3)(i) we have an explicit formula for the operator f_Φ (see [4], where such operators are of large importance). If $D_\Phi = [a, b]$, then for a piecewise monotone input ε we have

$$f_\Phi(\varepsilon)(t) = \begin{cases} \min \quad \{b,\ f_\Phi(\varepsilon)(t_0) + \varepsilon(t) - \varepsilon(t_0)\}, \quad t \in [t_0, t_1], \quad \text{if} \\ \qquad \varepsilon \quad \text{is nondecreasing in} \quad [t_0, t_1], \\ \max \quad \{a,\ f_\Phi(\varepsilon)(t_0) + \varepsilon(t) - \varepsilon(t_0)\}, \quad t \in [t_0, t_1], \quad \text{if} \\ \qquad \varepsilon \quad \text{is nonincreasing in} \quad [t_0, t_1] \end{cases}$$

with obvious modifications if $a = -\infty$ or $b = +\infty$. In particular, for $[a, b] = [-h, h]$ and $a(x) = \operatorname{sign} x \cdot \min\{|x|, h\}$ we obtain the Prandtl hysteresis operator f_h (cf. [5],[6]).

2. Properties of RH-operators

a) Monotonicity.

(2.1) Lemma. *Let $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$ be given. Using the notation from (1.12) we have*

(i) *for every convex continuously differentiable function G*

$$\frac{d}{dt} G(\xi_1 - \xi_2) \le (\xi_1' - \xi_2')\, G'(\varepsilon_1 - \varepsilon_2) \quad \textit{a.e.},$$

(ii) *for every increasing continuously differentiable function μ, nondecreasing function g and $\Phi = I_{D_\Phi}$*

$$(\mu(\xi_1) - \mu(\xi_2))'(g(\varepsilon_1 - \varepsilon_2) - g(\xi_1 - \xi_2)) \ge 0 \qquad \textit{a.e.}$$

PROOF : Both inequalities are consequences of (1.12)(i). For $\sigma_1 > \sigma_2$ we have $G'(\varepsilon_1 - \varepsilon_2) \geq G'(\xi_1 - \xi_2)$ and $\xi_1' \geq \xi_2'$ and analogously for $\sigma_1 < \sigma_2$, $\sigma_1 = \sigma_2$, so that (i) follows immediately. For $\Phi = I_{D_\Phi}$ we have especially $\xi_i'(\sigma_i - x) \geq 0$ for every $x \in D_\Phi$, hence $\mu(\xi_1)'(\sigma_1 - \sigma_2) \geq 0$, $\mu(\xi_2)'(\sigma_1 - \sigma_2) \leq 0$ and we use the same argument as above. ∎

(2.2) Remarks.

(i) For $G(x) = \frac{1}{2}x^2$ the inequality (2.1)(i) can be rewritten as

$$(\sigma_1 - \sigma_2)(\varepsilon_1' - \varepsilon_2') \geq \frac{1}{2}\frac{d}{dt}(\sigma_1 - \sigma_2)^2 \quad \text{a.e.}$$

(ii) Putting $g(x) = \operatorname{sign} x$ in (2.1)(ii) we obtain the Hilpert inequality (cf. [13])

$$\frac{d}{dt}|\mu(\xi_1) - \mu(\xi_2)| \leq (\mu(\xi_1) - \mu(\xi_2))' \operatorname{sign}(\varepsilon_1 - \varepsilon_2).$$

(iii) For $\Phi = I_{D_\Phi}$ we have $\xi'(\sigma - x) \geq 0$ $\forall x \in D_\Phi$.
Choosing $x = \sigma(t \pm h)$ the limit as $h \to 0+$ yields $\xi'\sigma' = 0$ a.e.

b) Energy inequalities.

(2.3) Lemma.

(i) *Let $\varepsilon \in W^{1,1}(0,T)$ be given, $\sigma = f_\Phi(\varepsilon)$. Then for every $0 \leq s < t \leq T$ we have*

$$\int_s^t \sigma\varepsilon' \, d\tau \geq \frac{1}{2}\sigma^2(t) - \frac{1}{2}\sigma^2(s).$$

(ii) *Let ε belong to $W^{2,1}(0,T)$. Then for almost every $0 \leq s < t \leq T$ we have*

$$\int_s^t \sigma'\varepsilon'' \, d\tau \geq \frac{1}{2}\sigma'^2(t) - \frac{1}{2}\sigma'^2(s).$$

PROOF : Part (i) is a special case of (2.2)(i) with $\varepsilon_2 = 0$. Part (ii) follows immediately from (1.7)(iii) and (1.10). ∎

c) Dependence on parameters.

We assume that the set of parameters $\Omega \subset R^N$ is a bounded open domain with a Lipschitzian boundary. For $1 \leq p < \infty$ we define the space $L^p(\Omega; C([0,T]))$
$= \{w \in L^p(\Omega; L^\infty(0,T)); w(x,\cdot)$ is continuous in $[0,T]$ for a.e. $x \in \Omega\}$, which is a closed subspace of $L^p(\Omega; L^\infty(0,T))$ and hence it is a Banach space with the norm

$$\left(\int_\Omega \|w(x,\cdot)\|^p_{[0,T]} \, dx\right)^{1/p}.$$

For $\varepsilon \in L^p(\Omega; C([0,T]))$ we can define the value of a RH-operator by the formula

$$f_\Phi(\varepsilon)(x,t) = f_\Phi(\varepsilon(x,\cdot))(t) \quad \text{for a.e.} \quad x \in \Omega. \tag{2.4}$$

By Theorem (1.12)(ii) the operator f_Φ defined by (2.4) maps $L^p(\Omega; C([0,T]))$ into itself Lipschitz continuously. The problem of differentiability with respect to parameters is solved in the following lemmas.

(2.5) Lemma. *Let* $\varepsilon, \frac{\partial \varepsilon}{\partial x_1} \in L^p(\Omega; C([0,T]))$. *Then* $\frac{\partial}{\partial x_1} f_\Phi(\varepsilon) \in L^p(\Omega; L^\infty(0,T))$ *and the inequality*

$$\left|\frac{\partial}{\partial x_1} f_\Phi(\varepsilon)(x,t)\right| \le \left\|\frac{\partial \varepsilon}{\partial x_1}(x,\cdot)\right\|_{[0,t]}$$

holds almost everywhere in $\Omega \times (0,T)$.

PROOF : We denote $\sigma = f_\Phi(\varepsilon)$, $\bar{x} = (x_2, \ldots, x_N)$, $x = (x_1, \bar{x}) \in \Omega$. For a.e. $\bar{x}$ and every $s, x_1, \hat{x}_1$ we have

$$|\varepsilon(x_1,\bar{x},s) - \varepsilon(\hat{x}_1,\bar{x},s)| \le \int_{\hat{x}_1}^{x_1} \left\|\frac{\partial \varepsilon}{\partial x_1}(\xi_1,\bar{x},\cdot)\right\|_{[0,s]},$$

hence (1.12)(ii) yields

$$|\sigma(x_1,\bar{x},t) - \sigma(\hat{x}_1,\bar{x},t)| \le \int_{\hat{x}_1}^{x_1} \left\|\frac{\partial \varepsilon}{\partial x_1}(\xi_1,\bar{x},\cdot)\right\|_{[0,t]} d\xi_1. \tag{2.6}$$

The function $\xi_1 \to \|\frac{\partial \varepsilon}{\partial x_1}(\xi_1,\bar{x},\cdot)\|_{[0,t]}$ is integrable for a.e. $\bar{x}$, hence $\sigma(\cdot,\bar{x},t)$ is absolutely continuous for a.e. $\bar{x}$ and every t. We choose x_1 to be a Lebesgue point of $\frac{\partial \sigma}{\partial x_1}(\cdot,\bar{x},t)$ and of $\xi_1 \to \|\frac{\partial \varepsilon}{\partial x_1}(\xi_1,\bar{x},\cdot)\|_{[0,t]}$. Dividing (2.6) by $(x_1 - \hat{x}_1)$ and taking the limit as $\hat{x}_1 \to x_1$ we obtain the assertion. ∎

(2.7) Lemma. *Let* $\varepsilon \in L^2(\Omega; C([0,t]))$ *be such that*

$$\frac{\partial \varepsilon}{\partial x_1}(\cdot,0) \in L^2(\Omega), \ \frac{\partial^2 \varepsilon}{\partial x_1 \partial t} \in L^2(\Omega; L^1(0,T)), \quad \sigma = f_\Phi(\varepsilon).$$

Then we have

$$\int_s^t \int_\Omega \frac{\partial \sigma}{\partial x_1}\frac{\partial^2 \varepsilon}{\partial x_1 \partial t}\, dx dt \ge \frac{1}{2}\int_\Omega \left[\left(\frac{\partial \sigma}{\partial x_1}(x,t)\right)^2 - \left(\frac{\partial \sigma}{\partial x_1}(x,s)\right)^2\right] dx$$

for almost every $0 \le s < t \le T$.

PROOF : By (2.5) $\frac{\partial \sigma}{\partial x_1}$ belongs to $L^2(\Omega; L^\infty(0,T))$, hence the integrals are meaningful for a.e. $0 \le s < t \le T$. For a.e. $(x_1,\bar{x}), (\hat{x}_1,\bar{x}) \in \Omega$ (2.2)(i) yields

$$\begin{aligned}(\sigma(x_1,\bar{x},\tau) - \sigma(\hat{x}_1,\bar{x},\tau))\left(\frac{\partial \varepsilon}{\partial t}(x_1,\bar{x},\tau) - \frac{\partial \varepsilon}{\partial t}(\hat{x}_1,\bar{x},\tau)\right) \ge \\ \ge \frac{1}{2}\frac{\partial}{\partial t}(\sigma(x_1,\bar{x},\tau) - \sigma(\hat{x}_1,\bar{x},\tau))^2\end{aligned}$$

for a.e. $\tau \in [0,T]$. Integrating both sides of this inequality $\int_s^t d\tau$ and dividing by $(x_1 - \hat{x}_1)^2$ we can pass to the limit as $\hat{x}_1 \to x_1$ for a.e. $(x_1,\bar{x}) \in \Omega$. We integrate over Ω and the proof is complete. ∎

d) **Parallel configurations of RH-operators.**

Let us consider a measurable space $(P, \mu), |\mu|(P) < \infty$. Let $\{\Phi_p, p \in P\}$ be a system of convex lower semicontinuous functions $R^1 \to [0, +\infty]$. Then the operator $F_P : C([0,T]) \to C([0,T])$ defined by the formula

$$F_P(\varepsilon)(t) = \int_P f_{\Phi_p}(\varepsilon)(t)\, d\mu(p) \tag{2.8}$$

is called *parallel configuration* (cf. [12]). A typical example is the Ishlinskii operator

$$F(\varepsilon)(t) = \int_0^\infty f_h(\varepsilon)(t)\eta(h)\, dh, \tag{2.9}$$

which is the parallel configuration of Prandtl's operators f_h (see (1.16)) with a density $\eta \in L^1(0,\infty)$, which is commonly assumed to be nonnegative.

We can imagine numerous nonlinear versions of (2.8). Let us mention e.g. the Preisach operator

$$W(\varepsilon)(t) = \mu_0(\varepsilon(t)) + \int_0^\infty \mu(h, \ell_h(\varepsilon)(t))\, dh, \tag{2.10}$$

where $\ell_h(\varepsilon) = \varepsilon - f_h(\varepsilon)$ and μ_0, μ are given functions.

The properties of Ishlinskii and Preisach operators are studied e.g. in [1], [2], [5], [6], [9], [11].

Very important are the memory effects which enable us to improve in this special case the inequality (2.3)(ii). We have (see [6]) for $\varepsilon \in W^{2,1}(0,T)$

$$\int_s^t F(\varepsilon)'\varepsilon''\, d\tau \geq \frac{1}{2}F(\varepsilon)'(t)\,\varepsilon'(t) - \frac{1}{2}F(\varepsilon)'(s)\,\varepsilon'(s) + \frac{1}{4}\gamma(\|\varepsilon\|_{[0,t]}) \int_s^t |\varepsilon'(\tau)|^3 d\tau \tag{2.11}$$

$$\text{for a.e. } 0 \leq s < t \leq T,$$

where F is the operator (2.9) and $\gamma(r) = \operatorname{infess}\{\eta(h); 0 < h \leq r\}$.

Notice that for $\gamma = 0$ (2.11) follows from (2.3)(ii) by integration, since (2.2)(iii) yields $\sigma'^2 = \sigma'\varepsilon'$.

The last term in (2.11) is an estimate from below for the dissipation of the "energy" $\frac{1}{2}F(\varepsilon)'\varepsilon'$ and the quantity γ expresses the "measure of convexity" of hysteresis loops generated by F.

3. One-phase Stefan problem as an equation with hysteresis

In the classical formulation (cf. [3]) the one phase Stefan problem is characterized by the system (for sake of simplicity we consider here only the one-dimensional case)

$$\begin{cases} \Theta_t - \Theta_{xx} = 0, & \text{if } T \geq t > s(x), \\ \Theta = 0, & \text{if } s(x) \geq t \geq 0, \end{cases} \qquad x \in (0,1), \tag{3.1}$$

(3.2) $$\Theta_x(x,s(x))\,s'(x) = -k(x), \qquad s(0)=0$$

(3.3) $$\Theta_x(0,t) = b(\Theta(0,t)-\Theta_1(t)),$$

where s,Θ are unknown functions, $k\geq 0, \Theta_1 \geq 0$ are given functions, $b>0$ a given constant.

This system represents a mathematical model for the evolution of the interface $t=s(x)$ between ice $(t<s(x))$ and water $(t>s(x))$ and the distribution of the temperature Θ in the water.

Putting $u(x,t) = \begin{cases} 0, & \text{if } t\leq s(x) \\ \int_{s(x)}^t \Theta(x,\tau)\,d\tau, & \text{if } t>s(x) \end{cases}$

we check easily that formally the inequality

(3.4) $$\int_0^1 [(u_t+k)(u-v)+u_x(u_x-v_x)]\,dx + \\ + b(u(t,0)-T_1(t))(u(t,0)-v(t,0)) \leq 0,$$

where $T_1(t) = \int_0^t \Theta_1(\tau)\,d\tau$, holds for every $v\in W^{1,2}(0,1)$ such that $v\geq 0$ a.e., with the initial condition

(3.5) $$u(x,0)=0.$$

In [3] we can find the proof of the existence and uniqueness of a solution $u,u_t \in L^\infty(0,T;L^2(0,1))$ such that $u_x,u_{xt}\in L^2((0,1)\times(0,T))$, $u\geq 0$ a.e., of (3.4), (3.5) (at least for k =const., Θ_1 =const.), so that it is natural to consider (3.4) as a weak formulation of (3.1) - (3.3).

Let us denote $K(x) = -\int_x^1 k(\xi)\,d\xi$, and $w(x,t) = \int_0^t u_x(x,\tau)\,d\tau - tK(x)$, where u is the solution of (3.4), (3.5). Formally we have

(3.6) $$w_t = u_x - K(x)$$

(3.7) $$w(x,0)=0, \qquad w(1,t)=0$$

(3.8) $$w_t(0,t) = -K(0)+b(u(0,t)-T_1(t)).$$

Putting (3.6), (3.7), (3.8) into (3.4) we obtain

(3.9) $$\int_0^1 [u_t(u-v)+w_t(u_x-v_x)]\,dx - [w_t(u-v)]_0^1 \leq 0$$

for every $0\leq v\in W^{1,2}(0,1)$. Comparing (3.9) to (1.1) we see that we have

(3.10) $$u = f(w_x) \qquad \text{in a weak sense,}$$

where $f=f_\Phi$ is the pure hysteresis operator corresponding to $\Phi = I_{D_\Phi}$, $D_\Phi = [0,+\infty)$ and the initial condition $f(\varepsilon)(0) = (\varepsilon(0))^+$.

From (3.6), (3.8), (3.10) we deduce the equation

(3.11) $$w_t - f(w_x)_x = -K(x)$$

with initial and boundary conditions (3.7) and

(3.12) $$w_t(0,t) = -K(0)+b(f(w_x)(0,t)-T_1(t))$$

Let us note that (1.16) yields an "almost explicit" formula for the operator f.

The theory developped in Sections 1 and 2 enables us to prove

(3.13) Theorem. *Let $\Theta_1 \in L^2(0,T), k \in L^2(0,1)$ be given. Then the problem (3.11), (3.7), (3.12) has a unique solution $w \in C([0,1] \times [0,T])$ such that $w_{xt}, w_{tt} \in L^2((0,1)\times(0,T)), f(w_x)_x, f(w_x)_t \in L^\infty(0,T;L^2(0,1))$ and (3.11) holds almost everywhere.*

Sketch of the proof. We can use for example the space-discretization method. We show here only formally how to derive the apriori estimates for the approximate solutions. The details concerning the convergence are omitted.

Notice that the uniqueness follows directly from (2.2) (i).

Multiplying (3.11) by w_t, integrating $\int_0^t \int_0^1 dx\, d\tau$ and using (3.12), (2.3)(i) we obtain

(3.14)
$$\int_0^t \int_0^1 w_t^2 dx\, d\tau + \frac{1}{2}\int_0^1 f^2(w_x)(x,t)\, dx +$$
$$+ \int_0^t f(w_x)(0,\tau)(-K(0) + b(f(w_x)(0,\tau) - T_1(\tau)))\, d\tau \le \int_0^t \int_0^1 -K(x) w_t\, dx\, d\tau.$$

Similarly, multiplying (3.11) by w_{xxt} we obtain after integration from (2.7), (3.12)

(3.15)
$$\int_0^t \int_0^1 w_{xt}^2\, dx\, dt + \frac{1}{2}\int_0^1 (f(w_x)_x(x,t))^2 dx + \int_0^t f(w_x)(0,\tau)\, w_{xt}(0,\tau)\, d\tau$$
$$\le - \int_0^t \int_0^1 k(x) w_{xt}(x,\tau)\, dx\, d\tau + \int_0^t T_1(\tau)\, w_{xt}(0,\tau)\, d\tau.$$

Since $w_{xt} \le f(w_x)_t$ a.e. and $T_1(\tau) \ge 0$, it follows from (3.15)

(3.16)
$$\int_0^t \int_0^1 w_{xt}^2 dx\, dt + \frac{1}{2}\int_0^1 (f(w_x)_x(x,t))^2 dx + \frac{1}{2} f^2(w_x)(0,t)$$
$$\le - \int_0^t \int_0^1 k(x)\, w_{xt}(x,\tau) dx\, d\tau - \int_0^t \Theta_1(\tau)\, f(w_x)(0,\tau)\, d\tau + T_1(t) f(w_x)(0,t).$$

Finally, we differentiate (3.11) with respect to t, multiply by w_{tt} and (2.3)(ii) yields after integration

(3.17)
$$\int_0^t \int_0^1 w_{tt}^2 dx\, dt + \frac{1}{2}\int_0^1 (f(w_x)_t(x,t))^2 dx$$
$$+ b \int_0^t f(w_x)_t(0,\tau)(f(w_x)_t(0,\tau) - \Theta_1(\tau))\, d\tau \le \frac{1}{2}\int_0^1 (f(w_x)_t(x,0))^2 dx = 0$$

since $w_{xt}(x,0) \le 0$ in $(0,1)$.

From (3.14) - (3.17) we derive the estimate

$$\int_0^T \int_0^1 (w_{xt}^2 + w_{tt}^2)dx\,dt \le \text{const.},$$

$$\max_t \int_0^1 (f(w_x)_x)^2 + (f(w_x)_t)^2 dx \le \text{const.},$$

which are sufficient, taking into account the compact embedding $\{v \in L^\infty(0,T;L^2(0,1)); v_t, v_x \in L^\infty(0,T;L^2(0,1))\} \hookrightarrow\hookrightarrow C([0,1]\times[0,T])$ and the monotonicity (2.2)(i) of f, for passing to the limit in (3.11).

4. Elasto-plastic vibrations

Forced longitudinal vibrations of a beam are governed by the equation of motion

$$u_{tt} - \sigma_x = g(x,t), \qquad x \in (0,\pi), \quad t \in (0,T) \tag{4.1}$$

where u, σ are the displacement and the stress, respectively, and g is a given forcing term. The material is assumed to obey the constitutive law for the parallel configuration of elasto-plastic elements

$$\sigma = \int_0^\infty \sigma_h\, \eta(h)\, dh, \qquad 0 \le \eta \in L^1(0,\infty), \tag{4.2}$$

$$(\sigma_h' - \varepsilon')(\sigma_h - x) \le 0, \qquad \forall |x| \le h, \tag{4.3}$$

$$|\sigma_h| \le h, \quad \sigma_h(0) = \operatorname{sign}\varepsilon(0)\min\{h, |\varepsilon(0)|\}, \tag{4.4}$$

$$\varepsilon = u_x. \tag{4.5}$$

For sake of simplicity we choose the boundary and initial conditions in the form

$$u(0,t) = u(\pi,t) = 0 \tag{4.6}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \tag{4.7}$$

The existence of a unique solution of (4.1) - (4.7) under appropriate assumptions in $[0,\pi]\times[0,T]$ for an arbitrary $T > 0$ follows from the classical theory of evolution differential inequalities (cf. [12] for further references). On the other hand, the system (4.1) - (4.5) can be written in the form of a single equation

$$u_{tt} - F(u_x)_x = g(x,t), \tag{4.8}$$

where F is the Ishlinskii operator (2.9). Making use of the estimate from below of the dissipation of "energy" (2.11) we can prove under the hypotheses

$$\operatorname{infess}\{\eta(h); 0 < h \le r\} > 0 \qquad \text{for every} \quad r > 0, \tag{4.9}$$

$$\int_0^\infty \int_\xi^\infty \eta(h)\, dh\, d\xi = +\infty \tag{4.10}$$

the following stronger results.

A. A unique ω-periodic solution of (4.8), (4.6) exists provided g is ω-periodic with respect to t and η decays sufficiently slowly (cf. [5], [7]).

B. For $g = 0$ (free vibrations) the solution of (4.6) - (4.8) decays to the equilibrium as $t \to \infty$ and the rate of decay is $|u_t(x,t)| + |F(u_x)(x,t)| \le c/t$ (cf. [8] for different boundary conditions). We cannot expect that $u_x \to 0$ as $t \to +\infty$: the deformation in the equilibrium depends on the initial conditions and need not vanish. This phenomenon can be easily observed in the case of ODE's with hysteresis operators as well as the relation between η and the rate of decay of the solution (see [6]).

C. We can introduce a nonlinear perturbation to the constitutive law, e.g. $\sigma = F(\varepsilon) + \psi(\varepsilon)$ or $\sigma = F(\psi(\varepsilon))$, where ψ is an increasing smooth odd function. This corresponds to different nonlinear-elasto-plastic constitutive laws. It can be shown that we can write them in the form $\sigma = W(\varepsilon)$ with a special Preisach operator (2.10). When solving corresponding problems we make use of an analogy of (2.11) which remains valid provided we guarantee the convexity of hysteresis loops. Typically this is true only for small amplitudes of vibrations, so that we have no existence results in such cases except for not too large data - see [9].

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