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Dedicated to the memory of Svatopluk Fučík

Abstract. The paper deals with conditions on the measure μ under which the fractional order integral T_γ and the fractional maximal function M_γ defined on the homogeneous measure space (X, ρ, μ) are operators acting continuously between Lebesgue, Lorentz and Orlicz spaces. The weighted as well as the non-weighted case is considered.

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0. Introduction.

In the paper one of the possible variants of an integral of fractional order on spaces of homogeneous type is proposed. For such integrals various estimates are obtained – of pointwise as well of integral character. One of the fundamental results is a full description of measures μ, for which the *fractional order integral*

$$T_\gamma f(x) = \int_X (\rho(x,y))^{\gamma-1} f(y)\, d\mu, \quad 0 < \gamma < 1, \tag{0.1}$$

defined on the homogeneous type space (X, ρ, μ) represents an operator acting continuously from $L^p(X,\mu)$ into $L^q(X,\mu)$ with $q^{-1} = p^{-1} - \gamma$, i.e., for which a result of the type of the well-known S.Sobolev theorem (see [1]) holds. Further, some analogues of the well-known weight theorems of B. Muckenhoupt and R.Wheeden [2] and of D.Adams [3] about classical Riesz potentials are proved.

1. Preliminaries and basic facts.

Let X be a space with measure μ, equipped with a quasimetric ρ, i.e., with a mapping

$$\rho : X \times X \to [0, \infty)$$

such that

(i) $\rho(x,y) > 0$ if and only if $x \neq y$;
(ii) $\rho(x,y) = \rho(y,x)$ for every pair of $x, y \in X$;
(iii) there exists a constant $\eta > 0$ such that for every x, y, z from X the following inequality holds:

$$\rho(x,z) \le \eta\{\rho(x,y) + \rho(y,z)\}.$$

Let all balls $B(x,r) = \{y \in X : \rho(x,y) < r\}$ be μ–measurable and assume that the measure μ fulfils the doubling condition

$$0 < \mu B(x,2r) \le c\mu B(x,r) < \infty$$

with c independent of x and r.

A space (X,ρ,μ) which satisfies all conditions mentioned above is called a *space of homogeneous type* (see, e.g., [4]).

Let $w : X \mapsto \mathbf{R}$ be positive a.e. and locally integrable. Such a function will be called a weight function. Denote by $L^p_w(X,\mu)\,(1 \le p \le \infty)$ the space of functions $f : X \to \mathbf{R}$ for which

$$\|f\|_{L^p_w(X,\mu)} = \left(\int_X |f(x)|^p\, w(x)\, d\mu\right)^{1/p} < \infty. \tag{1.1}$$

For $w \equiv 1$ we shall write $L^p_w(X,\mu) = L^p(X,\mu)$. Further, denote

$$w(E) = \int_E w(x)\, d\mu.$$

In the sequel, we shall consider *Lorentz spaces*. For $1 \le p \le \infty$, $1 \le s \le \infty$ denote

$$\|f\|_{L^{ps}_w(X,\mu)} = \begin{cases} \left(s\int_0^\infty (w\{x \in X : |f(x)| > \lambda\})^{s/p}\lambda^{s-1}\, d\lambda\right)^{1/s} \\ \qquad\qquad \text{for } 1 \le p < \infty,\ 1 \le s < \infty, \\ \sup\limits_{\lambda>0}\lambda(w\{x \in X : |f(x)| > \lambda\})^{1/p}, \\ \qquad\qquad \text{for } 1 \le p < \infty,\ s = \infty. \end{cases} \tag{1.2}$$

Obviously $L^{pp}_w(X,\mu) = L^p_w(X,\mu)$.

Further, let Φ be a *Young function*, i.e., $\Phi : \mathbf{R} \to [0,\infty)$ is an even, convex, continuous function such that $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \ne 0$ and $\lim\limits_{t\to 0}\Phi(t)/t = \lim\limits_{t\to\infty} t/\Phi(t) = 0$.

Denote by $L^\Phi_w(X,\mu)$ the *weighted Orlicz space*, i.e., the linear hull of the set

$$\{f : \int_X \Phi(f(x))\, w(x)\, d\mu < \infty\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^\Phi_w(X,\mu)} = \inf\{\lambda > 0 : \int_X \Phi(\lambda^{-1}f(x))\, w(x)\, d\mu \le 1\}. \tag{1.3}$$

The Young function

$$\Psi(t) \sim \sup_{s>0}(|t|s - \Phi(s)), \quad t \in \mathbf{R},$$

is the so-called *complementary function* (with respect to Φ).

The norm

$$\|f\|_{L^*_w(X,\mu)} = \sup\{\int_X f(x)\,g(x)\,d\mu : \int_X \Psi(g(x))\,w(x)\,d\mu \le 1\}$$

is equivalent to (1.3).

Now, let Φ be a Young function satisfying the so-called Δ_2-condition, i.e.,

$$\Phi(2t) \le c\Phi(t), \quad t \in \mathbf{R},$$

and set

$$i(\Phi) = \lim_{\lambda\to 0} \frac{1}{\log\lambda} \log \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)},$$
$$I(\Phi) = \lim_{\lambda\to\infty} \frac{1}{\log\lambda} \log \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}.$$

The numbers $i(\Phi)$ and $I(\Phi)$ will be called the lower and upper index of Φ, respectively.

The monographs [5], [6] and [7] are useful references for the theory of the spaces mentioned above.

Analogously to the well-known A_p classes of weight functions introduced by B.Muckenhoupt we consider, for $1 \le p < \infty$, the following classes:

1.1. Definition. The weight function w belongs to the class $A_p(X)$ for $1 < p < \infty$ if

$$\sup_{\substack{x\in X\\ r>0}} (\mu B(x,r))^{-1} \int_{B(x,r)} w(y)\,dy \cdot \left((\mu B(x,r))^{-1} \int_{B(x,r)} w^{-1/(p-1)}(y)\,dy \right)^{p-1} < \infty$$

and w belongs to $A_1(X)$ if there exists a positive constant c such that for any $x \in X$ and $r > 0$

$$\left(\mu B(x,r)\right)^{-1} \int_{B(x,r)} w(y)\,dy \le c \operatorname*{ess\,inf}_{y\in B(x,r)} w(y).$$

These classes have been considered in [9] and [10] in connection with establishing weighted estimates for maximal functions defined on spaces of homogeneous type. The properties of the class $A_p(X)$ are analogous to those of the B.Muckenhoupt classes. In particular, if $w \in A_p(X)$, then $w \in A_{p-\varepsilon}(X)$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1}$ for any $p_1 > p$. We shall use these properties in Section 3.

Example. It can be easily verified that for a fixed point $a \in X$

$$\left(\mu B\left(a, \rho(a,x)\right)\right)^{\delta} \in A_p(X) \Leftrightarrow -1 < \delta < p-1,$$
$$\left(\mu B\left(a, \rho(a,x)\right)\right)^{\delta} \in A_1(X) \Leftrightarrow -1 < \delta < 0.$$

For a locally integrable function $f : X \mapsto \mathbb{R}$ and for $0 \le \beta < 1$, we introduce the fractional maximal function

$$M_{\beta} f(x) = \sup_{r>0} \left(\mu B(x,r)\right)^{\beta-1} \int_{B(x,r)} |f(y)|\, d\mu.$$

Proposition A (see [10]). *Let $1 < p < \beta^{-1}$, $q^{-1} = p^{-1} - \beta$. Then the following two conditions are equivalent:*

(i) *There is a constant $c > 0$ such that for any $f \in L^p_w(X,\mu)$ the inequality*

$$\left(\int_X \left(M_{\beta}(f w^{\beta})(x)\right)^q w(x)\, d\mu\right)^{1/q} \le c \left(\int_X |f(x)|^p w(x)\, d\mu\right)^{1/p}$$

holds.

(ii) $w \in A_{1+\frac{q}{p'}}(X)$, $\quad p' = \frac{p}{p-1}$.

Proposition B (see [10]). *Let $q = (1-\beta)^{-1}$. Then the following two conditions are equivalent:*

(i) $w\{x : M_{\beta}(f w^{\beta})(x) > \lambda\} \le c\lambda^{-q} \left(\int_X |f(x)|\, d\mu\right)^q$

with a constant c independent of f and $\lambda > 0$.

(ii) $w \in A_1(X)$.

The proofs of these assertions are the same as for the case of fractional maximal functions defined in $\mathbb{R}^n$. One only has to use the properties of the class $A_p(X)$ and (instead of the Besicovitch covering lemma) the covering lemma from [4].

Finally, throughout this paper the letter c will be used to denote a positive constant, not necessarily the same at each occurrence.

2. The non–weighted case.

In this section we give a full description of such measures μ for which the operator T_{γ} acts continuously from $L^p(X,\mu)$ into $L^q(X,\mu)$ with $q^{-1} = p^{-1} - \gamma$. First, we will prove a lemma stating a pointwise estimate for the function $T_{\gamma} f(x)$. Estimates of this type for Riesz potentials were obtained in [11], [12].

Denote

$$\Omega(x) = \sup_{r>0} \frac{\mu B(x,r)}{r}.$$

2.1. Lemma. *Let us assume that $\Omega(x)$ is finite μ–a.e. Further, let $0 < \lambda < 1$, $1 \le p < \lambda\gamma^{-1}$. Then there exists a positive number c such that for every $r > 0$ and $x \in X$*

$$\text{(2.1)} \qquad |T_{\gamma} f(x)| \le c \left(r^{\gamma} M_0 f(x) \Omega(x) + r^{\gamma - \lambda/p} M_{\lambda/p} f(x) (\Omega(x))^{1-\lambda/p}\right).$$

PROOF : Let $r > 0$ be arbitrary and write $T_\gamma f(x)$ in the form

$$T_\gamma f(x) = \int_{B(x,r)} f(y)\,(\rho(x,y))^{\gamma-1}\,d\mu + \\ + \int_{X\setminus B(x,r)} f(y)\,(\rho(x,y))^{\gamma-1}\,d\mu = I_1 + I_2.$$

Denote

$$D_k(x,r) = B(x,2^{-k}r)\setminus B(x,2^{-k-1}r), \quad k = 0,1,2,\dots.$$

Then for I_1 we have the estimate

$$\begin{aligned}
|I_1| &\le \int_{B(x,r)} |f(y)|(\rho(x,y))^{\gamma-1}\,d\mu = \\
&= \sum_{k=0}^{\infty}\int_{D_k(x,r)} |f(y)|(\rho(x,y))^{\gamma-1}\,d\mu \le \\
&\le \sum_{k=0}^{\infty}(2^{-k-1}r)^{\gamma-1}\mu B(x,2^{-k}r)\frac{1}{\mu B(x,2^{-k}r)}\int_{B(x,2^{-k}r)} |f(y)|\,d\mu \le \\
&\le c_1\sum_{k=0}^{\infty} 2^{-k(\gamma-1)}r^{\gamma-1}(2^{-k}r)M_0 f(x)\,\Omega(x) \le \\
&\le c_2 r^{\gamma} M_0 f(x)\,\Omega(x).
\end{aligned}$$

Now let $V_k = B(x,2^{k+1}r)\setminus B(x,2^k r)$, $k = 0,1,2,\dots.$ Then estimating $|I_2|$ we obtain:

$$\begin{aligned}
|I_2| &\le \int_{X\setminus B(x,r)} |f(y)|(\rho(x,y))^{\gamma-1}\,d\mu = \\
&= \sum_{k=0}^{\infty}\int_{V_k(x,r)} |f(y)|(\rho(x,y))^{\gamma-1}\,d\mu \le \\
&\le \sum_{k=0}^{\infty}(2^k r)^{\gamma-1}(\mu B(x,2^{k+1}r))^{1-\lambda/p}(\mu B(x,2^{k+1}r))^{\lambda/p-1}\cdot \\
&\qquad\qquad \cdot\int_{B(x,2^{k+1}r)} |f(y)|\,d\mu \le \\
&\le c_3\sum_{k=0}^{\infty}(2^k r)^{\gamma-1}(2^{k+1}r)^{1-\lambda/p}\,M_{\lambda/p}f(x)(\Omega(x))^{1-\lambda/p} = \\
&= c_3 r^{\gamma-\lambda/p}\sum_{k=0}^{\infty}2^{k(\gamma-\lambda/p)}M_{\lambda/p}f(x)(\Omega(x))^{1-\lambda/p} \le \\
&\le c_4 r^{\gamma-\lambda/p}M_{\lambda/p}f(x)(\Omega(x))^{1-\lambda/p},
\end{aligned}$$

since by our assumption we have $\gamma-\lambda/p < 0$.

Now Lemma 2.1 follows immediately from these two estimates for I_1 and I_2. ∎

2.1. Theorem. *Let the function $\Omega(x)$ be finite μ-a.e. and let $0 < \lambda \leq 1$, $1 < p < \frac{\lambda}{\gamma}$, $1 \leq r \leq \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{\lambda} + \frac{\gamma p}{\lambda r}$. Then for every function $f \in L^p(x,\mu)$ such that $M_{\lambda/p} f \in L^r(X,\mu)$ the following estimate holds:*

$$\|\Omega^{1-\gamma} T_\gamma f\|_{L^q(X,\mu)} \leq c \|M_{\lambda/p} f\|^{\gamma p/\lambda}_{L^r(X,\mu)} \|f\|^{1-\gamma p/\lambda}_{L^p(X,\mu)}. \tag{2.2}$$

PROOF : Taking

$$r = r(x) = \left(\frac{M_{\lambda/p} f(x)}{M_0 f(x)}\right)^{p/\lambda} \frac{1}{\Omega(x)}$$

in (2.1), we obtain that

$$(\Omega(x))^{1-\gamma} |T_\gamma f(x)| \leq c (M_{\lambda/p} f(x))^{p\gamma/\lambda} (M_0 f(x))^{1-p\gamma/\lambda} \tag{2.3}$$

for every $x \in X$.

Now inequality (2.2) follows if we take the q-th power in (2.3) and apply Hölder's inequality to the right-hand side of the inequality obtained. ∎

One of the fundamental results of this section and, in fact, of the paper is given by

2.2. Theorem. *Let $1 < p < \gamma^{-1}$, $q^{-1} = p^{-1} - \gamma$. The following two conditions are equivalent:*

(i) *T_γ maps continuously $L^p(X,\mu)$ into $L^q(X,\mu)$.*

(ii) *There exists a constant $c > 0$ such that*

$$\mu B(x,r) \leq cr \tag{2.4}$$

for any $x \in X$ and $r > 0$.

PROOF : The implication (ii) $\Rightarrow$ (i) follows easily from Theorem 2.1 if we take there $\lambda = 1$, $r = \infty$. Indeed, from Hölder's inequality we have

$$M_{1/p} f(x) \leq \|f\|_{L^p(X,\mu)}$$

and the remaining conclusion follows from (2.2).

Now we show that (i) $\Rightarrow$ (ii). For an arbitrary ball $B(a,r)$ in X, take $f(x) \doteq \chi_{B(a,r)}(x)$. Then in view of (i), we have

$$\left(\int_{B(a,r)} \left(\int_{B(a,r)} (\rho(x,y))^{\gamma-1} \, d\mu\right)^q d\mu\right)^{1/q} \leq c(\mu B(a,r))^{1/p}$$

with a positive constant c independent of a and r. Since $x, y \in B(a,r)$, we have

$$r^{\gamma-1} \mu B(a,r) \mu B(a,r) (\mu B(a,r))^{1/q} \leq (\mu B(a,r))^{1/p}$$

and the equality $q^{-1} = p^{-1} - \gamma$ implies (ii).

So Theorem 2.2 is proved. ∎

Using the well-known Hunt's interpolation theorem for Lorentz spaces (see, e.g., [6]) and the considerations from the second part of the proof of Theorem 2.2, we prove a more general assertion.

2.3. Theorem. *Let $1 < p < \gamma^{-1}$, $q^{-1} = p^{-1} - \gamma$, $1 < s < \infty$. Then the operator T_γ acts continuously from $L^{ps}(X,\mu)$ into $L^{qs}(X,\mu)$ if and only if condition (2.4) is fulfilled.*

Moreover, it can be shown that condition (2.1) is equivalent with the continuity of T_γ as an operator from $L^{ps}(X,\mu)$ into $L^{q\infty}(X,\mu)$. Here we shall prove only a particular case.

2.4. Theorem. *Let $0 < \gamma < 1$, $q = 1/(1-\gamma)$. Then inequality (2.4) holds if and only if there is a constant $c > 0$ such that for each $f \in L^1(X,\mu)$ and $\lambda > 0$*

$$\mu\{x : |T_\gamma f(x)| > \lambda\} \le c\lambda^{-q}\|f\|^q_{L^1(X,\mu)}. \tag{2.5}$$

PROOF : Let us note that if (2.1) holds and if $p = 1$, $\lambda = 1$, and $r = \infty$, then by the argument used in the proofs of Lemma 2.1 and Theorem 2.1 it is possible to obtain instead of (2.3) the estimate

$$|T_\gamma f(x)| \le c_1 \left((M_0 f(x))^{1-\gamma}\|f\|^\gamma_{L^1(X,\mu)}\right), \tag{2.6}$$

where the constant c_1 is independent of x.

Further, form inequality (2.6) and from Proposition B we derive that

$$\mu\{x : |T_\gamma f(x)| > \lambda\} \le \mu\{x : M_0 f(x) > c_1^{-q}\lambda^{-q}\|f\|^{\gamma q}_{L^1(X,\mu)}\} \le$$
$$\le c_2\lambda^{-q}\|f\|^q_{L^1(X,\mu)}.$$

Theorem 2.4 is proved. ■

From the weak inequality (2.5) we conclude with help of the interpolation theorem of O'Neil [13], that the following assertion holds:

2.5. Theorem. *Let (2.1) hold and let $q = \frac{1}{1-\gamma}$. Suppose that f has a compact support. Then*

$$f \in L(\log^+ L)^s(X,\mu) \Rightarrow T_\gamma f \in L^{qs^{-1}}(X,\mu) \quad \textit{for } 0 < s \le 1$$

and

$$\int_1^\infty (u\{x : |T_\gamma f(x)| > \lambda\})^{1/q}(\log\lambda)^{s-1}\,d\lambda < \infty \quad \textit{for } 1 \le s < \infty.$$

3. The weighted case.

In this section we give a full description of measures for which weighted estimates for the fractional integral (0.1) hold, using the method of G. Welland [14].

We start with a lemma.

3.1. Lemma. *For any ε, $0 < \varepsilon < \min(\gamma, 1-\gamma)$, there exists a constant $c_\varepsilon > 0$ such that for any nonnegative function $\phi : X \to \mathbf{R}$ and for any point $x \in X$ the following inequality holds:*

$$T_\gamma\phi(x) \le c_\varepsilon\sqrt{M_{\gamma-\varepsilon}\phi(x)\, M_{\gamma+\varepsilon}(X)}\,(\Omega(x))^{1-\gamma}. \tag{3.1}$$

PROOF : Let r be an arbitrary positive real number. Similarly as in the proof of Lemma 2.1, we write the integral as the sum of two integrals:

$$T_\gamma\phi(x) = \int_{B(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu + \\ + \int_{X\setminus B(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu.$$

For $0 < \varepsilon < \gamma$ we then have

$$\int_{B(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu = \sum_{j=0}^{\infty}\int_{D_j(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu \le$$
$$\le \sum_{j=0}^{\infty}(2^{-j-1}r)^{\gamma-1}\int_{B(x,2^{-j}r)} \phi(y)\,dy \le$$
$$\le c_1(\varepsilon)r^\varepsilon \sum_{j=0}^{2} 2^{-\varepsilon j}(\mu B(x,2^{-j}r))^{\gamma-\varepsilon-1}\cdot$$
$$\cdot\int_{B(x,2^{-j}r)} \phi(y)\,d\mu(\Omega(x))^{(1-(\gamma-\varepsilon))} \le$$
$$\le c_2(\varepsilon)r^\varepsilon M_{\gamma-\varepsilon}\phi(x)(\Omega(x))^{(1-(\gamma-\varepsilon))}.$$

On the other hand, for $0 < \varepsilon < 1-\gamma$ we have

$$\int_{X\setminus B(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu =$$
$$= \sum_{j=0}^{\infty}\int_{V_j(x,r)} \phi(y)(\rho(x,y))^{\gamma-1}\,d\mu \le$$
$$\le \sum_{j=0}^{\infty}(2^j r)^{\gamma-1}\int_{B(x,2^{j+1}r)} \phi(y)\,d\mu \le$$
$$\le c_3(\varepsilon)r^{-\varepsilon}\sum_{j=0}^{\infty} 2^{-j\varepsilon}\left(\frac{\mu B(x,2^j r)}{2^j r}\right)^{1-(\gamma+\varepsilon)}(\mu B(x,2^j r))^{\gamma+\varepsilon-1}\cdot$$
$$\cdot\int_{B(x,2^j r)} \phi(y)\,d\mu \le$$
$$\le c_4(\varepsilon)r^{-\varepsilon}M_{\gamma+\varepsilon}\phi(x)(\Omega(x))^{(1-(\gamma+\varepsilon))}.$$

Consequently, we obtained that for any ε, $0 < \varepsilon < \min(\gamma, 1-\gamma)$, there exists a constant $c_\varepsilon > 0$ such that for every nonnegative function ϕ and for any $x \in X$ and $r > 0$ we have

$$T_\gamma\phi(x) \le c_\varepsilon(r^\varepsilon M_{\gamma-\varepsilon}\phi(x)v_1(x) + r^{-\varepsilon}M_{\gamma+\varepsilon}\phi(x)v_2(x)), \tag{3.2}$$

where

$$v_1(x) = (\Omega(x))^{1-(\gamma-\varepsilon)}$$

and

$$v_2(x) = (\Omega(x))^{1-(\gamma+\varepsilon)}.$$

Taking

$$r^\varepsilon = \left(\frac{M_{\gamma+\varepsilon}\phi(x)v_2(x)}{M_{\gamma-\varepsilon}\phi(x)v_1(x)}\right)^{1/2}$$

in (3.2), we obtain (3.1).

Lemma 3.1 is proved. ∎

3.1. Theorem. *Suppose that $1 < p < \gamma^{-1}, q^{-1} = p^{-1} - \gamma$ and the function $\Omega(x)$ is finite μ-a.e. Then for each $w \in A_\beta(X)$, $\beta = 1 + \frac{q}{p'}$, there exists such a constant $c > 0$ that for arbitrary f from $L^p_w(X,\mu)$ the following inequality holds:*

$$\left(\int_X |T_\gamma(fw^\gamma)(x)|^q(\Omega(x))^{q(\gamma-1)}w(x)\,dx\right)^{1/q} \le \tag{3.3}$$
$$\le c\left(\int_X |f(x)|^p w(x)\,dx\right)^{1/p}.$$

Proof : If $w \in A_\beta(X)$ then $w \in A_{\beta-\eta}(X)$ for sufficiently small positive η. Therefore it is possible to choose $\varepsilon, 0 < \varepsilon < \min(\gamma, 1-\gamma)$, in such a way that simultaneously $w \in A_{\beta_1}$ with $\beta_1 = 1 + \frac{p}{p'(1-p(\gamma+\varepsilon))}$ and $w \in A_{\beta_2}$ with $\beta_2 = 1 + \frac{p}{p'(1-p(\gamma-\varepsilon))}$. If we take

$$\frac{1}{q_\varepsilon} = \frac{1}{p} - (\gamma+\varepsilon),\ \frac{1}{\overline{q}_\varepsilon} = \frac{1}{p} - (\gamma-\varepsilon),$$

then we obtain that $w \in A_{1+q_\varepsilon/p'}$ and $w \in A_{1+\overline{q}_\varepsilon/p'}$.

Denoting

$$p_1 = \frac{2q_\varepsilon}{q} \quad\text{and}\quad p_2 = \frac{2\overline{q}_\varepsilon}{q}$$

we have

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Put

$$F_1(x) = (M_{\gamma+\varepsilon}(fw^\gamma)(x))^{q/2}(w(x))^{1/p_1}$$

and

$$F_2(x) = (M_{\gamma-\varepsilon}(fw^\gamma)(x))^{q/2}(w(x))^{1/p_2}.$$

Further, (3.1) together with Hölder's inequality implies the estimate

$$\int_X |T_\gamma(fw^\gamma)(x)|^q(\Omega(x))^{q(\gamma-1)}w(x)\,d\mu \le c_\varepsilon \int_X F_1(x)F_2(x)\,d\mu \le$$

$$\le c_\varepsilon \left(\int_X (M_{\gamma+\varepsilon}(fw)^\gamma(x))^{qp_1/2}w(x)\,d\mu\right)^{1/p_1}\cdot$$

$$\cdot\left(\int_X (M_{\gamma-\varepsilon}(fw)^\gamma(x))^{qp_2/2}w(x)\,d\mu\right)^{1/p_2} =$$

$$= c_\varepsilon \left(\int_X (M_{\gamma+\varepsilon}(fw)^\gamma(x))^{q_\varepsilon} w(x)\,d\mu\right)^{1/p_1}\cdot$$

$$\cdot\left(\int_X (M_{\gamma-\varepsilon}(fw)^\gamma(x))^{\bar{q}_\varepsilon} w(x)\,dx\right)^{1/p_2}.$$

Finally, using Proposition A we conclude that

$$\|T_\gamma(fw^\gamma)(\Omega(x))^{1-\gamma}\|_{L^q_w(X,\mu)} \le c\|f\|_{L^p_w(X,\mu)}.$$

Theorem 3.1 is proved. ■

3.2. Theorem. *Assume that there exist two positive numbers a_1 and a_2 such that for every $x \in X$ and $r > 0$*

$$a_1 r \le \mu B(x,r) \le a_2 r. \tag{3.4}$$

If $1 < p < \gamma^{-1}$, $q^{-1} = p^{-1} - \gamma$, then the inequality

$$\|T_\gamma(fw^\gamma)\|_{L^q_w(X,\mu)} \le c\|f\|_{L^p_w(X,\mu)} \tag{3.5}$$

holds for any $f \in L^p_w(X,\mu)$ with a constant $c > 0$ independent of f if and only if

$$w \in A_\beta,\ \beta = 1 + \frac{q}{p'}. \tag{3.6}$$

Proof : It is clear that the implication (3.6) ⇒ (3.5) holds if (2.4) and, a fortiori, (3.4) is fulfilled. The implication (3.5) ⇒ (3.6) follows from the pointwise inequality

$$M_\gamma(fw^\gamma)(x) \le c_1 T_\gamma(|f|w^\gamma)(x) \tag{3.7}$$

and Proposition A. ■

For Riesz potentials, Theorem 3.2 is due to B. Muckenhoupt and R.L. Wheeden [2]. It was proved by using the above described method by G. Welland [14]. For anisotropic potentials an analogous problem was solved by V. Kokilashvili and M. Gabidzashvili [15] where besides (3.6), also another necessary and sufficient condition for (3.5) was found.

3.3. Theorem. *Suppose that $\Omega(x)$ is finite μ-a.e. in X. Then assume that Φ_1 and Φ_2 are Young functions for which the following conditions are fulfilled:*

$$1 < i(\Phi_1) = p \le I(\Phi_1) = P < \infty$$

and

$$1 < i(\Phi_2) = q \le I(\Phi_2) = Q < \infty.$$

If $0 \le \gamma < 1, q^{-1} = p^{-1} - \gamma, Q^{-1} = P^{-1} - \gamma$ *and* $w \in A_{1+q/p'}$ *then*

$$\|\Omega^{\gamma-1} T_\gamma(f(\varepsilon w)^\gamma)\|_{L_w^{\Phi_2}(X,\mu)} \le c\|f\|_{L_w^{\Phi_1}(X,\mu)}$$

with c independent of f and $\varepsilon > 0$.

We omit the proof of Theorem 3.3 since it can be derived from Theorem 3.1 in the same way as for anisotropic potentials in [**16**].

It is obvious how we can obtain an analogue of Theorem 3.2 for weighted Orlicz spaces.

Proposition C (see [17]). *Suppose that $1 < p \le q < \infty$ and ν is a positive measure on X such that all balls are ν-measurable.*

Then for the validity of the inequality

$$\left(\int_X (M_\gamma f(x))^q \, d\nu\right)^{1/q} \le c\left(\int_X |f(x)|^p \, d\mu\right)^{1/p} \tag{3.8}$$

with a constant $c > 0$ independent of f, it is necessary and sufficient that

$$\nu B \le c_1(\mu B)^{(1/p-\gamma)q} \tag{3.9}$$

with c_1 independent of the ball B.

3.4. Theorem. *Let $1 < p < q < \infty$. If the function $\Omega(x)$ is finite ν-a.e. on X, then the following inequality holds*

$$\left(\int_X (\Omega(x))^{(\gamma-1)q} |T_\gamma f(x)|^q \, d\nu\right)^{1/q} \le c\left(\int_X |f(x)|^p \, d\mu\right)^{1/p}, \tag{3.10}$$

with a constant c independent of f.

Proof : Assume that $0 < \varepsilon, \min(\gamma, 1-\gamma)$. Obviously

$$q_1 = q\frac{1-p\gamma}{1-p(\gamma+\varepsilon)} > q > p.$$

Now choose the number ε such that also the inequality

$$q_2 = q\frac{1-p\gamma}{1-p(\gamma-\varepsilon)} \ge p$$

is fulfilled.

As above we put

$$p_1 = \frac{2q_1}{q} \quad \text{and} \quad p_2 = \frac{2q_2}{q}.$$

So we have

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Using Lemma 3.1 and Hölder's inequality we obtain the estimate

$$\int_X (\Omega(x))^{(\gamma-1)q} |T_\gamma f(x)|^q \, d\nu \le$$

$$\le c_1 \left(\int_X (M_{\gamma-\epsilon} f(x))^{qp_1/2} \, d\nu \right)^{1/p_1} \cdot \left(\int_X (M_{\gamma+\epsilon} f(x))^{qp_2/2} \, d\nu \right)^{1/p_2}.$$

Hence it follows that

$$\left(\int_X (\Omega(x))^{(\gamma-1)q} |T_\gamma f(x)|^q \, d\nu \right)^{1/q} \le$$

$$\le c_1 \sqrt{\|M_{\gamma-\epsilon} f\|_{L^{q_1}(X,\nu)} \|M_{\gamma+\epsilon} f\|_{L^{q_2}(X,\nu)}}.$$

Now, taking into account the choice of the numbers q_1 and q_2, we conclude by Proposition C and the last inequality that (3.10) is valid.

Theorem 3.4 is proved. ∎

From Theorem 3.4 we deduce

3.5. Theorem. *Let* $1 < p < q < \infty$ *and let the condition (3.4) be fulfilled. Then the following two conditions are equivalent:*

(i) T_γ *acts continuously from* $L^p(X, \mu)$ *into* $L^q(X, \nu)$.

(ii) *Inequality (3.9) holds.*

For the classical Riesz potentials, Theorem 3.5 was proved by D.Adams [3].

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