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# On $L_\infty$ – convergence of Rothe's method

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**Dedicated to the memory of Svatopluk Fučík**

*Abstract.* $L_\infty$ – convergence and $L_\infty$ – error estimates are proved for Rothe's method (method of lines or method of semidiscretization) applied to semilinear second order parabolic initial–boundary value problems.

*Keywords:* Parabolic boundary value problems, Rothe's method, $L_\infty$ – error estimates

*Classification:* 65N40, 65N59

**1. Introduction.** In this note we present a simple proof of $L_\infty$ – convergence and $L_\infty$ – error estimates for Rothe's method applied to semilinear second order parabolic equations (systems)

$$\partial_t u + Au = f(t,x,u) \qquad \text{in} \quad \Omega \times (0,T)$$

with linear boundary and initial conditions

$$\begin{aligned} Bu &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) &= u_0. \end{aligned}$$

We consider a corresponding variational formulation in the form

$$(\partial_t u(t), v) + ((u(t), v)) = (f(t, u(t)), v), \quad \forall v \in V \quad \text{a.e. } t \in I \equiv (0,T) \quad \text{with} \quad u(0) = u_0. \tag{1}$$

(see, e.g., [4], [5], [3]) where $V$ is a subspace of the Sobolev space $W_2^1(\Omega), \Omega \subset R^N$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega, (\cdot,\cdot)$ is the scalar product in $L_2(\Omega)$ and $((\cdot,\cdot))$ is a continuous bilinear form on $V \times V$ which corresponds to $A$ and $B$ (see [4]).

$C$ – convergence and $C$ – a priori error estimates for a modified Rothe's approximation have been studied in [2], see also [1]. In [2] a maximum principle have been used and stronger regularity of $u_0, \partial\Omega$ and $A$ have been required than in our concept.

**2. Assumptions.** We assume

$$((u,u)) + K|u|_2^2 \geq C\|u\|^2 \qquad \forall u \in V \tag{2}$$

where $K, C$ are positive constants and $|\cdot|_2, \|\cdot\|$ are the corresponding norms in $L_2, V$, respectively. Moreover, we assume

$$(3) \qquad ((u, u^p)) \geq -C_0|u|_{p+1}^{p+1}, \quad \forall u \in V \cap L_\infty(\Omega), \quad \forall p = 2k+1.$$

By $|u|_{p+1}$ we denote the norm in $L_{p+1}(\Omega)$. The function $f : I \times \Omega \times R \to R$ is continuous and satisfies

$$(4) \quad |f(t,x,s) - f(t',x,s')| \leq L_f(|t-t'|(1+|s|+|s'|) + |s-s'|) \qquad \forall t,t' \in I,\ x \in \Omega,\ s,s' \in R.$$

The only restrictive assumption concerning $u_0$ is: $u_0 \in V \cap L_\infty(\Omega)$ and there exists $z_0 \in L_\infty(\Omega)$ such that

$$(5) \qquad (z_0, v) + ((u_0, v)) = (f(0, u_0), v), \quad \forall v \in V$$

which requires more regularity of $u_0$.

Solving (1) we apply Rothe's method in the form

$$(6) \qquad (\delta u_i, v) + ((u_i, v)) = (f(t_i, u_{i-1}), v) \quad \forall v \in V$$

where $i = 1, \ldots, n$, $h = n^{-1}T$, $t_i = ih$ and $\delta u_i = h^{-1}(u_i - u_{i-1})$. The corresponding Rothe's function $u_n(t)$ is defined by

$$(7) \qquad u_n(t) = u_{i-1} + \delta u_i(t - t_{i-1}), \quad \forall t \in (t_{i-1}, t_i) \equiv I_i, \quad i = 1, \ldots, n.$$

Denote $\|u\|_\infty := \|u\|_{L_\infty(\Omega)}$ and $\|u\|_{\infty,Q} := \|u\|_{L_\infty(Q)}$ where $Q = Q_T = \Omega \times I$.

## 3. The proof of the main result.

Our main result is

**Theorem 1.** *Let the assumptions (2)-(5) be satisfied. Then the estimate*

$$\|u - u_n\|_{\infty,Q} \leq C\left(\frac{1}{n} + \sup_{|\tau| \leq n^{-1}} \|\partial_t u(t+\tau) - \partial_t u(t)\|_{\infty,Q}\right)$$

*takes place where $u$ is the solution of (1) and $u_n$ is the corresponding approximate solution from (6), (7).*

We note that the assumptions (2)–(5) imply $u \in L_\infty(I, V)$, $\partial_t u \in L_\infty(Q)$ – see Remark 10.

First we prove a priori estimates $\|\delta u_i\|_\infty \leq C$, $\|u_i\| \leq C$ uniformly for $n$, $i = 1, \ldots, n$ and then we prove Theorem 1.

**Lemma 1.** *The estimates* $\|\delta u_i\|_\infty \le C$, $\|u_i\| \le C$ *take place uniformly for* $n, i = 1, \dots, n$.

PROOF : First we prove the uniform a priori estimates $\|u_i\|_\infty \le C$, $\forall n$, $i = 1, \dots, n$ under the assumption $u_i \in L_\infty(\Omega)$. The existence of $u_i \in V$ satisfying (6) is a consequence of the Lax – Milgram Lemma. Testing (6) with $v = u_i^p (p = 2k+1)$ we estimate

$$|u_i|_{p+1}^{p+1} \le (u_{i-1}, u_i^p) + C_0 h |u_i|_{p+1}^{p+1} + h\{L_f(|u_{i-1}|, |u_i|^p) +$$
$$+ (|f_i|, |u_i|^p)\} \le (u_{i-1}, u_i^p) + C_0 h |u_i|_{p+1}^{p+1} + h \frac{1}{p+1} |f_i|_{p+1}^{p+1} +$$
$$+ h L_f \left(\frac{1}{p+1} |u_{i-1}|_{p+1}^{p+1} + 2 \frac{p}{p+1} |u_i|_{p+1}^{p+1}\right)$$

where the Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q} (p^{-1} + q^{-1} = 1)$ has been used and $f_i := f(t_i, 0)$. Hence we have

$$|u_i|_{p+1}^{p+1} \le (1 + (L + \varepsilon_n)h)(u_{i-1}, u_i^p) +$$
$$+ (1 + (L + \varepsilon_n)h) \left\{ \frac{h}{p+1} |f_i|_{p+1}^{p+1} + L_f \frac{h}{p+1} |u_{i-1}|_{p+1}^{p+1} \right\},$$

where $L := 2L_f + C_0 + 1$, $\varepsilon_n \to 0$ for $n \to \infty$.
Now we apply Young's inequality to the first term on the right hand side. We obtain

$$|u_i|_{p+1}^{p+1} \le (1 + (L + \varepsilon_n)h)^{p+1} \frac{1}{p+1} |u_{i-1}|_{p+1}^{p+1} + \frac{p}{p+1} |u_i|_{p+1}^{p+1} +$$
$$+ (1 + (L + \varepsilon_n)h) \left\{ \frac{h}{p+1} |f_i|_{p+1}^{p+1} + \frac{h}{p+1} L_f |u_{i-1}|_{p+1}^{p+1} \right\}$$

which implies

$$|u_i|_{p+1}^{p+1} \le 2(1 + (L + \varepsilon_n)h)^{p+1} \left\{ |u_{i-1}|_{p+1}^{p+1} + h |f_i|_{p+1}^{p+1} \right\}.$$

From this recurrent inequality we obtain successively

$$|u_i|_{p+1}^{p+1} \le 2^i (1 + (L + \varepsilon_n)h)^{(p+1)i} \left\{ |u_0|_{p+1}^{p+1} + \sum_{j=1}^{i} |f_i|_{p+1}^{p+1} h \right\}.$$

Taking $(p+1)$–th root and letting $p \to \infty$ we deduce

$$\|u_i\|_\infty \le e^{(L+\varepsilon_n)T} (\|u_0\|_\infty + \|f(t,0)\|_{\infty, Q}) \tag{8}$$

uniformly for $n$, $i = 1, \dots, n$ where $\varepsilon_n = \frac{L^2 T}{n}$ and $n \ge n_0(L_f, C_0)$.

We guarantee the boundedness of $u_i$ by the following arguments. Let us solve (6) by the Galerkin method where $u_{i,\lambda} \in V_\lambda$ and $V_\lambda = \text{span}(e_1, \dots, e_\lambda)$ stand in the place of $u_i, V$, respectively. Here, $\{e_i\}_1^\infty$ are linearly independent, $e_i \in V \cap L_\infty(\Omega)$ and the subspace spanned by these functions is dense in $V$. Then we obtain the estimate (8) with $u_{i,\lambda}$ ($\lambda$ is fixed) in the place of $u_i$. By standard arguments we obtain a priori estimates $|u_{i,\lambda}|_2 \leq C$, $\|u_{i,\lambda}\| \leq C(h)$ where $h$ is fixed, uniformly with respect to $\lambda, i = 1, \dots, n$. Hence $u_{i,\lambda} \to u_i$ in $L_2(\Omega)$ for $\lambda \to \infty, i = 1, \dots, n$. Then we conclude $u_i \in L_\infty(\Omega)$. To prove the a priori estimate $\|\delta u_i\|_\infty \leq C$ we subtract (6) for $i = j$ and $i = j - 1$ and put $v = (\delta u_i)^p$ where $p = 2k + 1$. We obtain

$$(\delta u_i - \delta u_{i_1}, (\delta u_i)^p) + h((\delta u_i, (\delta u_i)^p)) =$$
$$= (f(t_i, u_{i-1}) - f(t_{i-1}, u_{i-2}), (\delta u_i)^p) \leq hL_f(|u_{i-1}| + |u_{i-2}|, |\delta u_i|^p) +$$
$$+ hL_f(|\delta u_{i-1}|, |\delta u_i|^p).$$

Now, estimating $\|\delta u_i\|_\infty$ we proceed analogously as in the case $\|u_i\|_\infty$. Using (8) we successively obtain

$$|\delta u_i|_{p+1}^{p+1} \leq 2(1 + (L + \varepsilon_n)h)^{p+1}(|\delta u_{i-1}|_{p+1}^{p+1} + hL_f(|u_{i-1}|_{p+1}^{p+1} +$$
$$+ |u_{i-2}|_{p+1}^{p+1} + 1) \leq 2(1 + (L + \varepsilon_n)h)^{(p+1)}(|\delta u_{i-1}|_{p+1}^{p+1} +$$
$$+ Ch(\|u_0\|_\infty^{p+1} + \|f(t, 0)\|_{\infty,Q} + 1))$$

where $L := 4L_f + C_0$, $\varepsilon_n = \frac{L^2 T}{n}$, $n \geq n_0(L_f, C_0)$. From this recurrent inequality, analogously as in (8), we conclude (using also (5))

$$\|\delta u_i\|_\infty \leq e^{(L+\varepsilon_n)T}(\|z_0\|_\infty + \|u_0\|_\infty + \|f(t,0)\|_{\infty,Q} + 1) \tag{9}$$

for all $n$, $i = 1, \dots, n$. The estimate $\|u_i\| \leq C$ is a consequence of (8), (9) and (6). Thus the proof of Lemma 1 is complete. ∎

**10 Remark.** As a consequence of (8), (9) and (6) we have $\|u_i\|_{W^2_{2,\text{loc}}} \leq C$ for all $n$, $i = 1, \dots, n$ because of the interior regularity results for elliptic equations. Thus, the unique solution $u$ of (1) satisfies: $u \in L_\infty(I, V) \cap L_\infty(I, W^2_{2,\text{loc}}(\Omega)), \partial_t u \in L_\infty(Q_T)$.

Now let us denote $\widetilde{u}_i = h^{-1} \int_{I_i} u, \overline{u}_i = u(t_i), e_i = \widetilde{u}_i - u_i$, for $i = 1, \dots, n$ where $I_i = (t_{i-1}, t_i)$.

**Proof** of Theorem 1: Let us integrate (1) over $I_i (1 \leq i \leq n)$. We obtain

$$(\delta \overline{u}_i, v) + ((\widetilde{u}_i, v)) = (\widetilde{f}_i, v) \quad \forall v \in V \tag{11}$$

where $\widetilde{f}_i := h^{-1} \int_{I_i} f(t, u)$. Subtracting (11) and (6) for $v = e_i^p$ we obtain

$$(e_i - e_{i-1}, e_i^p) + h((e_i, e_i^p)) =$$
$$= h(z_i, e_i^p) - h(f(t_i, u_{i-1}), e_i^p) + h(\widetilde{f}_i, e_i^p) \tag{12}$$

for $i = 1, \dots, n$ where $p = 2k+1$, $e_0 \equiv 0, u := u_0$ for $t \in (-h, 0)$ and

$$z_i := \delta\widetilde{u}_i - \delta\overline{u}_i = h^{-2}\int_{I_i}(u(s) - u(s-h))\,ds - h^{-1}\int_{I_i}\partial_t u =$$
$$= h^{-1}\int_{I_i}(h^{-1}\int_{s-h}^{s}\partial_t u(\tau)\,d\tau - \partial_t u(s))\,ds.$$

Now we estimate

$$|z_i| \le h^{-2}\int_{I_i}\int_{s-h}^{s}|\partial_t u(s) - \partial_t u(\tau)|\,d\tau\,ds \le \tag{13}$$
$$\le \sup_{|\tau|\le h} h^{-1}\int_{I_i}|\partial_t u(s+\tau) - \partial_t u(s)|\,ds$$

and

$$|\widetilde{f}_i - f(t_i, u_{i-1})| \le |\widetilde{f}_i - f(t, \widetilde{u}_i)| + |f(t, \widetilde{u}_i) - f(t, u_i)| + \tag{14}$$
$$+|f(t, u_i) - f(t_i, u_{i-1})| \le L_f(h^{-2}\int_{I_i}\int_{I_i}|u(s) - u(\tau)|\,d\tau\,ds +$$
$$+|e_i| + h(|\delta u_i| + C)) \le L_f(\int_{I_i}|\partial_t u| + |e_i| + h(|\delta u_i| + C))$$

where $C := \max_{n,i}\|u_i\|_\infty$ - see (8). We proceed in (12) analogously as in the proof of Lemma 1. Using the estimates (13), (14) in (12) we have

$$|e_i|_{p+1}^{p+1} \le (e_{i-1}, e_i^p) + h(C_0 + L_f)|e_i|_{p+1}^{p+1} + 3hL_f\frac{p}{p+1}|e_i|_{p+1}^{p+1} +$$
$$+\frac{1}{p+1}h\int_\Omega \sup_{|\tau|\le h} h^{-1}\int_{I_i}|\partial_t u(s+\tau) - \partial_t u(s)|\,ds)^{p+1}\,dx +$$
$$+\frac{L_f}{p+1}(h^{p+1}|\delta u_i|_{p+1}^{p+1} + h(\int_{I_i}\partial_t u)^{p+1})\,dx + C^{p+1}h^{p+1}.$$

Here, we use the estimates

$$(\sup_{|\tau|\le h} h^{-1}\int_{I_i}|\partial_t u(s+\tau) - \partial_t u(s)|\,ds)^{p+1} \le$$
$$\le h^{-1}\sup_{|\tau|\le h}\int_{I_i}|\partial_t u(s+\tau) - \partial_t u(s)|^{p+1}\,ds,$$
$$(\int_{I_i}\partial_t u)^{p+1} \le h^p\int_{I_i}|\partial_t u|^{p+1}\,ds.$$

Then, analogously as in the proof of Lemma 1 we obtain

$$|e_i|_{p+1}^{p+1} \le 2^i(1 + (L + \varepsilon_n)h)^{(p+1)i}\{|e_0|_{p+1}^{p+1} + \tag{15}$$
$$+h^{p+1}(\int_0^{t_i}\int_\Omega(|\partial_t u_n|^{p+1} + |\partial_t u|^{p+1}) + C^{p+1}) +$$
$$+\sup_{|\tau|\le h}\int_0^{t_i}\int_\Omega|\partial_t u(s+\tau) - \partial_t u(s)|^{p+1}\,dx\,ds$$

where $e_0 \equiv 0$, $L = 4L_f + C_0$, $\varepsilon_n = \frac{L^2T}{n}$, $n \geq n_0(L_f, C_0)$. Then (15) implies

$$\|e_i\|_\infty \leq e^{(L+\varepsilon_n)T}(h(\|\partial_t u_n\|_{\infty,Q_T} + \|\partial_t u\|_{\infty,Q_T} + C) + \\ + \sup_{|\tau|\leq h} \|\partial_t u(s+\tau) - \partial_t u(s)\|_{\infty,Q_T})$$

for $i = 1, \dots, n$. For $t \in I_i$ we estimate

$$\|u - u_n\|_\infty \leq \|u - \overline{u_i}\|_\infty + \|\widetilde{u}_i - \overline{u}_i\|_\infty + \|\widetilde{u}_i - u_i\|_\infty + \\ +2h\|\delta u_i\|_\infty \leq C(2h(\|\partial_t u\|_{\infty,Q_T} + 2\|\delta_t u_n\|_{\infty,Q_T}) + \\ + \sup_{|\tau|\leq h} \|\partial_t u(s+\tau) - \partial_t u(s)\|_{\infty,Q_T})$$

and finally

$$\|u - u_n\|_{\infty,Q_T} \leq C(\frac{1}{n} + \sup_{|\tau|\leq h} \|\partial_t u(s+\tau) - \partial_t u(s)\|_{\infty,Q_T})$$

which is the required estimate. ∎

As a consequence we have

**Theorem 2.** *Suppose (2)-(5). Let $u$ be the solution of (1) and let $u_n$ be the Rothe's function defined by (7).*

i) *If $\partial_t u \in C(I, L_\infty(\Omega))$ then $u_n \to u$ in $L_\infty(Q_T)$;*
ii) *If $\partial_t^2 u \in L_\infty(Q_T)$ then $\|u_n - u\|_{\infty,Q_T} = \mathcal{O}(\frac{1}{n})$.*

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