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Optimal control in coefficients for elliptic variational inequalities and optimality conditions

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Dedicated to the memory of Svatopluk Fučík

Abstract. This paper concerns an optimal control problem for an elliptic variational inequality with controls appearing in coefficients, right hand sides and convex sets of states as well. The existence of an optimal control is verified and necessary optimality conditions are derived. The application to the optimal design of an elastic plate with an obstacle is presented.

Keywords: Optimal control problem, elliptic variational inequality, convex set, elastic plate, thickness–function

Classification: 49A29, 49A27,49B34

We shall deal with an optimal control problem for an elliptic variational inequality with controls appearing in coefficients, right hand sides and convex sets of states as well. To the results presented in [2] we add here the necessary conditions of optimality in the generalized form. We use the method of penalization in a similar way as it has been done in [1]. A special type of the convergence of sequences of sets and functionals introduced by Mosco plays an important role in our considerations.

1. On the convergence of sets and functionals. Let V be a normed linear space. Following Mosco ([5]), we introduce the convergence of sequences of subsets of V.

Definition 1.1. A sequence $\{K_n\}$ of subsets of the space V converges to a set $K \subset V$, if

i) K contains all weak limits of sequences $\{u_k\}, u_k \in K_{n_k}$, where $\{K_{n_k}\}$ are arbitrary subsequences of $\{K_n\}$;

ii) every element $v \in K$ is the strong limit of some sequence $\{v_n\}, v_n \in K_n$.

Notation: $K = \operatorname*{Lim}_{n\to\infty} K_n$.

Let $j : V \to (-\infty, \infty]$ be a functional. The set

$$\text{epi } j := \{(v,\beta) \in V \times R : j(v) \le \beta\}$$

is called the epigraph of j.

Definition 1.2. A sequence $\{j_n\}$ of functionals from V into $(-\infty,\infty]$ converges to $j : V \to (-\infty,\infty]$ in V, if epi $j = \operatorname*{Lim}_{n\to\infty}$ epi j_n. Notation: $j = \operatorname*{Lim}_{n\to\infty} j_n$

Let us recall the following lemma of Mosco on the convergence of functionals in V.

Lemma 1.1. *Let $j_n : V \to (-\infty, \infty]$, $n = 1, 2, \ldots$ Then $j = \operatorname{Lim}_{n\to\infty} j_n$ if and only if the following conditions hold:*

i) *For every $v \in V$ there exists a sequence $\{v_n\} \subset V$ such that $\lim_{n\to\infty} v_n = v$ (strongly) in V and $\limsup_{n\to\infty} j_n \leq j(v)$;*

ii) *for every subsequence $\{j_{n_k}\}$ of $\{j_n\}$ and every sequence $\{v_k\} \subset V$ weakly convergent to $v \in V$ holds*

$$j(v) \leq \liminf_{k\to\infty} j_{n_k}(v_k).$$

Remark 1.1. Due to the previous lemma the condition $j = \operatorname{Lim}_{n\to\infty} j_n$ implies that for every $v \in V$ there exists a sequence $\{v_n\} \subset V$ such that $\lim_{n\to\infty} v_n = v$ (strongly) in V and $\lim_{n\to\infty} j_n(v_n) = j(v)$.

2. Optimal control problem for a variational inequality. Let U be a reflexive Banach space of controls with a norm $\|\cdot\|_U$. Let $U_{ad} \subset U$ be a set of admissible controls compact in U. Further, denote as V, a real Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$, V^* its dual space with a norm $\|\cdot\|_*$ and with the duality pairing $\langle\cdot,\cdot\rangle$. We introduce the systems $\{K(e)\}$, $\{A(e)\}$ of convex closed subsets $K(e) \subset V$ and linear bounded operators $A(e) \in L(V, V^*)$, $e \in U_{ad}$, satisfying the following assumptions:

$$\bigcap_{e\in U_{ad}} K(e) \neq \emptyset, \tag{2.1}$$

$$e_n \to e \text{ in } U \Rightarrow K(e) = \operatorname{Lim}_{n\to\infty} K(e_n), \tag{2.2}$$

$$\|A(e)\|_{L(V,V^*)} \leq c_1 \text{ for all } e \in U_{ad}, \tag{2.3}$$

$$\langle A(e)v, v\rangle \geq \alpha\|v\|^2, \alpha > 0; \quad \text{for all } e \in U_{ad} \text{ and } v \in V, \tag{2.4}$$

$$e_n \to e \text{ in } U \Rightarrow A(e_n) \to A(e) \text{ in } L(V, V^*). \tag{2.5}$$

Let $B \in L(U, V^*)$, $f \in V^*$. It is well known ([3]) that there exists for every $e \in U_{ad}$ a unique solution

$$u(e) \in K(e) \tag{2.6}$$

of the elliptic variational inequality

$$\langle A(e)u(e), v - u(e)\rangle \geq \langle f + B(e), v - u(e)\rangle \quad \text{for all} \quad v \in K(e). \tag{2.7}$$

Further, consider the functional $J : U \times V \to R$ for which the following condition holds:

$$\begin{cases} e_n \to e \text{ in } U,\ u_n \rightharpoonup u \text{ in } V \text{ (weakly)} \Rightarrow \\ \Rightarrow J(e, u) \leq \lim_{n\to\infty} J(e_n, u_n). \end{cases} \tag{2.8}$$

We shall formulate the optimal control problem in the following way:

Problem $\mathbf{P}_0$. Find a control $e_0 \in U_{ad}$ such that

$$\langle A(e_0)u(e_0), v - u(e_0)\rangle \geq \langle f + B(e_0), v - u(e_0)\rangle \quad \text{for all } v \in K(e_0), \tag{2.9}$$

$$J(e_0, u(e_0)) = \min_{e \in U_{ad}} J(e, u(e)). \tag{2.10}$$

In a similar way as in [2], the following existence theorem can be verified:

Theorem 2.1. *Let the assumptions (2.1) - (2.5), (2.8) be satisfied. Then there exists at least one solution e_0 of the Optimal control problem P_0.*

PROOF : As the solution $u(e)$ of the variational inequality (2.7) is uniquely determined for every $e \in U_{ad}$, we can introduce the functional j as

$$j(e) = J(e, u(e)), \qquad e \in U_{ad}. \tag{2.11}$$

Due to the compactness of U_{ad} in U, there exists a sequence $\{e_n\} \subset U_{ad}$ such that

$$\lim_{n\to\infty} j(e_n) = \inf_{e \in U_{ad}} j(e), \tag{2.12}$$

$$\lim_{n\to\infty} e_n = e_0 \quad \text{in} \quad U, \quad e_0 \in U_{ad}. \tag{2.13}$$

Denoting $u(e_n) := u_n \in K(e_n)$ we obtain the inequality

$$\langle A(e_n)u_n, v - u_n\rangle \geq \langle f + B(e_n), v - u_n\rangle \quad \text{for all} \quad v \in K(e_n). \tag{2.14}$$

Inserting $v = v_0 \in \bigcap_{e \in U_{ad}} K(e)$ in (2.14) we arrive at

$$\langle A(e_n)u_n, u_n\rangle \leq \langle A(e_n)u_n, v_0\rangle + \langle f + B(e_n), u_n - v_0\rangle, \qquad n = 1, 2, \dots \tag{2.15}$$

The assumptions (2.3), (2.4) and the continuity of B imply

$$\|u_n\| \leq C, \quad n = 1, 2, \dots \tag{2.16}$$

It follows the existence of a subsequence (again denoted by $\{u_n\}$) and the element $u_0 \in V$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } V. \tag{2.17}$$

As $u_n \in K(e_n)$, the assumption (2.2) implies

$$u_0 \in K(e_0). \tag{2.18}$$

Using (2.5), (2.13), (2.16), (2.17) we obtain

$$A(e_n)u_n \rightharpoonup A(e_0)u_0 \quad \text{in } V^*. \tag{2.19}$$

Let $v \in K(e_0)$ be an arbitrary element and $\{v_n\}$ be a sequence for which

$$v_n \to v \quad \text{in } V,\ v_n \in K(e_n), \quad n = 1, 2, \ldots \tag{2.20}$$

Using (2.13), (2.14), (2.18), (2.19) we obtain the inequality

$$\langle A(e_0), v - u_0 \rangle \geq \langle f + B(e_0), v - u_0 \rangle.$$

As the element $v \in K(e_0)$ is chosen arbitrarily we get

$$u_0 \equiv u(e_0) \tag{2.21}$$

and

$$u(e_n) \rightharpoonup u(e_0) \quad \text{in } V. \tag{2.22}$$

Then (2.8), (2.12) yield

$$J(e_0, u(e_0)) \leq \liminf_{n \to \infty} J(e_n, u(e_n)) = \inf_{e \in U_{ad}} J(e, u(e)).$$

Hence

$$J(e_0, u(e_0)) = \inf_{e \in U_{ad}} J(e, u(e)),$$

which completes the proof. ∎

3. Necessary optimality conditions. We proceed now to obtain optimality conditions for the Optimal control problem P_0. A similar problem was solved in [4], where the convex set K of admissible states did not depend on a control parameter e. We can formulate the state inequality (2.7) in the form

$$\begin{aligned} &\langle A(e)u(e), v - u(e) \rangle + \Phi(e, v) - \Phi(e, u(e)) \geq \\ &\geq \langle f + B(e), v - u(e) \rangle \quad \text{for all} \quad e \in U_{ad}, \quad v \in V \end{aligned} \tag{3.1}$$

where

$$\Phi(e, v) = \begin{cases} 0, & \text{if } v \in K(e), \\ +\infty, & \text{if } v \notin K(e). \end{cases} \tag{3.2}$$

Rewrite now the Optimal control problem (2.9), (2.10) in the following way:

Problem P. Find a couple $[e_0, u_0] \in W$ such that

$$J(e_0, u_0) = \min_{[e,u] \in W} J(e, u) \tag{3.3}$$

with

$$W = \{[e, u] \in U_{ad} \times V : u \equiv u(e) \quad \text{from (3.1)}\}. \tag{3.4}$$

The existence Theorem 2.1 can be rewritten as

Theorem 3.1. *There exists at least one solution* $[e_0, u_0]$ *of the Problem P.*

In order to derive the optimality condition for (3.3), (3.4), we regularize the functional Φ and change the cost functional J. For each $\varepsilon > 0$ we consider the approximated

Problem P$_\varepsilon$. Find $[e_\varepsilon, u_\varepsilon] \in W_\varepsilon$ such that

$$J(e_\varepsilon, u_\varepsilon) + \frac{1}{2}\|e_\varepsilon - e_0\|_U^2 = \min_{[e,u]\in W_\varepsilon} [J(e,u) + \frac{1}{2}\|e - e_0\|_U^2], \tag{3.5}$$

where

$$W_\varepsilon = \{[e,u] \in U_{ad} \times V : A(e)u + \frac{\partial}{\partial u}\Phi^\varepsilon(e,u) = f + B(e)\},$$

$\Phi^\varepsilon(e,\cdot) : V \to R$ is a convex Fréchet differentiable functional for every $e \in U_{ad}$.

We assume that there exists a family of such functionals with the following properties:

$$\Phi^\varepsilon(e,v) \geq -c(\|v\| + 1) \quad \text{for all } \varepsilon > 0,\ e \in U_{ad},\quad v \in V, \tag{3.6}$$

$$\lim_{\varepsilon\to 0} \Phi^\varepsilon(e,v) = \Phi(e,v) \quad \text{for all } e \in U_{ad},\quad v \in V, \tag{3.7}$$

$$e_n \to e \quad \text{in } U \Rightarrow \Phi^\varepsilon(e,\cdot) = \operatorname*{Lim}_{n\to\infty} \Phi^\varepsilon(e_n,\cdot), \tag{3.8}$$

$$e_n \to e \quad \text{in } U,\ \varepsilon_n \to 0 \Rightarrow \Phi(e,\cdot) = \operatorname*{Lim}_{n\to\infty} \Phi^{\varepsilon_n}(e_n,\cdot), \tag{3.9}$$

$$\|\frac{\partial}{\partial u}\Phi^\varepsilon(e,u_1) - \frac{\partial}{\partial u}\Phi^\varepsilon(e,u_2)\|_* \leq M_1(\varepsilon)\|u_1 - u_2\| \quad \text{for all } \varepsilon > 0,\ e \in U_{ad};\quad u_1, u_2 \in V, \tag{3.10}$$

$$\|\frac{\partial}{\partial u}\Phi^\varepsilon(e,v_0)\|_* \leq M_2 \quad \text{for any } v_0 \quad \text{and all } e \in U_{ad}, \varepsilon > 0. \tag{3.11}$$

As the functionals $\Phi^\varepsilon(e,\cdot)$ are convex, the operators $\frac{\partial}{\partial u}\Phi^\varepsilon(e,\cdot) : V \to V^*$ are monotone, i.e.

$$\langle\frac{\partial}{\partial u}\Phi^\varepsilon(e,u_1) - \frac{\partial}{\partial u}\Phi^\varepsilon(e,u_2), u_1 - u_2\rangle \geq 0 \quad \text{for all } e \in U_{ad},\quad u_1, u_2 \in V. \tag{3.12}$$

The theory of monotone operators yields the following result (see e.g. [3]):

Theorem 3.2. *For every* $\varepsilon > 0$, $e \in U_{ad}$ *there exists a unique solution* $u_\varepsilon(e) \in V$ *of the equation*

$$A(e)u_\varepsilon(e) + \frac{\partial}{\partial u}\Phi^\varepsilon(e, u_\varepsilon(e)) = f + B(e) \tag{3.13}$$

which is equivalent to the variational inequality

$$\langle A(e)u_\varepsilon(e), v - u_\varepsilon(e)\rangle + \Phi^\varepsilon(e,v) - \Phi^\varepsilon(e,u_\varepsilon(e)) \geq \geq \langle f + B(e), v - u_\varepsilon(e)\rangle \qquad \textit{for every } v \in V. \tag{3.13'}$$

Let us return now to the above formulated Optimal control problem P_ε.

Theorem 3.3. *For every $\varepsilon > 0$ there exists at least one optimal pair $[e_\varepsilon, u_\varepsilon]$ for the Problem P_ε.*

PROOF : We shall use a similar approach as in the proof of Theorem 2.1. Using the assumptions (3.10), (3.11), (3.12) we obtain the sequence of pairs $[e_n^\varepsilon, u_n^\varepsilon] \in U_{ad} \times V$ and a pair $[e_\varepsilon, u_\varepsilon] \in U_{ad} \times V$ such that

$$\lim_{n\to\infty} [J(e_n^\varepsilon, u_n^\varepsilon) + \frac{1}{2}\|e_n^\varepsilon - e_0\|_U^2] = \inf_{[e,u]\in W_\varepsilon} [J(e,u) + \frac{1}{2}\|e - e_0\|_U^2], \tag{3.14}$$

$$e_n^\varepsilon \to e_\varepsilon \quad \text{in} \quad U, \tag{3.15}$$

$$u_n^\varepsilon \rightharpoonup u_\varepsilon \quad \text{in} \quad V. \tag{3.16}$$

Each pair $[e_n^\varepsilon, u_n^\varepsilon]$ satisfies the variational inequality

$$\begin{aligned} \langle A(e_n^\varepsilon)u_n, v - u_n^\varepsilon\rangle + \Phi^\varepsilon(e_n^\varepsilon, v) - \Phi^\varepsilon(e_n^\varepsilon, u_n^\varepsilon) \geq \\ \geq \langle f + B(e_n^\varepsilon), v - u_n^\varepsilon\rangle \quad \text{for every } v \in V. \end{aligned} \tag{3.17}$$

The assumption (3.8) implies

$$\Phi^\varepsilon(e_\varepsilon, \cdot) = \operatorname*{Lim}_{n\to\infty} \Phi^\varepsilon(e_n^\varepsilon, \cdot) \tag{3.18}$$

and due to Lemma 1.1

$$\Phi^\varepsilon(e_\varepsilon, u_\varepsilon \leq \lim_{n\to\infty} \inf \Phi^\varepsilon(e_n^\varepsilon, u_n^\varepsilon). \tag{3.19}$$

Further, for every $v \in V$ there exists a sequence $\{v_n\} \subset V$ such that

$$v_n \to v \quad \text{in } V, \tag{3.20}$$

$$\lim_{n\to\infty} \sup \Phi^\varepsilon(e_n^\varepsilon, v_n) \leq \Phi^\varepsilon(e_\varepsilon, v). \tag{3.21}$$

The inequality (3.17) then implies

$$\begin{aligned} \langle A(e_\varepsilon)u_\varepsilon, v - u_\varepsilon\rangle + \Phi^\varepsilon(e_\varepsilon, v) - \Phi^\varepsilon(e_\varepsilon, u_\varepsilon) \geq \\ \geq \langle f + B(e_\varepsilon, v - u_\varepsilon\rangle \quad \text{for every } v \in V. \end{aligned} \tag{3.22}$$

Hence, $u \equiv u_\varepsilon(e_\varepsilon)$ and we get the assertion of Theorem 3.3 using the assumption (2.8) and a weak lower semicontinuity of the norm $\|\cdot\|_U^2$. ∎

Theorem 3.4. *Let $\{[e_{\varepsilon_n}, u_{\varepsilon_n}]\}$, $\varepsilon_n \to 0$, be a sequence of solutions (optimal pairs) to the Problem P_{ε_n}. Then there exists a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon_n\}$ such that*

$$e_{\varepsilon_k} \to e_0 \quad \textit{in } U, \tag{3.23}$$

$$u_{\varepsilon_k} \to u_0 \equiv u(e_0) \quad \textit{in } V, \tag{3.24}$$

where $[e_0, u_0]$ *is the solution of the Optimal control problem P.*

PROOF : The estimate

$$J(e_{\varepsilon_n}, u_{\varepsilon_n}) + \frac{1}{2}\|e_{\varepsilon_n} - e_0\|_U^2 \leq J(e_0, u_0^{\varepsilon_n}), \quad n = 1, 2, \ldots \tag{3.25}$$

takes place, where $u_0^{\varepsilon_n} \equiv u_{\varepsilon_n}(e_0)$ is a solution of the state equation (3.13) or, equivalently, of the inequality (3.13') with $e \equiv e_0$.

Let z be an arbitrary element of $K(e_0)$. The inequality (3.13') implies

$$\begin{aligned} \langle A(e_0)u_0^n, u_0^n - z\rangle + \Phi^{\varepsilon_n}(e_0, u_0^{\varepsilon_n}) - \Phi^{\varepsilon_n}(e_0, z) \geq \\ \geq \langle f + B(e_0), u_0^{\varepsilon_n} - z\rangle. \end{aligned} \tag{3.26}$$

Using the assumptions (2.4), (3.6), (3.7) and the form of $\Phi(e_0, \cdot)$ we obtain the boundedness of the sequence $\{u_0^{\varepsilon_n}\}$ in V. Then there exist $u_0 \in V$ and a subsequence $u_0^{\varepsilon_k}$ such that

$$u_0^{\varepsilon_k} \rightharpoonup u_0 \quad \text{in } V. \tag{3.27}$$

Since the function $v \to \langle A(e_0)v, v\rangle$ is convex and continuous on V, it is weak lower continuous and

$$\langle A(e_0)u_0, u_0\rangle \leq \lim_{k\to\infty} \inf \langle A(e_0)u_0^{\varepsilon_k}, u_0^{\varepsilon_k}\rangle.$$

The last inequality together with (3.1), (3.7), (3.8), (3.26), (3.27) yields

$$\begin{aligned} \langle A(e_0)u_0, u_0 - v\rangle + \Phi(e_0, u_0) \leq \Phi(e_0, v) + \langle f + B(e_0), u_0 - v\rangle \\ \text{for all } v \in V. \end{aligned}$$

Hence $u_0 \equiv u(e_0)$ is a solution of (3.13) and $[e_0, u_0]$ is an optimal pair for the Optimal control problem P.

It remains to verify (3.23), (3.24). We have the subsequence of $\{\varepsilon_k\}$ (denoted again by $\{\varepsilon_k\}$) and the element $e \in U_{ad}$ such that

$$e_{\varepsilon_k} \to e \quad \text{in } U. \tag{3.28}$$

Corresponding states $u_{\varepsilon_k} \equiv u^{\varepsilon_k}(e_{\varepsilon_k})$ satisfy the inequality

$$\begin{aligned} \langle A(e_{\varepsilon_k})u_{\varepsilon_k}, u_{\varepsilon_k} - v\rangle + \Phi^{\varepsilon_k}(e_{\varepsilon_k}, u_{\varepsilon_k}) - \Phi^{\varepsilon_k}(e_{\varepsilon_k}, v) \leq \\ \leq \langle f + B(e_{\varepsilon_k}), u_{\varepsilon_k} - v\rangle \quad \text{for all } v \in V. \end{aligned} \tag{3.29}$$

The assumptions (2.4), (3.6), (3.7) imply the boundedness of the sequence $\{u_{\varepsilon_k}\}$. Hence there exists a subsequence of $\{\varepsilon_k\}$ (denoted again by $\{\varepsilon_k\}$)such that

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } V. \tag{3.30}$$

Using (2.5), (3.9), (3.28), (3.29) and weak lower semicontinuity of $v \to \langle A(e)v, v\rangle$ we obtain

$$\langle A(e)u, u-v\rangle + \Phi(e,u) - \Phi(e,v) \le \langle f + B(e), u-v\rangle \quad \text{for all } v \in V. \tag{3.31}$$

Thus $u \equiv u(e)$ is a solution of the inequality (3.1). Simultaneously

$$J(e,u) \le \lim_{k\to\infty} \inf J(e_{\varepsilon_k}, u_{\varepsilon_k})$$

and

$$\frac{1}{2}\lim_{k\to\infty} \sup \|e_{\varepsilon_k} - e_0\|_U^2 \le \lim_{k\to\infty} \sup[J(e_{\varepsilon_k}, u_{\varepsilon_k}) + \frac{1}{2}\|e_{\varepsilon_k} - e_0\|_U^2] + \\ + \lim_{k\to\infty} \sup[-J(e_{\varepsilon_k}, u_{\varepsilon_k})] \le J(e_0, u_0) - J(e,u) \le 0,$$

which implies

$$\lim_{k\to\infty} \|e_{\varepsilon_k} - e_0\|_U^2 = 0. \tag{3.32}$$

Comparing with (3.28), (3.30), we see that $e = e_0, u(e) = u_0$ and

$$u_{\varepsilon_k} \rightharpoonup u_0 \quad \text{in } V. \tag{3.33}$$

Further, we shall verify the strong convergence of the sequence $\{u_{\varepsilon_k}\}$. Using the uniform coerciveness of the operators $\{A(e)\}$, we obtain the inequality

$$\alpha\|u_{\varepsilon_k} - u_0\|^2 \le \langle A(e_0)u_{\varepsilon_k} - A(e_{\varepsilon_k})u_{\varepsilon_k}, u_{\varepsilon_k} - u_0\rangle + \\ + \langle A(e_{\varepsilon_k})u_{\varepsilon_k}, u_{\varepsilon_k} - u_0\rangle + \langle A(e_0)u_0, u_{\varepsilon_k} - u_0\rangle. \tag{3.34}$$

Due to (2.5), (3.33) the strong convergence will follow after the relation

$$\lim_{k\to\infty} \langle A(e_{\varepsilon_k})u_{\varepsilon_k}, u_{\varepsilon_k} - u_0\rangle = 0 \tag{3.35}$$

is proved.

From the assumption (3.9) and Remark 1.1 follows the existence of a sequence $\{v_k\}$ such that

$$v_k \to u_0 \quad \text{in } V \tag{3.36}$$

and

$$\lim_{k\to\infty} \Phi^{\varepsilon_k}(e_{\varepsilon_k}, v_k) = \Phi(e_0, u_0) \tag{3.37}$$

Monotonicity of $A(e_{\varepsilon_k})$ together with the inequality (3.22) imply

$$(3.38)\qquad \begin{aligned}&\langle A(e_{\varepsilon_k})v_k, u_{\varepsilon_k} - v_k\rangle \le \langle A(e_{\varepsilon_k})u_{\varepsilon_k}, u_{\varepsilon_k} - v_k\rangle \le \\ &\le \langle f + B(e_{\varepsilon_k}), u_{\varepsilon_k} - v_k\rangle + \Phi^{\varepsilon_k}(e_{\varepsilon_k}, v_k) - \Phi^{\varepsilon_k}(e_{\varepsilon_k}, u_{\varepsilon_k}).\end{aligned}$$

According to (2.5), (3.28), (3.33), (3.36), we have the relations

$$(3.39)\qquad \lim_{k\to\infty} \langle A(e_{\varepsilon_k})v_k, u_{\varepsilon_k} - v_k\rangle = 0,$$

$$(3.40)\qquad \lim_{k\to\infty} \langle f + B(e_{\varepsilon_k}), u_{\varepsilon_k} - v_k\rangle = 0.$$

Further, the condition (3.9) implies

$$\Phi(e_0, u_0) \le \lim_{k\to\infty} \inf \Phi^{\varepsilon_k}(e_{\varepsilon_k}, u_{\varepsilon_k}).$$

This, together with (3.37) - (3.40), implies the relation (3.35). So, the proof of our theorem is complete. ■

In order to derive optimality conditions for the approximating Optimal control P_ε, we add some differentiability assumptions. We assume that there exist Fréchet derivatives

$$\begin{gathered}A'(e) \in L(U, L(V, V^*)),\ \frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi^\varepsilon(e,u) \in L(U, V^*),\\ \frac{\partial^2}{\partial u^2}\Phi^\varepsilon(e,u) \in L(V, V^*),\ \frac{\partial}{\partial e}J(e,u) \in U^*,\ \frac{\partial}{\partial u}J(e,u) \in V^*\\ \text{for all } [e,u] \in U \times V.\end{gathered}$$

Moreover, we assume that U is the HIlbert space with the inner product $(\cdot,\cdot)_U$.

Theorem 3.5. *Let $[e_\varepsilon, u_\varepsilon]$ be an optimal pair for the Problem P_ε. Then there exists $p_\varepsilon \in V$ satisfying the system*

$$(3.41)\qquad \begin{aligned}&i)\quad A(e_\varepsilon)u_\varepsilon + \frac{\partial}{\partial u}\Phi^\varepsilon(e_\varepsilon, u_\varepsilon) = f + B(e_\varepsilon),\\ &ii)\quad A(e_\varepsilon)p_\varepsilon + \frac{\partial^2}{\partial u^2}\Phi^\varepsilon(e_\varepsilon, u_\varepsilon)p_\varepsilon = \frac{\partial J}{\partial u}(e_\varepsilon, u_\varepsilon),\\ &iii)\quad \langle B^* p_\varepsilon + \frac{\partial J}{\partial e}(e_\varepsilon, u_\varepsilon), e - e_\varepsilon\rangle_U + (e_\varepsilon - e_0, e - e_\varepsilon)_U \ge\\ &\qquad \ge \langle [A'(e_\varepsilon)(e - e_\varepsilon)]u_\varepsilon + [\frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi^\varepsilon(e_\varepsilon, u_\varepsilon)](e - e_\varepsilon), p_\varepsilon\rangle\\ &\qquad \textit{for all } e \in U_{ad}.\end{aligned}$$

Proof : The map $u_\varepsilon(\cdot) : U \to V$ defined by the equation (3.13) is Fréchet differentiable due to the differentiability assumptions written above. The derivative $u'_\varepsilon(e_\varepsilon) \in L(U, V)$ solves the equation

$$(3.42)\qquad \begin{aligned}&[A(e_\varepsilon) + \frac{\partial^2}{\partial u^2}\Phi^\varepsilon(e_\varepsilon, u_\varepsilon)][u'_\varepsilon(e_\varepsilon)h] + [A'(e_\varepsilon)h]u_\varepsilon +\\ &+[\frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi^\varepsilon(e_\varepsilon, u_\varepsilon)]h = Bh \quad \text{for every } h \in U.\end{aligned}$$

The condition (3.5) implies the variational inequality

$$(3.43)\qquad \begin{aligned}&\langle\frac{\partial}{\partial e}J(e_\varepsilon,u_\varepsilon),e-e_\varepsilon\rangle_U+\langle\frac{\partial}{\partial u}J(e_\varepsilon,u_\varepsilon),u'_\varepsilon(e)(e-e_0)\rangle_V+\\ &+(e_\varepsilon-e_0,e-e_\varepsilon)_U\geq 0\quad\text{for every } e\in U_{ad}.\end{aligned}$$

The operator $\frac{\partial^2}{\partial u^2}\Phi^\varepsilon(e_\varepsilon,u_\varepsilon)\in L(V,V^*)$ is symmetric and positive, because of convexity of the function $\Phi^\varepsilon(e_\varepsilon,\cdot)$. Then there exists a unique solution (adjoint state) $p_\varepsilon\in V$ of the equation (3.41 ii). Using the symmetry of $A(e_\varepsilon)$ we obtain from (3.42), (3.43) the inequality (3.41 iii). This completes the proof. ∎

Remark 3.1. It can be verified in the same way as above that the set $\{p_\varepsilon\}$ is bounded in V. Therefore, we may conclude that there exist an element $p_0\in V$ and a sequence $\{\varepsilon_k\},\varepsilon_k>0$ such that

$$p_{\varepsilon_k}\rightharpoonup p_0\quad\text{in } V.$$

Simultaneously, $u_{\varepsilon_k}\rightarrow u_0$ in V, $e_{\varepsilon_k}\rightarrow e_0$ in U and we can formulate the generalized first order order necessary optimality conditions as the limits of the conditions (3.41):

$$\begin{aligned}
&i)\quad A(e_0)u_0+\frac{\partial}{\partial u}\Phi(e_0,u_0)\ni f+B(e_0),\\
&ii)\quad A(e_0)p_0+\frac{\partial^2}{\partial u^2}\Phi(e_0,u_0)p_0=\frac{\partial J}{\partial u}(e_0,u_0),\\
&iii)\quad \langle B^*p_0+\frac{\partial J}{\partial e}(e_0,u_0),e-e_0\rangle\geq\\
&\qquad\geq\langle[A'(e_0)(e-e_0)]u_0+[\frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi(e_0,u_0)](e-e_0),p_0\rangle\\
&\qquad\text{for all } e\in U_{ad}.
\end{aligned}$$

The elements $\frac{\partial^2}{\partial u^2}\Phi(e_0,u_0)p_0$ and $[\frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi(e_0,u_0)](e-e_0)$ are considered as the weak limits in V^* of the sequences

$$\frac{\partial^2}{\partial u^2}\Phi^{\varepsilon_k}(e_{\varepsilon_k},u_{\varepsilon_k})p_{\varepsilon_k}\quad\text{and}\quad[\frac{\partial}{\partial e}\frac{\partial}{\partial u}\Phi^{\varepsilon_k}(e_{\varepsilon_k},u_{\varepsilon_k})](e-e_{\varepsilon_k}).$$

4. Optimal design of a plate with an obstacle. The previous theory can be applied to the optimal design of an elastic plate with respect to its variable thickness. We assume a thin plate whose middle surface is a bounded region $\Omega\subset R^2$ with a Lipschitz boundary $\partial\Omega$. We put $U=H^2(\Omega)$ and

$$\begin{aligned}U_{ad}=\{&e\in H^3(\Omega):0<e_{\min}\leq e(x)\leq e_{\max}\quad\text{for all } x\in\Omega,\\ &\|e\|_3\leq C_1,\iint_\Omega e(x)\,dx=C_2,\ e|_{\partial\Omega}=\varphi_0,\ \frac{\partial e}{\partial n}|_{\partial\Omega}=\varphi_1\}.\end{aligned}$$

The plate is assumed to be clamped on the boundary. Put $V = H_0^2(\Omega)$. The set of possible deflection (admissible states) of the plate is

$$K(e) = \{v \in H_0^2(\Omega) : v(x) \geq g(x) + \frac{1}{2}e(x) \quad \text{for all } x \in \Omega\},$$

where the function $g : \overline{\Omega} \to R$ represents the obstacle for the deflection of the plate and satisfies the conditions

$$g \in C(\overline{\Omega}),\ g(s) < -\frac{1}{2}\varphi_0(s) \quad \text{for all } s \in \partial\Omega.$$

Then $\{K(e)\}$, $e \in U_{ad}$, is a family of nonempty closed convex subsets of V satisfying the assumptions (2.1), (2.2).

The system of operators $A(e) \in L(V, V^*)$ is defined by

$$\langle A(e)u, v\rangle = \frac{E}{12(1-\sigma^2)} \iint_\Omega e^3(x)[u_{11}v_{11} + \sigma(u_{11}v_{22} + u_{22}v_{11}) + \\ + 2(1-\sigma)u_{12}v_{12} + u_{22}v_{22}]\,dx$$

where

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, i, j = 1, 2; \quad E > 0, 0 < \sigma < 1.$$

For the perpendicular load we take the functional

$$\langle f, v\rangle = \sum_{j=1}^{N} P_j v(X_j) + \iint_\Omega f_0 v\,dx, \quad v \in V,$$

where

$$P_j \in R, X_j \in \Omega, j = 1, \ldots, N; \quad f_0 \in L^2(\Omega).$$

We can include the own weight of the plate represented by the operator $B : U \to V^*$:

$$\langle Be, v\rangle = k \iint_\Omega e(x)v(x)\,dx, \quad e \in U_{ad}, \quad v \in V.$$

Regular functionals Φ^ε approximating the indicatrix function ϕ from (3.2) can have the form

$$\Phi^\varepsilon(e, v) = \frac{1}{\varepsilon} \iint_\Omega \varphi_e(x, v(x))\,dx, \quad v \in V, \quad e \in U_{ad},$$

where $\varphi_e : \Omega \times R \to R$ is defined by

$$\varphi_e(x,t) = \begin{cases} 3[t - g(x) - \frac{e(x)}{2}]^2 + 3[t - g(x) - \frac{e(x)}{2}] + 1 \\ \qquad \text{for } t \leq g(x) + \frac{e(x)}{2} - 1, \\ -[t - g(x) - \frac{e(x)}{2}]^3 \\ \qquad \text{for } g(x) + \frac{e(x)}{2} - 1 \leq t \leq g(x) + \frac{e(x)}{2}, \\ 0 \quad \text{for } t \geq g(x) + \frac{e(x)}{2}. \end{cases}$$

Cost functionals in the optimal design problem can have either the form

$$J_1(e,v) = \|Tv - z_d\|_H^2 + N\|e\|_U^2, \quad N \geq 0,$$

where H is a Hilbert space, $T \in L(V,H)$, or

$$J_2(e,v) = \iint_\Omega e^2(x) S[v,v]\, dx,$$

where

$$S[v,v] = (v_{11}^2 + v_{22}^2)(1 - \sigma + \sigma^2) + v_{11}v_{22}(-1 + 4\sigma - \sigma^2) + 3(1-\sigma)^2 v_{12},$$

which corresponds to the minimization of the intensity of the shear stress at the extreme fibers of the plate.

Similar optimal design problems can be formulated for beams or shells (see for inst. [2], [4]).

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