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An extension of the Borel lemma

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Abstract. The Borel lemma is shown to hold true with the independence assumption replaced by a slightly weaker one.

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The Borel lemma (BL) for independent events $D_n, n \geq 1$, states that

$$(1) \qquad [\sum_1^\infty P(D_n) = \infty] \Rightarrow [P(\sum_1^\infty 1_{D_n} = \infty) = 1],$$

where 1_D denotes the indicator of D. We shall show that the implication (1) remains true, if the independence assumption is weakened in a specific way.

Extended Borel lemma. *Let $D_n = A_n B_n, n \geq 1$, where*

(i) *$A_n, n \geq 1$, are independent events,*
(ii) *$B_n, n \geq 1$, are events such that $\lim_{n\to\infty} P(B_n|A_n) = 1$.*

Then the assertion (1) holds.

Remark 1. With $B_n = \Omega, n \geq 1$, the extended BL reduces to the usual one.

Remark 2. Equivalently, the extended BL can be formulated as follows: Let $A_n, n \geq 1$, be independent events, $\sum P(A_n) = \infty$. Let $D_n \subset A_n, n \geq 1$, be events such that $P(D_n) \sim P(A_n)$. Then $P(\sum 1_{D_n} = \infty) = 1$. (Here, $\sim$ means that the ratio of the left and right hand sides tends to 1.)

Remark 3. Another version is obtained for the cross–independence case: Let $A_n, n \geq 1$, be independent events, $\sum P(A_n) = \infty$. Let $B_n, n \geq 1$, be events such that, for each n, A_n and B_n are independent and that $\lim_{n\to\infty} P(B_n) = 1$. Then

$$P(\sum 1_{A_n B_n} = \infty) = 1.$$

Lemma 1. *Let $\langle a_n \rangle, \langle b_n \rangle$ be sequences of reals from $[0,1]$ such that $\sum_1^\infty a_n = \infty, b_n \to 0$. Then there exists a sequence $\langle c_n \rangle$ of reals from $(0,1]$ such that*

$$(2) \qquad \sum_1^\infty a_n c_n = \infty, \qquad \sum_1^\infty a_n b_n c_n < \infty.$$

PROOF : Put $s_0 = 1$ and determine integers

$$1 < r_1 \le s_1 < r_2 \le s_2 < \dots$$

so that

$$1 \le \sum_{s_{k-1} < n \le r_k} a_n \le 2, \quad b_n < 2^{-k} \text{ for } n > s_k, \quad k \ge 1.$$

For $n \ge 1$ define

$$c_n = \begin{cases} 2^{-(n-r_k)} & \text{for } r_k < n \le s_k, \quad k \ge 1, \\ 1 \text{ otherwise.} \end{cases}$$

It is easy to check that $\langle c_n \rangle$ satisfies (2). ∎

PROOF of the extended BL: Assume $\sum P(A_n B_n) = \infty$. Hence, $\sum P(A_n) = \infty$. Define $a_n = P(A_n), b_n = P(B_n^c | A_n)$; they satisfy the assumptions of Lemma 1, hence, there is a sequence $\langle c_n \rangle, c_n \in (0, 1]$, such that (2) holds. Let $\langle C_n \rangle$ be a sequence of independent events, independent also of $\langle A_n \rangle$ and of $\langle A_n B_n \rangle$, and such that $P(C_n) = c_n$. Put $\overline{A}_n = A_n C_n$; $\langle \overline{A}_n \rangle$ is a sequence of independent events. We have

$$P(\overline{A}_n B_n^c) = P(\overline{A}_n) P(B_n^c | \overline{A}_n) = P(A_n) P(C_n) P(B_n^c | A_n) = a_n b_n c_n,$$

i.e., $\sum P(\overline{A}_n B_n^c) < \infty$, hence $P(\sum 1_{\overline{A}_n B_n^c} < \infty) = 1$.
At the same time, $\sum P(\overline{A}_n) = \sum a_n c_n = \infty$, hence $P(\sum 1_{\overline{A}_n} = \infty) = 1$. Combining both probability 1 statements, we get

$$P(\sum 1_{\overline{A}_n B_n} = \infty) = 1 \text{ and, consequently, } P(\sum 1_{A_n B_n} = \infty) = 1.$$

∎

The extended BL was formulated for the benefit of some time series studies; see [1], e.g. The authors tried to find a result of this kind in literature, but without success.

REFERENCE

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