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Orlicz lattices with modular topology II.

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Abstract. The paper is a continuation of the author's paper [14], where the so-called modular topology on Orlicz lattices is investigated. Here the modular topology on an Orlicz lattice endowed with a convex modular is studied. We prove that sequentially modular continuous functionals have extension property. The different spaces of linear mappings on Orlicz lattices are considered. Applications to the theory of Orlicz spaces are given.

Keywords: Orlicz lattices, Orlicz spaces, locally solid topologies

Classification: 46E30

1. Modular topology on Orlicz lattices with convex modulars.

We shall use here the same notation and terminology as in the first part [14].

In this section we assume that (X,ρ) is an Orlicz lattice, where ρ is a convex modular which satisfies the σ–Lebesgue property, the σ–Fatou property and the σ–Levi property. Moreover, we assume that normal dual $X_n^\sim$ of X separates the points of X.

Then, according to [14, Theorem 2.5], the modular topology $\tau_\rho^\wedge$ coincides with the Mackey topology $(\tau(X,X_n^\sim)$. Moreover, by [5, Ch.X,§2, Theorem 5], the Riesz space X is order isomorphic to some ideal of $L^0(\Omega,\sum,\mu)$ for some measure space $(\Omega,\sum,\mu)$.

On the other hand, if X is an ideal of $L^0(\Omega,\sum\mu)$, where the measure μ is locally finite, then by [14, Theorem 3.1] $\tau_\rho^\wedge$ is a Hausdorff locally convex topology, hence the normal dual $X_n^\sim$ separates the points of X.

The usual topology $\tau_\rho^\vee$ can be generated by the so-called Luxemburg B–norm $\|x\|_{(\rho)}=\inf\{\lambda>0:\rho(x/\lambda)\le 1\}$. It is easy to see that $\|\cdot\|_{(\rho)}$ satisfies the σ–Fatou property, and since X has the countable sup property (see [15, Theorem 1.7]), $\|\cdot\|_{(\rho)}$ has also the Fatou property. Moreover, since ρ satisfies the Levi property (see [14, Theorem 2.1(vi)]), $\|\cdot\|_{(\rho)}$ satisfies the Levi property, because $Bd(\rho)=Bd(\tau_\rho^\vee)$ (see [14, Theorem 1.2]). Moreover, by [15, Theorem 2.6], $(X,\|\cdot\|_{(\rho)})$ is a Banach lattice. According to [5, Ch.X,§4, Theorem 8] we have the following result.

Theorem 1.1. *X is a perfect Riesz space, i.e., $\pi(X)=(X_n^\sim)_n^\sim$, where $\pi:X\to(X_n^\sim)_n^\sim$ and $\pi(x)(f)=f(x)$ for $x\in X$ and $f\in X_n^\sim$.*

We shall need the following result.

Theorem 1.2. *The equality $Bd(\tau_\rho^\wedge)=Bd(\tau_\rho^\vee)$ holds.*

PROOF : It suffices to show that $Bd(\sigma(X,X_n^\sim))\subset Bd(\tau_\rho^\vee)$. Since $\|\cdot\|_{(\rho)}$ has the Fatou property, by the Nakano–Amemiya–Mori theorem (see [5, Ch.X,§4,Theorem

7]) for every $x \in X$ we have $\|x\|_{(\rho)} = \sup\{|f(x)| : f \in X_n^\sim, \|f\| \leq\}$, where $\|f\| = \sup\{|f(x)| : x \in X, \|x\|_{(\rho)} \leq 1\}$. Let us assume that a subset A of X is $\sigma(X, X_n^\sim)$ – bounded, i.e. that $\sup\{|f(x) : x \in A\} < \infty$ for every $f \in X_n^\sim$. For each $x \in A$ let us put $F_x(f) = f(x)$ for $f \in X_n^\sim$. Then $\sup\{F_x(f)| : x \in A\} < \infty$, for every $f \in X_n^\sim$. Since $(X_n^\sim, \|\cdot\|)$ is a Banach lattice, by the Banach –Steinhaus theorem we get $\sup\{\|F_x\| : x \in A\} < \infty$, where $\|F_x\| = \sup\{|f(x)| : f \in X_n^\sim, \|f\| \leq 1\}$. Thus, $\sup\{\|x\|_{(\rho)} : x \in A\} < \infty$, and the proof is finished. ∎

From the last theorem it follows that the bornological topology associated to $\tau_\rho^\wedge$ is the usual topology $\tau_\rho^\vee$ (see [19, p. 15]). Hence we have the following.

Theorem 1.3. *The space $(X, \tau_\rho^\wedge)$ is bornological if and only if the modular ρ is metrizing.*

The next important properties of $\tau_\rho^\wedge$ are included in the following theorems.

Theorem 1.4. *The modular topology $\tau_\rho^\wedge$ satisfies the Levi property.*

PROOF : We know that the modular ρ satisfies the Levi property. Combining [14, Theorem 1.2] with Theorem 1.2 we get $Bd(\rho) = Bd(\tau_\rho^\wedge)$, and thus the proof is finished. ∎

Theorem 1.5. *The space $(X, \tau_\rho^\wedge)$ is complete.*

PROOF : We know that $\tau_\rho^\wedge$ is a Hausdorff Lebesgue topology and by the previous theorem $\tau_\rho^\wedge$ is also a Levi topology. In view of [1, Theorem 13.9] X endowed with $\tau_\rho^\wedge$ is complete. ∎

Since the space $(X, \tau_\rho^\vee)$ is complete, by the last theorem we have the following.

Corollary 1.6. *If the modular ρ is not metrizing, then the modular topology $\tau_\rho^\wedge$ is not metrizable.*

In [20], J.H.Webb defines and studies sequentially barreled spaces. A locally convex Hausdorff space (X, τ) is called <u>sequentially barrelled</u> if every $\sigma(X^*, X)$–convergent to zero sequence in X^* is equicontinuous.

Theorem 1.7. *The space $(X, \tau_\rho^\wedge)$ is sequentially barrelled.*

PROOF : By [14, Theorem 2.4 and Theorem 2.5] we have $\tau_\rho^\wedge = \tau(X, (X, \tau_\rho^\wedge)^+)$. Since the space $(X, \tau_\rho^\wedge)$ is complete, according to [20, Theorem 4.3] the space $(X, \tau_\rho^\wedge)$ is sequentially barrelled. ∎

Theorem 1.8.. *The space $(, X, \tau_\rho^\wedge)$ is barrelled if and only if the modular ρ is metrizing.*

PROOF : If ρ is metrizing, then $\tau_\rho^\wedge = \tau_\rho^\vee$. Assume that the space $(X, \tau_\rho^\wedge)$ is barrelled . Since the norm topology $\tau_\rho^\vee$ has the Fatou property, it has a base $\mathcal{B} = \{V\}$ of neighbourhoods of zero consisting of solid and order closed sets. Then by [1, Theorem 12.7], every set $V \in \mathcal{B}$ is closed for $\tau_\rho^\wedge$, because $\tau_\rho^\wedge$ is a Hausdorff Lebesgue topology. Thus, by [3, Theorem 6.2.1] we get $\tau_\rho^\vee \subset \tau_\rho^\wedge$, and this means that ρ is metrizing. ∎

2. Extension of sequentially modular continuous functionals.

In the theory of Orlicz lattices the problem of extension of sequentially modular continuous functionals is of interest.

We recall some notations. Assume that $(\Omega, \sum, \mu)$ is a σ-finite measure space. Let X be an ideal of L^0 such that $\operatorname{supp} X = \Omega$, and let $\|\cdot\|$ be a Riesz norm on X. Then the pair $(X, \|\cdot\|)$ is called a normed function space.

The associated space X^* of X is defined by

$$X^\times = \{y \in X' : f_y \in X^*\},$$

where X' and X^* denote the Köthe dual and the Banach dual of X respectively, and $f(x) = \int_\Omega x(t)y(t)\,d\mu$ for $x \in X$.

The associated norm $\|\cdot\|^\times$ on $X^\times$ is given by

$$\|y\|^\times = \|f_y\|_{X^*} = \sup\{|\int_\Omega x(t)y(t)\,d\mu| : x \in X, \|x\| \le 1\}$$

(see [5, Ch.VI, §1]).

If X_0 is an ideal of X, we shall denote by $(X_0)^\times$ the associated space of $(X_0, \|\cdot\|)$, i.e., $(X_0)^\times = \{y \in (X_0)' : g_y \in (X_0)^*\}$, where $g_y(x) = \int_\Omega x(t)y(t)\,d\mu$ for $x \in X_0$. We shall denote by $\|\cdot\|^\times_{X_0}$ the associated norm on $(X_0)^\times$.

The following result will be needed.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed function space, and let X_0 be an ideal of X such that $\operatorname{supp} X_0 = \Omega$. The $(X_0)^\times = X^\times$ and $\|y\|^\times_{X_0} = \|y\|^\times$ for every $y \in X^\times$.*

PROOF : See [16, Theorem 0.1.]. ■

If a convex modular ρ is defined on X then we put $\|x\| = \|x\|_{(g)} = \inf\{\lambda > 0 : \rho(x/\lambda) \le 1\}$ for $x \in X$.

We are now ready to prove that sequentially modular continuous functionals on Orlicz lattices in L^0 have the extension property. The details follow.

Theorem 2.2. *Let X be an ideal of L^0 such that $\operatorname{supp} X = \Omega$, and let ρ be a convex modular on X that satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property. Let X_0 be an ideal of X such that $\operatorname{supp} X_0 = \Omega$. If f_0 is sequentially modular continuous functional on (X_0, ρ), then there exists a sequentially modular continuous functional f on (X, ρ) such that $f(x) = f_0(x)$ for $x \in X_0$ and $\|f_0\| = \|f\|$.*

PROOF : Let f_0 be a sequentially modular continuous functional on (X_0, ρ). Since ρ satisfies the σ-Lebesgue property on X_0, f_0 is sequentially order continuous on X_0. Therefore, by [5, Ch.VI, §1, Theorem 1] there exists a unique $y \in (X_0)'$ such that $f_0(x) = \int_\Omega x(t)y(t)\,d\mu$ for $x \in X_0$. Since $f_0 \in (X_0)^*$, we obtain that $y \in (X_0)^\times$. But according to Theorem 2.1, we have that $y \in X^\times$ and $\|y\|^\times_{X_0} = \|y\|^\times$. Therefore, putting $f(x) = \int_\Omega x(t)y(t)\,d\mu$ for $x \in X$, we obtain by [14, Theorem 3.3] that f is a sequentially modular continuous functional on the whole space (X, ρ). Moreover, we have $f(x) = f_0(x)$ for $x \in X_0$ and $\|f_0\| = \|y\|^\times_{X_0} = \|y\|^\times = \|f\|$. ■

3. The modular topology $\tau_\rho^\wedge$ on Orlicz lattices endowed with convex orthogonal additive modulars.

In this section, we assume that (X,ρ) is an Orlicz lattice, where ρ is a convex orthogonal additive modular (i.e., $\rho(x_1+x_2)=\rho(x_1)+\rho(x_2)$ if $|x_1|\wedge|x_2|=0$) that satisfies both the σ–Fatou property and the σ–Levi property. Then ρ satisfies also the σ–Lebesgue property (see [16, Theorem 4.2]). Moreover, in view of [21, Theorem 5.1], the Riesz space X is order isomorphic to some Musielak–Orlicz space L^φ.

As in the section 1 we assume that the normal dual $X_n^\sim$ of X separates the points of X. By [15, Theorem 1.6], $(X,\|\cdot\|_{(\rho)})$ is a Banach lattice, and this means that the modular ρ is complete (see [11, p. 171]).

Moreover, we shall assume that $\lim_{\lambda\to 0}\lambda^{-1}\rho(\lambda x)=0$ for $x\in X$. Then, in view of [11, Theorem 38.9, Theorem 38.10 and Theorem 42.17] on $X_n^\sim$, the so–called <u>conjugate modular</u> $\widetilde{\rho}$ can be defined as follows

$$\widetilde{\rho}=\sup\{f(x)-\rho(x):x\in X\}\qquad \text{for } f\in X_n^\sim.$$

By [11, Theorem 38.9], the conjugate modular $\widetilde{\rho}$ is convex and orthogonal additive, satisfies the σ–Fatou property and is monotone complete (for definition see [11, p. 157]). Hence $\widetilde{\rho}$ satisfies also the σ–Lebesgue property (see [17, Theorem 4.2]) and the σ–Levi property, because for a modular bounded subset A of $X_n^\sim$ we have $\sup\{\widetilde{\rho}(kf):f\in A\}<\infty$ for some number $k>0$ (see [4, Proposition 1.3]).

In X another B–norm $\|\cdot\|_\rho^0$ can be defined as follows: $\|x\|_\rho^0=\sup\{|f(x)|: f\in X_n^\sim,\widetilde{\rho}(f)\le 1\}$. It is known that if the modular ρ is continuous (i.e., $\rho(x)=\sup\{\rho(a):|a|\le|x|,\rho(a)<\infty\}$ [11, p.182]), then B–norm $\|\cdot\|_\rho^0$ is equivalent to $\|\cdot\|_{(\rho)}$ (see [11, Theorem 43.6 and Theorem 40.9]).

Henceforth, we shall assume that the modulars ρ and $\widetilde{\rho}$ are continuous on X and $X_n^\sim$.

The following lemma will be needed.

Lemma 3.1. *The equality $Bd(\sigma(X_n^\sim,X))=Bd(\tau_{\widetilde{\rho}}^\vee)$ holds.*

PROOF : We first show that $Bd(\sigma(X_n^\sim,(X_n^\sim)_n^\sim)=Bd(\tau_{\widetilde{\rho}}^\vee)$. We known that the topology $\tau_{\widetilde{\rho}}^\vee$ on $X_n^\sim$ satisfies the Fatou property, because $\widetilde{\rho}$ satisfies the Fatou property. Moreover, the space $(X_n^\sim,\tau_{\widetilde{\rho}}^\vee)$ is complete, because the modular $\widetilde{\rho}$ is complete on $X_n^\sim$. Thus $(X_n^\sim,\tau_{\widetilde{\rho}}^\vee)^*=(X_n^\sim)^\sim$. Then ,by Nakano–Amemiya–Mori theorem, for every $f\in X_n^\sim$ we have $\|f\|_{(\widetilde{\rho})}=\sup\{|F(f)|:F\in(X_n^\sim)_n^\sim,\|F\|\le 1\}$, where $\|F\|=\sup\{|F(f)|:f\in X_n^\sim,\|f\|_{(\widetilde{\rho})}\le 1\}$. Arguing similarly as in the proof of Theorem 1.2 we obtain that $Bd(\sigma(X_n^\sim,(X_n^\sim)_n^\sim)=Bd(\tau_{\widetilde{\rho}}^\vee)$. In view of the perfectness of the Riesz space X (see Theorem 1.1) the proof is finished. ∎

As an application of Lemma 3.1 we have the following.

Theorem 3.2. *The strong topology $\beta(X,X_n^\sim)$ coincides with the norm topology $\tau_\rho^\vee$ on X, i.e., $\beta(X,X_n^\sim)=\tau_\rho^\vee$.*

PROOF : By Lemma 3.1, the polar sets $(B_{(\widetilde{\rho})}(r))^0$ of the balls $B_{(\widetilde{\rho})}(r) = \{f \in X_n^\sim : \|f\|_{(\widetilde{\rho})} \le r\}, r > 0$, constitute a base of neighbourhoods of zero for the strong topology $\beta(X, X_n^\sim)$ on X. Since $\|f\|_{(\widetilde{\rho})} \le 1$ iff $\widetilde{\rho}(f) \le 1$, we have

$$\begin{aligned} B_{(\widetilde{\rho})}(r))^0 &= \{x \in X : |f(x) \le 1 \text{ for } f \in X_n^\sim \text{ with } \|f\|_{(\widetilde{\rho})} \le r\} \\ &= \{x \in X : \|x\|_\rho^0 \le r^{-1}\}. \end{aligned}$$

Thus the proof is completed. ∎

The next theorem characterizes semireflexivity of $(X, \tau_\rho^\wedge)$ in terms of the conjugate modular $\widetilde{\rho}$.

Theorem 3.3. *The space $(X, \tau_\rho^\wedge)$ is semireflexive if and only if the conjugate modular $\widetilde{\rho}$ on $X_n^\sim$ is metrizing.*

PROOF : Since $\tau_\rho^\wedge$ is a Lebesgue topology, in view of [1, Theorem 22.4] the space $(X, \tau_\rho^\wedge)$ is semireflexive iff $\beta(X_n^\sim, X)$ is a Lebesgue topology. By the perfectness of X (see Theorem 1.1) and Theorem 3.2 we have $\beta(X_n^\sim, X) = \tau_{\widetilde{\rho}}^\vee$. But $\tau_{\widetilde{\rho}}^\vee$ is a Lebesgue topology iff $\widetilde{\rho}$ is metrizing. ∎

As an application of Theorem 1.8 and Theorem 3.3, we get a characterization of reflexivity of the space $(X, \tau_\rho^\wedge)$.

Corollary 3.4. *The space $(X, \tau_\rho^\wedge)$ is reflexive if and only if both ρ and $\widetilde{\rho}$ are metrizing.*

4. Some properties of the modular topology $\tau_\rho^\wedge$ on Orlicz spaces L^φ.

In this section we apply the results form Sections 1 and 3 for an investigation of the modular topology $\tau_\rho^\wedge$ on Orlicz spaces L^φ generated by a convex Orlicz function φ.

By <u>an Orlicz function</u> we mean a function $\varphi : [0, \infty) \to [0, \infty]$ which is nondecreasing, left continuous, continuous at zero and $\varphi(u) = 0$ iff $u = 0$. An Orlicz function φ is called <u>convex</u> if $\varphi(\alpha u + \beta v) \le \alpha\varphi(u) + \beta\varphi(v)$ for $\alpha, \beta \ge 0, \alpha + \beta = 1$.

If φ is a convex Orlicz function, then the functional

$$\rho_\varphi(x) = \int_\Omega \varphi(|x(t)|)\, d\mu$$

is a convex orthogonal additive modular on the Orlicz space L^φ satisfying the σ–Lebesgue property, the σ–Fatou property and the σ–Levi property. (see [21, §1]).

If φ is an N–function (see [7]), the modular topology $\tau_\rho^\wedge$ has been investigated in the author's papers [**12**], [**13**]. We shall show that the results of [**12**] and [**13**] are valid also for an arbitrary convex Orlicz function.

To the end of this section we assume that $(\Omega, \sum, \mu)$ is a σ–finite measure space.

Let φ^* denote <u>the function complementary to φ in the sense of Young</u> (see [**8**]). Let us note that φ^* vanishes only at zero iff $\varphi(u)/u \to 0$ as $u \to 0$. In general, the functional $\rho_{\varphi^*}(x) = \int_\Omega \varphi^*(|x(t)|)\, d\mu$ is a semimodular on the space $L^{\varphi^*} = \{x \in L^0 : \rho_{\varphi^*}(\lambda x) < \infty$ for some $\lambda > 0\}$, i.e., $\rho_{\varphi^*}(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$ (cf. [**9**]). It is well known that the Köthe dual of L^φ is the space L^{φ^*} (cf. [**8**]).

Thus, according to [14, Theorem 3.3], we get the following.

Theorem 4.1. *Let φ be a convex Orlicz function. For a linear functional f on L^{φ} the following statements are equivalent:*

(i) *f is continuous for $\tau_{\rho}^{\wedge}$.*
(ii) *f is sequentially modular continuous.*
(iii) *f is sequentially order continuous.*
(iv) *There exists a unique $y \in L^{\varphi^*}$ such that*

$$f(x) = \int_{\Omega} x(t)y(t)\,d\mu \qquad \text{for } x \in L^{\varphi}.$$

Remark 1. For φ being an N–function and μ being a finite measure (resp. a σ–finite measure), the equivalence of (ii) and (iv) of the last theorem has been proved in a different way in [9,Theorem 4.11] (resp. [10, Theorem 1]).

Combining [14, Theorem 2.5] ,with Theorem 3.1, we have the following result.

Theorem 4.2. *Let φ be a convex Orlicz function. Then the modular topology $\tau_{\rho}^{\wedge}$ coincides with the Mackey topology $\tau(L^{\varphi}, L^{\varphi^*})$.*

Remark 2. If φ is an N function, then the equality $\tau_{\rho}^{\wedge} = \tau(L^{\varphi}, L^{\varphi^*})$ is proved in the author's paper [**13**].

At last, we give a characterization of the barrelledness and the semireflexivity of $(L^{\varphi}, \tau_{\rho}^{\wedge})$ in terms of Orlicz functions φ and φ^*.

Theorem 4.3. *Let φ be a convex Orlicz function such that $\varphi(u)/u \to 0$ as $u \to 0$. Then the space $(L^{\varphi}, \tau_{\rho}^{\wedge})$ is barrelled (resp. semireflexive) if and only if φ (resp. φ^*) satisfies the Δ_2–condition*

(i) *for all u, if μ is an infinite atomless measure,*
(ii) *for large u, if μ is a finite atomless measure,*
(iii) *for small u, if μ is a purely atomic measure with measure of atoms b_n satisfying $0 < \inf_n b_n \leq \sup_n b_n < \infty$.*

PROOF : For every $x \in L^{\varphi}, \rho_{\varphi}(\lambda x)/\lambda \to 0$ as $\lambda \to 0$, because $\varphi(u)/u \to 0$ as $u \to 0$. According to Theorem 4.1 and [8, p. 66], we have $\widetilde{\rho}_{\varphi}(f) = \rho_{\varphi^*}(y)$, when $f(x) = \int_{\Omega} x(t)y(t)\,d\mu$ for $x \in L^{\varphi}$, where $y \in L^{\varphi^*}$. The result follows from Theorem 1.8 and Theorem 3.3. ∎

Remark 3. For φ being an N function, the above result is proved in [12, Theorem 2.2 and Theorem 2.4].

5. Linear mappings on Orlicz lattices.

In this section we shall show the various relations between different spaces of linear mappings between Orlicz lattices.

Let (X_1, ρ_1) and (X_2, ρ_2) be Orlicz lattices.

Given linear topologies τ_1 and τ_2 on X_1 and X_2 respectively, we will denote by $\mathcal{L}(\tau_1, \tau_2)$ (resp. $\mathcal{L}_s(\tau_1, \tau_2)$) the collection of all linear (τ_1, τ_2)–continuous (resp. sequentially (τ_1, τ_2)–continuous) mappings of X_1 into X_2.

A linear mapping A of X_1 into X_2 is said to be order continuous (resp. sequentially order continuous, resp. sequentially order star-continuous if $x_\alpha \xrightarrow{(o)} 0$ (reps. $x_n \xrightarrow{(o)} 0$, resp. $X_n \xrightarrow{(o)^*} 0$) in X_1 implies $Ax_\alpha \xrightarrow{(o)} 0$ (reps. $Ax_n \xrightarrow{(o)} 0$, resp. $Ax_n \xrightarrow{(o)^*} 0$) in X_2 (see [18, p.50]). The collection of all order continuous (resp. sequentially order continuous, resp. sequentially order star-continuous) linear mappings of X_1 into X_2 will be denoted by $\mathcal{L}^0$ (resp. $\mathcal{L}^{S0}, \mathcal{L}_*^{S0}$). The collection of all order bounded linear mappings of X_1 into X_2 will be denoted by $\mathcal{L}^\sim$.

A linear mapping A of X_1 into X_2 is said to be modular continuous (resp. sequentially modular continuous, resp. sequentially modular star-continuous) if $x_\alpha \xrightarrow{(\rho_1)} 0$ (resp. $x_n \xrightarrow{(\rho_1)} 0$, resp. $x_n \xrightarrow{(\rho_1)^*} 0$) in X_1 implies $Ax_\alpha \xrightarrow{(\rho_2)} 0$ (resp. $Ax_n \xrightarrow{(\rho_2)} 0$, resp. $Ax_n \xrightarrow{(\rho_2)^*} 0$) in X_2. The collection of all modular continuous (resp. sequentially modular continuous, resp. sequentially modular star-continuous) linear mappings of X_1 into X_2 will be denoted by $\mathcal{L}^\rho$ (resp. $\mathcal{L}^{S\rho}$, resp. $\mathcal{L}_*^{S\rho}$).

The following relations are well known:

$$\mathcal{L}^0 \subset \mathcal{L}^{S0} \subset \mathcal{L}_*^{S0}$$

(see [18, p.50]), and similarly one can show that:

$$\mathcal{L}^\rho \subset \mathcal{L}^{S\rho} \subset \mathcal{L}_*^{S\rho}.$$

Henceforth, we assume that (X_1, ρ_1) and (X_2, ρ_2) are Orlicz lattices with convex modulars ρ_1 and ρ_2 satisfying the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property on X_1 and X_2 respectively. Moreover, we assume that the normal duals $(X_1)_n^\sim$ and $(X_2)_n^\sim$ separate the points of X_1 and X_2 respectively.

We are now ready to prove the following theorems.

Theorem 5.1. *The spaces $\mathcal{L}_S(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$ and $\mathcal{L}(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$ coincide, i.e., $\mathcal{L}_S(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge) = \mathcal{L}(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$.*

Proof : It suffices to show that $\mathcal{L}_S(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge) \subset \mathcal{L}(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$. Indeed, let $A \in \mathcal{L}_S(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$. Since $(X_1, \tau_{\rho_1}^\wedge)^* = (X_1, \tau_{\rho_1}^\wedge)^+$ (see [14, Theorem 2.4]), in view of [20, Corollary 1.10] every sequentially $\tau_{\rho_1}^\wedge$-closed hyperplane in X_1 is $\tau_{\rho_1}^\wedge$-closed. Hence, by [6, §32, 1 (11)], A is $(\sigma(X_1, (X_1)_n^\sim), \sigma(X_2, (X_2)_n^\sim))$ - continuous, so according to [2, Ch.IV, §4, Theorem 7] A is $(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$ - continuous, because we know that $\tau_{\rho_i}^\wedge = \tau(X, (X_i)_n^\sim)$ for $i = 1, 2$. Thus the proof is finished. ∎

Theorem 5.2. *The following relations hold:*

$$\mathcal{L}^0 \subset \mathcal{L}^{S0} \subset \mathcal{L}_*^{S0} \subset \mathcal{L}(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge) \subset \mathcal{L}(\tau_{\rho_1}^\vee, \tau_{\rho_2}^\vee)$$

and

$$\mathcal{L}^\rho \subset \mathcal{L}^{S\rho} \subset \mathcal{L}_*^{S\rho} \subset \mathcal{L}(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge) \subset \mathcal{L}(\tau_{\rho_1}^\vee, \tau_{\rho_2}^\vee).$$

Moreover, every positive $(\tau_{\rho_1}^\wedge, \tau_{\rho_2}^\wedge)$ - continuous linear mapping of X_1 into X_2 is order continuous.

PROOF : In view of Theorem 1.1 we have $\mathcal{L}(\tau^{\wedge}_{\rho_1}, \tau^{\wedge}_{\rho_2}) \subset \mathcal{L}(\tau^{\vee}_{\rho_1}, \tau^{\vee}_{\rho_2})$.
Now, we shall show that

$$\mathcal{L}^{S0}_* \subset \mathcal{L}(\sigma(X_1, (X_1)^{S0}_*), \sigma(X_2, (X_2)^{S0}_*)).$$

Indeed, a linear mapping A of X_1 into X_2 is $(\sigma(X_1, (X_1)^{S0}_*), \sigma(X_2, (X_2)^{S0}_*))$ – continuous if and only if $f \circ A \in (X_1)^{S0}_*$ for every $f \in (X_2)^{S0}_*$. But it holds if $A \in \mathcal{L}^{S0}_*$.

We have $\tau^{\wedge}_{\rho_i} = \tau(X_i, (X_i)^{S0}_*)$ for $i = 1,2$ (see [14, Theorem 2.5]). Therefore, using [2, Ch.IV, §4, Theorem 7], we obtain the inclusions of our theorem, because $\mathcal{L}^{S0}_* = \mathcal{L}^{S\rho}_*$ (see [15, Corollary 2.4]).

Now let A be a positive $(\tau^{\wedge}_{\rho_1}, \tau^{\wedge}_{\rho_2})$ – continuous linear mapping of X_1 into X_2, and let $x_\alpha \downarrow 0$ in X_1. Then $x_\alpha \to 0$ for $\tau^{\wedge}_{\rho_1}$, so $Ax_\alpha \to 0$ for $\tau^{\wedge}_{\rho_2}$. Since $0 \leq Ax_\alpha \downarrow$ in X_2, by [1, Theorem 5.6] $Ax_\alpha \downarrow 0$ in X_2, and this means that A is order continuous. ■

Theorem 5.3. *Suppose that the modular ρ_1 is metrizing. Then*

$$\mathcal{L}^{\rho} = \mathcal{L}^{S\rho} = \mathcal{L}^{S\rho}_* = \mathcal{L}(\tau^{\wedge}_{\rho_1}, \tau^{\wedge}_{\rho_2}) = \mathcal{L}(\tau^{\vee}_{\rho_1}, \tau^{\vee}_{\rho_2}).$$

PROOF : In view of Theorem 5.2 it suffices to show that

$$\mathcal{L}(\tau^{\vee}_{\rho_1}, \tau^{\vee}_{\rho_2}) \subset \mathcal{L}^{\rho}.$$

Indeed, if $A \in \mathcal{L}(\tau^{\vee}_{\rho_1}, \tau^{\vee}_{\rho_2})$, then A is $(\tau^{\vee}_{\rho_1}, \tau^{\vee}_{\rho_2})$ – bounded. Hence, by [14, Theorem 1.2], A is (ρ_1, ρ_2) – bounded (see [4, Definition 3.1]). Therefore, according to [4, Theorem 3.2], there exist constants k, $M > 0$ such that $\rho_2(k\,Ax) \leq M\|x\|_{(\rho_1)}$ for all $x \in X_1$ satisfying $\rho_1(x) \leq 1$.

To prove that A is modular continuous, assume that $x_\alpha \xrightarrow{(\rho_1)} 0$. Then $\|x_\alpha\|_{(\rho_1)} \to 0$, because ρ_1 is metrizing. Given $\varepsilon > 0$ there exists α_0 such that $\|x_\alpha\|_{(\rho_1)} \leq (\varepsilon M^{-1} \wedge 1)$ for $\alpha \geq \alpha_0$, so $\rho_1(x) \leq 1$ for $\alpha \geq \alpha_0$. Hence $\rho_2(k\,Ax) \leq \varepsilon$ for $\alpha \geq \alpha_0$, and this means that $Ax \xrightarrow{(\rho_2)} 0$. Thus, $A \in \mathcal{L}^{\rho}$, and the proof is finished. ■

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