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Abstract. In this paper we investigate some linear topology on a σ–Dedekind complete Riesz space X that is determined by some functional ρ defined on X and called a modular . The pairs (X,ρ) will be called Orlicz lattices, and this topology will be called a modular topology and denoted by $\tau_\rho^\wedge$. The modular topology $\tau_\rho^\wedge$ is the finest linear topology on X under which every modular convergent sequence is convergent. Under some additional conditions relating properties of a modular ρ to the order structure of a Riesz space X, the modular topology $\tau_\rho^\wedge$ has a number of interesting properties. It is proved that $\tau_\rho^\wedge$ is the finest Lebesgue topology on X. It is shown that a linear functional on X is continuous for $\tau_\rho^\wedge$ if and only if it is order continuous. The Mackey topology of $(X,\tau_\rho^\wedge)$ is described. Examples of Orlicz lattices with modular topology are given.

Keywords: Orlicz lattices, Orlicz spaces, locally solid topologies

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1. Topological properties of modular convergence in Orlicz lattices. Given a linear topological space (X,τ), we shall denote:

$(X,\tau)^*$ – the topological dual.

$(X,\tau)^+$ – the sequential topological dual = the collection of all sequentially τ–continuous linear functionals on X.

$Bd(\tau)$ – the collection of all τ–bounded subsets of X.

For notation and terminology concerning Riesz spaces and locally solid topologies we refer to ([**1**], [**10**]).

We start with the definition of an Orlicz lattice (see [**18**], [**16**]). Let X be a σ–Dedekind complete Riesz space. A functional $\rho : X \to [0,\infty]$ is called <u>a modular</u>, if the following conditions hold:

(ρ1) $\rho(x)=0$ iff $x=0$.
(ρ2) $|x|\le|y|$ implies $\rho(x)\le\rho(y)$.
(ρ3) $\rho(x_1\vee x_2)\le\rho(x_1)+\rho(x_2)$ for $x_1\ge 0, x_2\ge 0$.
(ρ4) $\rho(\lambda x)\to 0$ if $\lambda\to 0$.

A pair (X,ρ) will be called <u>an Orlicz lattice</u>. Let us note that ρ is a modular in the sense of ([**12**]). A modular ρ is said to be <u>convex</u>, if $\rho(\alpha x_1+\beta x_2)\le\alpha\rho(x_1)+\beta\rho(x_2)$ for $\alpha,\beta\ge 0$ and $\alpha+\beta=1$. A modular ρ is said to be <u>metrizing</u>, whenever $\rho(x_n)\to 0$ implies $\rho(2x_n)\to 0$ for a sequence (x_n) in X.

A net (x_α) in X is said to be <u>modular convergent</u> to $x\in X$, in symbols $x_\alpha \xrightarrow{(\rho)} X$, if there exists a number $\lambda>0$ such that $\rho(\lambda(x_\alpha-x))\underset{\alpha}{\to}0$ (cf. [**12**], p. 50).

We say that a sequence (x_n) in X <u>is order star-convergent</u> (resp. <u>relatively uniform star-convergent</u>; resp. <u>modular star-convergent</u>) to $x\in X$, in symbols

$x_n \overset{(0))^*}{\to} x$ (resp. $x_n \overset{(ru)^*}{\to} x$, resp. $x_n \overset{(\rho)^*}{\to} x$) if every subsequence of (x_n) contains a subsequence which is order convergent (resp. relatively uniform convergent; resp. modular convergent) to x.

A subset A of X is said to be modular bounded, if $\lambda_n x_n \overset{(\rho)}{\to} 0$ for each sequence (x_n) in A and each sequence (λ_n) of numbers that converges to 0. We will denote by $Bd(\rho)$ the collection of all modular bounded subsets of X.

Now, let (X, ρ) be an Orlicz lattice. Then on X one can define two linear topologies, closely associated with modular convergence in X.

First, the family $\mathcal{B}_\rho^\vee = \{\alpha U_\rho(\varepsilon) : \alpha \neq 0, \varepsilon > 0\}$, where $U_\rho(\varepsilon) = \{x \in X : \rho(x) < \varepsilon\}$, constitutes a base of neighbourhoods of zero for the well-known topology $\tau_\rho^\vee$ on X, generated by the Riesz F-norm $\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq \lambda\}$. It is known that $\|x_n - x\|_\rho \to 0$ iff $\rho(\lambda(x_n - x)) \to 0$ for all $\lambda > 0$. The topology $\tau_\rho^\vee$ has the following important property which can be found in ([**6**], Theorem 3.4).

Theorem 1.1. *$\tau_\rho^\vee$ is the coarsest of all linear topologies τ on X under which $x_\alpha \overset{\tau}{\to} 0$ implies $x_\alpha \overset{(\rho)}{\to} 0$ for a net (x_α) in X.*

In ([**6**], Theorem 7.4), the following result is proved.

Theorem 1.2. *A subset A of A is $\tau_\rho^\vee$-bounded iff it is modular bounded, i.e., $Bd(\tau_\rho^\vee) = Bd(\rho)$.*

Next, in ([**6**], Theorem 4.1) it is proved that the family:

$$\mathcal{B}_\rho^\wedge = \left\{ \bigcup_{N=1}^{\infty} \left(\sum_{n=1}^{N} U_\rho(\varepsilon_n)\right) : (\varepsilon_n) - \text{a sequence of positive numbers} \right\},$$

constitutes a base of neighbouroods of zero for some linear topology on X which will be called a modular topology and denoted by $\tau_\rho^\wedge$. The following basic property of $\tau_\rho^\wedge$ can be found in [**6**], Theorem 4.2).

Theorem 1.3. *$\tau_\rho^\wedge$ is the finest of all linear topologies τ on X under which $x_\alpha \overset{(\rho)}{\to} 0$ implies $x_\alpha \overset{\tau}{\to} 0$ for a net (x_α) in X.*

Moreover, arguing as in the proof of ([**14**], Theorem 1.2) one can show that the modular topology $\tau_\rho^\wedge$ has the following stronger property which will be the key tool for the discussion on topological structure of Orlicz lattices.

Theorem 1.4. *$\tau_\rho^\wedge$ is the finest of all linear topologies τ on X under which $x_n \overset{(\rho)}{\to} 0$ implies $x_n \overset{\tau}{\to} 0$ for a sequence (x_n)inX.*

It is seen that $\tau_\rho^\wedge \subset \tau_\rho^\vee$, and according to ([**7**], Theorem 3.5) we have the following.

Theorem 1.5. *$\tau_\rho^\wedge = \tau_\rho^\vee$ iff a modular ρ is metrizing.*

In view of ([**7**], Section 5.1) and ([**6**], Theorem 9.3) we have:

Theorem 1.6. *If a modular ρ is convex, then the topologies $\tau_\rho^\wedge$ and $\tau_\rho^\vee$ are locally convex.*

The next theorem characterizes the linear topologizability of modular convergence of nets.

Theorem 1.7. *Modular convergence of nets in X is generated by some linear topology iff a modular ρ is metrizing.*

PROOF : The result follows from Theorems 1.1, 1.3 and 1.4. ∎

Moreover, we have the following.

Theorem 1.8. *Sequential modular convergence in X is generated by some linear metrizable topology iff a modular ρ is metrizing.*

PROOF : If ρ is metrizing, then by Theorem 1.5, $x_n \xrightarrow{(\rho)} 0$ iff $x_n \xrightarrow{\tau_\rho^\vee} 0$. Assume the there exists a metrizable linear topology τ on X such that $x_n \xrightarrow{(\rho)} 0$ iff $x_n \xrightarrow{\tau} 0$. Then in view of ([7], Theorem 1.3) and ([6], Theorem 3.4) we have $\tau_\rho^\vee \subset \tau$. In turn, by Theorem 1.4 we have $\tau \subset \tau_\rho^\wedge$. Thus $\tau_\rho^\wedge = \tau_\rho^\vee$, and this means that ρ is metrizing. ∎

In Section 3 we give an example of Orlicz lattice (X, ρ), where a modular ρ is not metrizing, but sequential modular convergence is generated by the modular topology $\tau_\rho^\wedge$.

2. The modular topology $\tau_\rho^\wedge$ on Orlicz lattices.

For an Orlicz lattice (X, ρ) we shall consider some further conditions relating properties of a modular ρ to the order structure of X, described in the following definition. We say that:

(i) ρ satisfies the σ–Lebesgue (resp. the Lebesgue) property if $x_n \downarrow 0$ in X with $\rho(\lambda x_1) < \infty$ for some $\lambda > 0$ implies $\rho(\lambda x_n) \downarrow 0$ (resp. $x_\alpha \downarrow 0$ in X with $\rho(\lambda x_\alpha) < \infty$ for some indices α_0 and $\lambda > 0$ implies $\rho(\lambda x_\alpha) \downarrow 0$).

(ii) ρ satisfies the σ–Fatou property, if $x_n \uparrow x$ in X implies $\rho(x_n) \uparrow \rho(x)$.

(iii) ρ satisfies the σ–Levi property, if $0 \le x_n \uparrow$ in X and the set $\{x_n\}$–modular bounded implies that $x_n \uparrow x$ holds in X for some $x \in X$.

By replacing the word "sequence" with "net" in the definitions (ii) and (iii) we obtain the Fatou property and the Levi property respectively.

We shall need the following results concerning the order structure of Orlicz lattices.

Theorem 2.1. *Let (X, ρ) be an Orlicz lattice. Then the following statements hold:*

(i) If ρ satisfies the σ–Lebesgue property, then X is a super Dedekind complete Riesz space and ρ satisfies also the Lebesgue property.

(ii) If ρ satisfies the ρ–Fatou property and the σ–Levi property, then $x_n \xrightarrow{(\rho)} 0$ implies $x_n \xrightarrow{(0)^*} 0$ for a sequence (x_n) in X.

(iii) If ρ satisfies the σ–Lebesgue property, the σ–Fatou property and the σ–Levi property, then $x_n \xrightarrow{(\rho)^*} 0$ iff $x_n \xrightarrow{(0)^*} 0$ for a sequence (x_n) in X.

(iv) If ρ satisfies the σ–Levi property, then X endowed with the F–norm topology $\tau_\rho^\vee$ is complete.

(v) If ρ satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property, then ρ satisfies also the Levi property.

PROOF : (i) –(iv). See ([**16**]).

(v) Assume that $0 \le x_\alpha \uparrow$ and the set $\{x_\alpha\}$ is modular bounded. By (i) it suffices to show that the set $\{x_\alpha\}$ is order bounded. Conversely, let us assume that the set $\{x_\alpha\}$ is not order bounded. Then, in view of ([**16**], Theorem 3.1) for every $\lambda > 0$ there exist $\varepsilon > 0$ and an increasing sequence of natural numbers (k_n) such that

$$\sup\{\rho(\lambda k_n^{-1} \bigvee_{j=1}^{m} x_{\alpha_j}) : x_{\alpha_j} \in \{x_\alpha\}, m \in N\} > \varepsilon$$

for all n. Hence, since $0 \le x_\alpha \uparrow$, there exists an increasing sequence of indices $\{\alpha_n\} \subset \{\alpha\}$ such that $\rho(\lambda k_n^{-1} x_{\alpha_n}) > \varepsilon$ for all n, and this means that the set $\{x_\alpha\}$ is not modular bounded, which contradicts our assumption. Thus the set $\{x_\alpha\}$ is order bounded, and the proof is finished. ∎

These additional conditions imposed on a modular ρ will impose also a number of interesting properties of the modular topology $\tau_\rho^\wedge$ on X. The basic properties of $\tau_\rho^\wedge$ are included in the following theorem.

Theorem 2.2. *Let (X, ρ) be an Orlicz lattice.*

(i) Suppose that modular ρ satisfies the σ-Lebesgue property. Then $\tau_\rho^\wedge$ is a Lebesgue topology.

(ii) Suppose that a modular ρ satisfies the σ-Lebesgue property and the σ-Levi property. Then $\tau_\rho^\wedge$ is finer that any σ-Lebesgue topology on X.

PROOF : (i) Using ([**1**], Theorem 1.2), it is easy to show that $\tau_\rho^\wedge$ is a locally solid topology. Combining Theorem 2.1(i) and Theorem 1.3 we have that $\tau_\rho^\wedge$ is a Lebesgue topology.

(ii) Let us assume that τ is a σ-Lebesgue topology on X, and let $x_n \xrightarrow{(\rho)} 0$. Then by Theorem 1.4, we get $\tau \subset \tau_\rho^\wedge$. ∎

As an application of the last theorem we get the following:

Theorem 2.3. *Let (X, ρ) be an Orlicz lattice. Suppose that a modular ρ satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property. Then the following statement are equivalent:*

(i) ρ is metrizing.

(ii) $\tau_\rho^\wedge = \tau_\rho^\vee$.

(iii) The Riesz F-norm $\|\cdot\|_\rho$ is order continuous on X, i.e., $\tau_\rho^\vee$ is a Lebesgue topology.

(iv) The Riesz space X has the diagonal property for order convergence.

(iv) In the Riesz space X order convergence is stable.

PROOF : (i)⇔ (ii) See Theorem 1.5. (ii)⇒ (iii) See Theorem 2.2.

(iii)⇒(iv) Assume that $\tau_\rho^\vee$ is a Lebesgue topology. Since X endowed with $\tau_\rho^\vee$ is complete (see Theorem 2.1), according to ([**3**], Ch. 7.1.3, Proposition 8) and ([**3**], Ch.5.2.8, Proposition 3) The Riesz space X has the diagonal property.

(iv)$\Rightarrow$ (v) See ([10], Theorem 70.2).

(v)$\Rightarrow$ (i) Let $x_n \stackrel{(0)}{\to} 0$. Then $x_n \stackrel{(ru)^*}{\to} 0$, because in X order convergence and relatively order convergence coincide (see [10], Theorem 16.3). Since $(X, \tau_\rho^\vee)$ is complete, in view of ([3], Ch. 7.1.3, Proposition 7) $x_n \stackrel{\tau_\rho^\vee}{\to} 0$, but this means that $\tau_\rho^\vee$ is a σ-Lebesgue topology. According to Theorem 2.2, we get $\tau_\rho^\wedge = \tau_\rho^\vee$; so by Theorem 1.5 ρ is metrizing. ■

In order to characterize the topological dual of $(X, \tau_\rho^\wedge)$ we now recall some definitions.

Let $X, \rho)$ be an Orlicz lattice. A linear functional f on X is said to be modular continuous (resp. sequentially modular continuous; resp. sequentially modular star–continuous) if $x_\alpha \stackrel{(\rho)}{\to} 0$ (resp. $x_n \stackrel{(\rho)}{\to} 0$; resp. $x_n \stackrel{(\rho)^*}{\to} 0$) in X implies $f(x_\alpha) \to 0$ (resp. $f(x_n) \to 0$; resp. $f(x_n) \to 0$).

The collection of all modular continuous (resp. sequentially modular continuous; resp. sequentially modular star–continuous) linear functional on X will be denoted by X^ρ (resp. $X^{S\rho}$; resp. $X_*^{S\rho}$).

We will denote by X^0 (resp. X^{S0} ,resp. X_*^{S0}) the collection of all order continuous (resp. sequentially order continuous; resp. sequentially order star–continuous) linear functionals on X. In view of ([4], Ch.VII,§2) we have

$$X^0 \subset X^{S0} = X_*^{S0},$$

and similarly one can get

$$X^\rho \subset X^{S\rho} = X_*^{S\rho}.$$

We will also write $X_n^\sim$ and $X_c^\sim$ (see [1], Definition 3.8) instead of X^0 and X^{S0}, because $X^{S0} \subset X^\sim$, where $X^\sim$ denotes the collection of all order bounded linear functionals on X ([19], Proposition 5.22).

The next theorem characterizes the topological dual of $(X, \tau_\rho^\wedge)$.

Theorem 2.4. *If a modular ρ satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property, then for a linear functional f on X the following statements are equivalent:*

(i) *f is continuous for $\tau_\rho^\wedge$.*
(ii) *f is sequentially continuous for $\tau_\rho^\wedge$.*
(iii) *f is modular continuous.*
(iv) *f is sequentially modular continuous.*
(v) *f is sequentially modular star–continuous.*
(vi) *f is order continuous.*
(vii) *f is sequentially order continuous.*
(viii) *f is sequentially order star–continuous.*

Proof : We shall show that $(X, \tau_\rho^\wedge)^* = X, \tau_\rho^\wedge)^+ = X^\rho = X^{S\rho} = X_*^{S\rho} = X^0 = X^{S0} = X_*^{S0}$. Since $\tau_\rho^\wedge$ is a Lebesgue topology, we have $(X, \tau_\rho^\wedge)^+ \subset X^{S0}$. We have that $X^0 = X^{S0}$, because X has the countable sup property (see Theorem

2.1(i)). Thus $(X,\tau_\rho^\wedge)^+ \subset X^0 = X^{S0} = X^{S0}$. On the other hand, we have $(X,\tau_\rho^\wedge)^* \subset X^\rho \subset X^{S\rho} = X_*^{S\rho}$ (see Theorem 1.3). According to Theorem 2.1 (iii) we have $X_*^{S0} = X_*^{S\rho}$ and since $(X,\tau_\rho^\wedge)^* \subset (X,\tau_\rho^\wedge)^+$, it suffices to show that $X^{S\rho} \subset (X,\tau_\rho^\wedge)^*$. For this purpose, we shall show that $\sigma(X, X^{S\rho}) \subset \tau_\rho^\wedge$. In view of Theorem 1.4, it will be sufficient to show that for a sequence (x_n), $x_n \xrightarrow{(\rho)} 0$ implies $f(x_n) \to 0$ for each $f \in X^{S\rho}$. But this is obvious, and thus the proof is finished. ∎

Under the assumption imposed on the Orlicz lattice (X,ρ) as in Theorem 2.4, we have that the Mackey topology $\tau_\rho^\wedge$ of $(X,\tau_\rho^\wedge)$ (see [**20**]) coincides with the Mackey topology $\tau(X, X_n^\sim)$. The next theorem characterizes the Mackey topology $\tau(X, X_n^\sim)$.

Theorem 2.5. *Let (X,ρ) be an Orlicz lattice with the normal dual $X_n^\sim$ separating the points of X. Suppose that a modular ρ satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property. Then the following statements holds:*

(i) $\tau(X,X_n^\sim)$ is the finest locally convex topology on X which is weaker that $\tau_\rho^\wedge$.

(ii) $\tau(X,X_n^\sim)$ has a base of neighbourhoods of zero consisting of all sets of the form : $\bigcup_{N=1}^\infty (\sum_{n=1}^N \operatorname{conv} U_\rho(\varepsilon_n))$, where (ε_n) is a sequence of positive numbers.

(iii) If a modular ρ is convex, then $\tau_\rho^\wedge = \tau(X,X_n^\sim)$.

PROOF : (i) It is known that the Mackey topology $\tau(X,X_c^\sim)$ is locally solid ([**1**]. Ex. 4,p. 163). Since $X_n^\sim = X_c^\sim$, by ([**1**], Theorem 9.1) $\tau(X,X_n^\sim)$ is a Lebesgue topology, and according to Theorem 2.2 $\tau(X,X_n^\sim) \subset \tau_\rho^\wedge$. Now, let ξ be a locally convex topology on X weaker than $\tau_\rho^\wedge$. Then $(X,\xi)^* \subset X_n^\sim$; hence $\sigma(X,(X,\xi)^*) \subset \sigma(X,X_n^\sim)$. In view of ([**2**], Theorems 6 and 7), we get $\xi \subset \tau(X,(X,\xi)^*) \subset \tau(X,X_n^\sim)$.

(ii) Write $W(\varepsilon_n) = \bigcup_{N=1}^\infty(\sum_{n=1}^N U_\rho(\varepsilon_n))$, where (ε_n) is a sequence of positive numbers. It is easy to check that the system of all sets of the form $\operatorname{conv} W(\varepsilon_n)$, is a base of neighbourhoods of zero for some locally convex topology τ_c on X weaker than $\tau_\rho^\wedge$. Suppose that f is a linear functional on X continuous for $\tau_\rho^\wedge$. Then f is bounded on some neighbourhood $W(\varepsilon_n)$, and has the same bound on $\operatorname{conv} W(\varepsilon_n)$. Therefore f is continuous for τ_c. Moreover, in view of ([**6**], Theorem 10.4), also the system of all sets of the form: $\bigcup\limits_{N=1}^{\infty} (\sum_{n=1}^N \operatorname{conv} U_\rho(\varepsilon_n))$ is a base of neighbourhoods of zero for τ_c. Since $\tau_c \subset \tau_\rho^\wedge$, by (i) we have $\tau_c \subset \tau(X,X_n^\sim)$. On the other hand, since $\tau(X,X_n) \subset \tau_\rho^\wedge$, it is easy to see that $\tau(X,X_n^\sim) \subset \tau_c$. Thus $\tau_c = \tau(X,X_n^\sim)$.

(iii) If ρ is convex, then by Theorem 1.6 the modular topology $\tau_\rho^\wedge$ is locally convex. Therefore, by (i) $\tau_\rho^\wedge = \tau(X,X_n^\sim)$. ∎

For a convex modular ρ the modular topology $\tau_\rho^\wedge$ will be investigated in detail in the second part of this paper ([**17**]).

As an application of Theorem 2.4 we have the following result.

Theorem 2.6. *Let (X,ρ) be an Orlicz lattice with the normal dual $X_n^\sim$ separating the points of X. Suppose that a modular ρ satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property. Then for a linear functional f on X the following statements are equivalent:*

(i) *f is order continuous.*

(ii) *f is continuous for the weak topology $\sigma(X, X_n^\sim)$.*
(iii) *f is sequentially continuous for $\sigma(X, X_n^\sim)$.*
(iv) *f is sequentially continuous for the absolute weak topology $|\sigma|(X, X_n^\sim)$.*
(v) *f is continuous for $|\sigma|(X, X_n^\sim)$.*

Proof : Since $(X, |\sigma|(X, X_n^\sim))^* = X_n^\sim$ (see [1], Theorem 6.6), it suffices to show that (iv)$\Rightarrow$ (v). Indeed, in view of ([1], Theorems 6.6 and 9.1) $|\sigma|(X, X_n^\sim)$ is a Lebesgue topology. Therefore according to Theorem 2.2, $|\sigma|(X, X_n^\sim) \subset \tau_\rho^\wedge$. Thus f is sequentially continuous for $\tau_\rho^\wedge$, so by Theorem 2.4 f is continuous for $|\sigma|(X, X_n^\sim)$. ■

3. Modular topology on Orlicz lattices of measurable functions.

Assume that $(\Omega, \sum, \mu)$ is a measure space. Let L^0 denote the set of equivalence classes of all real valued μ-measurable functions defined and finite a.e. on Ω. Then L^0 is σ-universally complete Riesz space under the ordering $x \leq y$ whenever $x(t) \leq y(t)$ a.e. on Ω ([1], Definition 23.17). The family of Riesz pseudonorms $\{p_A : A \in \sum, \mu(A) < \infty\}$, where

$$p_A(x) = \int_A |x(t)|(1 + |x(t)|)^{-1}\, d\mu \quad \text{for } x \in L^0,$$

generates the σ-Lebesgue topology τ_0 on L^0 ([1], Theorem 24.1). It is known that τ_0 is the topology of convergence in measure on the μ-measurable subsets of Ω whose measure is finite. Let us note that if μ is locally finite (i.e., for every $A \in \sum, \mu(A) > 0$ there exists $B \in \sum$ such that $B \subset A$ and $0 < \mu(B) < \infty$), then τ_0 is a Hausdorff topology. It is well known that if the measure μ is σ-finite, then τ_0 is generated by the Riesz F-norm

$$\|x\|_0 = \int_\Omega |x(t)|(1 + |x(t)|)^{-1} f(t)\, d\mu \quad \text{for } x \in L^0,$$

where a function $f : \Omega \to (0, \infty)$ is μ-measurable and $\int_\Omega f(t)\, d\mu = 1$ ([7], Ch.I., §6). We have the following result.

Theorem 3.1. *Let X be an ideal of L^0 and let ρ be a modular on X that satisfies the σ-Fatou property and the σ-Levi property. Then the modular topology $\tau_\rho^\wedge$ on X is finer than the topology τ_0 restricted to X. In particular, if the measure μ is locally finite, then $\tau_\rho^\wedge$ is a Hausdorff topology.*

Proof : Assume that $x_n \xrightarrow{(\rho)} 0$ in X. Then by Theorem 2.1 (ii) we have $x_n \xrightarrow{(0)} 0$ in X, so also $x_n \xrightarrow{(0)^*} 0$ in L^0, and this means that $x_n \xrightarrow{\tau_0} 0$. According to Theorem 1.4 we get $\tau_\rho^\wedge \supset \tau_{0|X}$. ■

Given an ideal X of L^0 we will denote by X' the Köthe dual of X, i.e.,

$$X' = \{y \in L^0 : \operatorname{supp} y \subset \operatorname{supp} X (\operatorname{mod} \mu) \text{ and } xy \in L^1 \quad \text{for } x \in X\}.$$

According to Theorem 2.4 and ([5], Ch.VI.§1, Theorem 1) we have:

Theorem 3.2. *Let X be an ideal of L^0 and let the measure μ be σ-finite. Suppose that a modular σ on X satisfies the σ-Lebesgue property, the σ-Fatou property and the σ-Levi property. Then for a linear functional f on X the following statements are equivalent:*

(i) *f is continuous for $\tau_\rho^\wedge$.*
(ii) *f is sequentially modular continuous.*
(iii) *f is sequentially order continuous.*
(iv) *There exists a unique $y \in X'$ such that*

$$f(x) = \int_\Omega x(t)y(t)\,d\mu \quad \text{for all } x \in X.$$

Examples.

A. The well known examples of Orlicz lattices of measurable functions are Orlicz spaces (see [**9**], [**11**]) and Musielak–Orlicz spaces (see [**13**], [**22**]). It is known that the usual modular

$$\rho_\varphi(x) = \int_\Omega \varphi(|x(t)|, t)\,d\mu$$

has on the Musielak–Orlicz space L^φ the σ–Fatou property, and since $Bd(\tau_\rho^\vee) = Bd(\rho_\varphi)$ (see Theorem 1.2) in view of ([**22**], Theorem 1.1), ρ_φ has the σ–Levi property. Moreover, by the Lebesgue Dominated Convergence theorem, ρ_φ has on L^φ the σ–Lebesgue property. Therefore, according to Theorem 2.1(v), ρ_φ has the Levi property.

The modular topology $\tau_\rho^\wedge$ on Orlicz spaces L^φ is investigated in detail in the author's papers ([**14**], [**15**]).

B. Let $(\Omega, \sum, \mu)$ be a σ–finite measure space and let L^∞ be the space of essentially bounded μ–measurable functions. We denote by τ_∞ the topology on L^∞ generated by the B–norm: $\|x\|_\infty = \operatorname*{esssup}_{t\in\Omega}|x(t)|$. It is well known that $\|\cdot\|_\infty$ is not order continuous, whenever Ω does not consist of only finite number of atoms. Let us put

$$\rho_\infty(x) = \begin{cases} \|x_0\| & \text{if } x \in L^\infty \text{ and } \|x\|_\infty \le 1, \\ \infty & \text{if } x \in L^\infty \text{ and } \|x\|_\infty > 1. \end{cases}$$

It is easy to verify that ρ_∞ is a modular on L^∞. The following result is known ([8], p.922).

Theorem 3.3. *The modular topology $\tau_{\rho_\infty}^\wedge$ coincides with the mixed topology $\gamma(\tau_\infty, \tau_{0|L^\infty})$ ([**21**]).*

The next theorem characterizes the sequential modular convergence in (L^∞, ρ_∞).

Theorem 3.4. *For a sequence (x_n) in L^∞ the following statements are equivalent:*

(i) $x_n \xrightarrow{(\rho_\infty)} 0$.
(ii) *$\|x_n\|_0 \to 0$ and $|x_n(t)| \le K$ a.e. on Ω for some $K > 0$.*
(iii) *$x_n \to 0$ for $\tau_{\rho_\infty}^\wedge$.*

PROOF : (i)$\Leftrightarrow$ (ii) Obvious.

(iii)$\Leftrightarrow$ (ii) It is known that $\|\cdot\|_\infty$ satisfies both the σ–Fatou property and the σ–Levi property ([5], Ch.IV,§3.3). Hence by ([5] Ch.IV, §3, Lemma 5) the balls $B(r)=\{x\in L^\infty : \|x\|_\infty \le r\}, r>0$ are closed in (L^0,τ_0). Therefore, according to ([22], Theorem 2.6.1) and Theorem 3.3 we have (ii)$\Leftrightarrow$ (iii). ■

On the other hand, we have the following result.

Theorem 3.5. *The modular ρ_∞ is not metrizing.*

PROOF : Since $\tau_{0|L^\infty}$ is strictly coarser than τ_∞, we can choose a sequence (x_n^0) in L^∞ such that $\|x_n^0\|_0 \to 0$ and $\|x_n^0\|_\infty \not\to 0$. Therefore there exists a number $\varepsilon_0>0$ and a subsequence $(x_{k_n}^0)$ of (x_n^0) such that $\|x_{k_n}^0\|_\infty > \varepsilon_0$. Taking $y_n^0=(1/\varepsilon_0)x_{k_n}^0$ we have $\|y_n^0\|_\infty > 1$. Denoting by $z_n^0=(1/\|y_n^0\|_\infty)y_n^0$ we have $\|z_n^0\|_\infty=1$, and since $\|z_n^0\|_0 \le \|y_n^0\|_0$, we get $\rho_\infty(z_n^0)=\|z_n^0\|_0 \to 0$. On the other hand, $\rho_\infty(2z_n^0)=\infty$, and this means that the modular ρ_∞ is not metrizing. ■

Theorem 3.6. *The modular ρ_∞ satisfies the σ–Lebesgue property, the σ–Fatou property and the σ–Levi property.*

PROOF : Since τ_0 is Lebesgue topology, it is easy to see that ρ_∞ satisfies the σ–Lebesgue property. We shall now show that ρ_∞ satisfies the σ–Fatou property. Indeed, let $x_n \uparrow x$ in L^∞. If $\|x\|_\infty \le 1$, then $\rho_\infty(x_n)\uparrow \rho_\infty(x)$, because τ_0 is a σ–Fatou topology. If $\|x\|_\infty > 1$, then $\rho_\infty(x)=\infty$. Assume that $\rho_\infty(x_n)\le K$ for some $K>0$ and $n=1,2,\ldots$. Then $\|x_n\|_\infty \le 1$. Since $\|x_n-x\|_0 \to 0$, by ([5], Ch.IV, §3, Lemma 4) we have $\liminf\limits_{n\to\infty}\|x_n\|_\infty \ge \|x\|_\infty > 1$, because $\|\cdot\|_\infty$ satisfies the σ–Fatou property. This contradiction establishes that ρ_∞ satisfies the σ–Fatou property. At last, w shall show that ρ_∞ satisfies the σ–Levi property. Combining Theorem 3.3, Theorem 3.4 and ([21], Theorem 2.4.1), we get $Bd(\rho)=Bd(\tau_\rho^\wedge)=Bd(\gamma(\tau_\infty,\tau_{0|L^\infty}))=Bd(\tau_\infty)$. Since τ_∞ is a σ–Levi topology, the prof is finished. ■

Combining Theorems 2.2, 3.3 and 3.6, we have:

Corollary 3.7. *The mixed topology $\gamma(\tau_\infty,\tau_{0|L^\infty})$ is the finest Lebesgue topology on L^∞.*

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