Lubica Kossaczká

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Abstract. The integral manifold for the problem

$$\frac{du}{dt} + Au = Lu_t + \varepsilon F(t, u_t)$$
$$u_0 = \varphi$$
$$u(0) = x$$

is constructed, where L is a linear operator, u_t denotes the deviation of u and A is a sectorial operator.

Keywords: Functional differential equations, parabolic equations with delay, sectorial operator, integral manifold

Classification: 35R10, 34K30

§1. Introduction and results.

The appropriate tool for studying the dynamical systems are invariant manifolds. Their main meaning is in the fact that they generally enable us to reduce the infinite dimension of the investigated problem to the finite one. Invariant (or integral) manifolds were studied, e.g. in [**Ha**, 1], [**Ke**], [**Pl**],[**Mit**]. Invariant manifolds of the ordinary differential equations with deviation were also investigated in [**Ha 2**], [**Fo**]. The present paper deals with the functional differential equation

$$\text{(E)} \qquad \begin{aligned} \frac{du}{dt} + Au &= Lu_t + \varepsilon F(t, u_t) \\ u_0 &= \varphi \\ u(0) &= x \end{aligned}$$

We denote u_t the translation of u, given by $u_t(s) = u(t+s)$ for $s \in R^- = (-\infty, 0), t \in R^+ = \langle 0, \infty)$. The mild solutions $(u(t), u_t)$ under some assumptions on F can be expressed in the form (see [**Pe-Mi**])

$$z(t) = (u(t), u_t) = T(t+\sigma)z(\sigma) + \varepsilon \int_\sigma^t T(t-s)[F(s, u_s), 0]\, ds.$$

We shall investigate the existence of integral manifold of (E) in the form

$$\widetilde{S} = \{(t, y, \xi); y \in R^n, t \in R, \xi = g(t, y)\},$$

where g will be some appropriate function. This problem is similar to that of [Fo]. There the integral manifold of ODE with deviating argument

$$\frac{dx}{dt} = f(x_t) + \varepsilon F(t, x_t)$$

has been constructed, where f is a linear continuous operator.

So, applying the technique similar as in [Fo] and the result of [Pe-Mi], we get the desirable integral manifold.

§2. Assumptions and denotations.

We shall deal with the problem (E). Suppose that X is a Banach space and A is a sectorial operator in X with compact resolvent (for the definition see [He] or [Mi,1]).

Let $R^- = (-\infty, 0), R^+ = \langle 0, \infty)$.

Let Y be a space of strongly measurable functions on R^- with values in X such that $\int_{-\infty}^0 e^{\gamma s}|\varphi(s)|\,ds$ for some fixed $\gamma > 0$ is finite. Then we define $\|\varphi\|_Y = \int_{-\infty}^0 e^{\gamma s}|\varphi(s)|\,ds$. Let $L : Y \to X$ be a continuous linear operator. Denote, similarly as [Mi,2], $T(t) : Z \to Z$ a C_0 semigroup on the space $Z = X \times Y$ such that $T(t) : (x, \varphi) \to (u(t), u_t)$, where u is a solution of (E) with $\varepsilon = 0$. Let $F : R \times Y \to X$ be a continuous function such that $|F(t, u)|_X \leq K$ for $(t, u) \in R \times Y$ and $|F(t, u_1) - F(t, u_2)|_X \leq \operatorname{Lip} F|u_1 - u_2|_Y$ for each $t \in R, u_1, u_2 \in Y$.

§3.

Let B be an infinitesimal generator of the semigroup $T(t)$. According to ([Mi,2], [Pe-Mi])

$\sigma^1 = \{\lambda, \operatorname{re}\lambda > -\gamma + \varepsilon_1\} \cap \sigma(B)$ is a finite spectral set and

$\sigma^2 = \{\lambda, \operatorname{re}\lambda \leq -\gamma + \varepsilon_1\} \cap \sigma(B)$ is a spectral set. Let P_1, P_2 denote the corresponding projections and $Z_1 = P_1 Z, Z_2 = P_2 Z$. The projection P_1 is finite dimensional and we have $Z = Z_1 \oplus Z_2$. So, according to [Mi,2] we can choose such $0 < a < b$ that

$$\operatorname{re}\sigma(B)/Z_1 > -a + \varepsilon_0 > -b > \operatorname{re}\sigma(B)/Z_2. \tag{1}$$

There exists such a base of the finite dimensional space Z_1 that

$$\begin{array}{ll} B\varphi_1^1 = \lambda_1\varphi_1^1 & B\varphi_1^2 = \lambda_2\varphi_1^2 \\ B\varphi_2^1 = \lambda_1\varphi_2^1 + \varphi_1^1 & B\varphi_2^2 = \lambda_2\varphi_2^2 + \varphi_1^2 \ldots \\ \vdots & \\ B\varphi_k^1 = \lambda_1\varphi_k^1 + \varphi_{k-1}^1 & \end{array}$$

which means that this base corresponds to Jordan decomposition of the subspace Z_1.

Hence, if we generally denote this base as $(\varphi_1, \ldots \varphi_n) = \Phi$, then there exists such a matrix $\widetilde{B}$ that

$$B\Phi = \Phi\widetilde{B}, \text{ with } \sigma(\widetilde{B}) = \sigma(B/Z_1). \tag{2}$$

From the fact that $BT(t)\Phi = T(t)B\Phi = T(t)\Phi\widetilde{B}$ we have

$$T(t)\Phi = \Phi e^{\widetilde{B}t}. \tag{3}$$

From (1) and (3) the following estimates take place

$$\begin{aligned} |P_2 T(t)| &\le Ce^{-bt} \quad \text{for } t \ge 0 \\ |e^{\widetilde{B}t}| &\le Ce^{-at} \quad \text{for } t \le 0. \end{aligned} \tag{4}$$

Definition of a mild solution.

A function $u : (-\infty, T) \to X$ is called a mild solution on the interval $\langle 0, T)$, if u satisfies the initial condition, its restriction to $\langle 0, T)$ belongs to the space $C(\langle 0, T), X)$ and $u(t) = e^{-At}x + \int_0^t e^{-A(t-s)}(Lu_s + \varepsilon F(s, u_s))\,ds$ holds for $t \in \langle 0, T)$. According to [**Pe**,Mi] under assumptions on F for the solution $u(t)$, the following formula takes place

$$z(t) = (u(t), u_t) = T(t)(x, \varphi) + \varepsilon \int_0^t T(t-s)[F(s, u_s), 0]\,ds \tag{5}$$

where $[F(s, u_s), 0] \in Z$.

The expression of the element of Z.

Let $\Psi_1, \ldots, \Psi_n$ be linear continuous operators from Z into R such that $\operatorname{Ker}\Psi_i = \varphi_1 \oplus \ldots \varphi_{i-1} \oplus \varphi_{i+1} \oplus \ldots \varphi_n \oplus Z_2$ and $\Psi_i(\varphi_i) = 1$ for each $i \in \{1, \ldots n\}$. Each element $z \in Z$ can be expresses in the form $z = z_1 + z_2$, where $z_1 = \Phi b, b \in R^n, z_1 \in Z_1, z_2 \in Z_2$. Then $\Psi(z) = \Psi(\Phi b) + \Psi(z_2) = b + 0$, $\quad P_1 z = \Phi(\Psi(z))$. Then from (5) we get the following formulas

$$\begin{aligned} z_1(t) &= T(t-\sigma)z_1(\sigma) + \varepsilon \int_\sigma^t T(t-s)P_1[F(s, u_s), 0]\,ds \\ z_2(t) &= T(t-\sigma)z_2(\sigma) + \varepsilon \int_\sigma^t T(t-s)P_2[F(s, u_s), 0]\,ds \end{aligned} \tag{6}$$

Denoting $y(t) = \Psi(z(t))$ (so $y \in R^n$), from the first equation in (6) we have

$$\begin{aligned} \Phi y(t) &= T(t-\sigma)\Phi\Psi(z(\sigma)) + \varepsilon \int_\sigma^t T(t-s)\Phi(\Psi[F(s, u_s), 0])\,ds = \\ &= \Phi e^{\widetilde{B}(t-\sigma)}\Psi(z(\sigma)) + \varepsilon \int_\sigma^t \Phi e^{\widetilde{B}(t-s)}\Psi[F(s, u_s), 0]\,ds. \end{aligned} \tag{7}$$

We define

$$\begin{aligned} &h_1 : R \times X \times Y \to R^n \qquad \text{and} \\ &h_2 : R \times X \times Y \to Z_2 \qquad \text{such that} \\ &h_1(s, x, y) = \Psi[F(s, y), 0] \qquad \text{and} \\ &h_2(s, x, y) = P_2[F(s, y), 0]. \end{aligned}$$

As for as Ψ, P_2 are continuous and linear and F is bounded and Lipschitz continuous in the second variable, h_1 and h_2 are bounded and Lipschitz continuous in the third variable, too. Let H_1 be a constant such that $|h_1(s,x_1,y_1)-h_1(s,x_2,y_2)|_{R^n} \leq H_1\|y_1-y_2\|_Y$ and $|h_1(s,x,y)| \leq H_1$; and similarly h_2.

(7) implies that

$$y(t)=e^{\widetilde{B}(t-\sigma)}y(\sigma)+\varepsilon\int_\sigma^t e^{\widetilde{B}(t-s)}h_1(s,z(s))\,ds. \tag{8}$$

Hence $y(t)=\Psi(z(t))$ satisfies the following conditions

$$\begin{aligned} \frac{dy}{dt} &= \widetilde{B}y(t)+\varepsilon h_1(t,z(t)),\\ y(0) &= \Psi(z(0))=\Psi([x,\varphi]). \end{aligned} \tag{9}$$

Definition (analogous to [Fo]). Let $L(\rho,\gamma)=\{g : R\times R^n \to Z_2; |g(t,y_1)-g(t,y_2)|_{Z_2} \leq \gamma|y_1-y_2|_{R^n}$ and $|g(t,y)| \leq \rho$ for each $t\in R, y_1,y_2,y\in R^n\}$. Let $y(t;\sigma,y_0)$ be a solution of the problem

$$\begin{aligned} \frac{dy}{ds} &= \widetilde{B}y+\varepsilon h_1(s,\Phi y(s)+g(s,y(s)),\\ y(\sigma,\sigma,y_0) &= y_0, \sigma\in(-\infty,\infty), y_0\in R^n. \end{aligned}$$

We say that $\widetilde{S}=\{(t,y,\xi); t\in(-\infty,\infty), y\in R^n, \xi=g(t,y)\}$ is an integral manifold for the system

$$\begin{aligned} \frac{dy}{dt} &= \widetilde{B}y+h_1(t,\Phi y(t)+z_2(t))\\ z_2(t) &= T(t-\sigma)z_2(\sigma)+\varepsilon\int_\sigma^t T(t-s)[h_2(s,\Phi y(s)+z_2(s))]\,ds, \end{aligned} \tag{10}$$

if from the fact that $y(t,\sigma,y_0)$ is a solution of the equation

$$\frac{dy}{dt}=\widetilde{B}y+\varepsilon h_1(t,\Phi y(t)+g(t,y)) \text{ for each } \sigma\in(-\infty,\infty), y(\sigma,\sigma,y_0)=y_0,$$

it follows that $y(t)=y(t,\sigma,y_0)$, $z_2(t)=g(t,y(t))$ is a solution of the system (10).

Remark 1. (similar to [Fo])

If $\widetilde{S}$ is an integral manifold for the system (10), described by the function g, then

$$S=\{(t,\Phi y+\xi); t\in(-\infty,\infty), y\in R^n, \xi=g(t,y), \xi\in Z_2\}$$

is an integral manifold for the equation (6):

$$z(t)=T(t-\sigma)z(\sigma)+\varepsilon\int_\sigma^t T(t-s)[F(s,x_s),0]\,ds.$$

Now we want to get a simpler expression for the integral manifold $\widetilde{S}$ of the system (10), described by the function g. Denote $Y(s) = y(s,t,y_0,g)$.

Then $g(t,y_0) = T(t-\sigma)z_2(\sigma) + \varepsilon \int_\sigma^t T(t-s)h_2(s, \Phi Y(s) + g(s,Y(s))\, ds$. From (4) and the fact that $|z_2(\sigma)|_Z$ is bounded, we have $g(t,y_0) = \varepsilon \int_{-\infty}^t T(t-s)h_2(s,\Phi Y(s) + g(s,Y(s)))\, ds$. Let U be the operator such that

$$Ug(t,y_0) = \varepsilon \int_{-\infty}^t T(t-s)[h_2(s,\Phi Y(s) + g(s,Y(s)))]\, ds.$$

We show that for sufficiently small ε the operator U maps $L(\rho,\gamma)$ into itself and that $U : L(\rho,\gamma) \to L(\rho,\gamma)$ has a unique fixed point.

Theorem 1. *Let the assumption from §2 be fulfilled. For each $\rho > 0, \gamma > 0$, there exists an $\varepsilon_0 > 0$ such that to each $\varepsilon, 0 < \varepsilon < \varepsilon_0$, there exists a unique $g \in L(\rho,\gamma)$ with the property that $\widetilde{S}_g$ is an integral manifold of the system (10).*

PROOF : The proof is very similar to those of [**Fo**]. We have already shown that if there exists an integral manifold of the system (10) generated with the function g, then this function is a solution of the equation $Ug = g$.

Let g be a solution of this equation. We show that the set $\widetilde{S}_g = \{(t,y,g(t,y)), y \in R^n, t \in R\}$ fulfils the definition of an integral manifold for the system (10) and, thus, $S = \{(t, \Phi y + g(t,y)), t \in R, y \in R^n\}$ is a manifold for the equation (6). To that aim, it is necessary to show that $z_2(t) = g(t, y(t,r,y_0,g))$ is a solution of the equation

$$z_2(t) = T(t-\sigma)z_2(\sigma) + \varepsilon \int_\sigma^t T(t-s)h_2(s, \Phi y(s,r,y_0,g) + z_2(s))\, ds.$$

According to the definition of z_2 we have

$$\begin{aligned} z_2(t) = g(t,y(t,r,y_0,g)) &= \varepsilon \int_{-\infty}^t T(t-s)h_2(s, \Phi y(s,t,y(t,r,y_0,g),g) + \\ &\quad + g(s, y(s,t,y(t,r,y_0,g),g)))\, ds = \\ &= \varepsilon T(t-\sigma) \int_{-\infty}^\sigma T(\sigma - s)h_2[s, \Phi y(s,\sigma,y(t,r,y_0,g),g) + \\ &\quad + g(s,y(s,\sigma,y(\sigma,r,y_0,g))]\, ds + \varepsilon \int_\sigma^t T(t-s)h_2[s, \Phi y(s,r,y_0,g) + \\ &\quad + g(s,y(s,r,y_0,g))]\, ds, \end{aligned}$$

which we have to prove.

On the other hand, we show that to each $\rho > 0, \gamma > 0$ there exists an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, we have a unique solution of $Ug = g$.

First we must estimate $|y(s,t,y_0,g) - y(s,t,\overline{y}_0,\overline{g})|$ for $s \le t$: Denote $\overline{y}(s) =:$

$y(s,t,\overline{y}_0,\overline{g})$, then we have

$$|y(s)-\overline{y}(s)| = |e^{\widetilde{B}(s-t)}(y_0-\overline{y}_0)+\varepsilon\int_t^s e^{\widetilde{B}(s-r)}[h_1(r,\Phi y(r)+$$
$$+g(r,y)) - h_1(r,\Phi\overline{y}(r)+\overline{g}(r,\overline{y}(r))]\,dr|_{R^n} \leq$$
$$\leq Ce^{-a(s-t)}|y_0-\overline{y}_0| + C\varepsilon H_1|\int_t^s e^{-a(s-r)}[(|\Phi|+\gamma)(|y(r)-\overline{y}(r)|)+\|g-\overline{g}\|]\,dr =$$
$$= Ce^{-a(s-t)}|y_0-\overline{y}_0| + C\varepsilon H_1|\int_t^s e^{-a(s-r)}[(|\Phi|+\gamma)(|y(r)-\overline{y}(r)|)\,dr+$$
$$+C\varepsilon H_1|\int_t^s e^{-a(s-r)}\,dr|\,\|g-\overline{g}\| = Ce^{-a(s-t)}|y_0-\overline{y}_0|+$$
$$+\frac{C\varepsilon H_1}{a}\|g-\overline{g}\|(e^{a(t-s)}-1)+C\varepsilon H_1(|\Phi|+\gamma)|\int_t^s e^{-a(s-r)}|y(r)-\overline{y}(r)|\,dr|$$

and hence

$$|y(s)-\overline{y}(s)|\cdot e^{a(s-t)} \leq C|y_0-\overline{y}_0| + \frac{C\varepsilon H_1}{a}\|g-\overline{g}\| - \frac{C\varepsilon H_1}{a}e^{a(s-t)}\|g-\overline{g}\|+$$
$$C\varepsilon H_1(|\Phi|+\gamma)|\int_t^s e^{a(r-t)}|y(r)-\overline{y}(r)|\,dr|.$$

Put $|y(s)-\overline{y}(s)|\cdot e^{a(s-t)} = w(s)$ and

$$X = C(|y_0-\overline{y}_0|) + \frac{C\varepsilon H_1}{a}\|g-\overline{g}\|)$$
$$Y = \frac{C\varepsilon H_1\|g-\overline{g}\|}{a}$$
$$Z = C\varepsilon H_1(|\Phi|+\gamma).$$

Then we have $w(s) \leq X - Ye^{a(s-t)} + Z|\int_t^s w(r)\,dr|$. Solving this we come to the inequality

$$w(s)+k_1e^{a(s-t)} \leq (X-\frac{Zk_1}{a}) + Z|\int_t^s w(r)+k_1e^{-a(t-r)}\,dr|,$$

where $k_1 = \frac{Ya}{a+Z}$. Thus by the Gronwall inequality for $s \leq t$ we get $w(s)+k_1e^{-a(t-s)} \leq (X-\frac{Zk_1}{a})e^{Z(t-s)}$, from where it follows that

$$|y(s,t,y_0,g)-y(s,t,\overline{y}_0,\overline{g})| \leq (X-\frac{Zk_1}{a})e^{(-Z-a)(s-t)} \leq \tag{11}$$
$$Ce^{-C\varepsilon H_1[(|\Phi|+\gamma)-a](s-t)}\cdot(|y_0-\overline{y}_0|+\frac{C\varepsilon H_1}{a}\|g-\overline{g}\|),$$

$$|Ug(t,y_0)-U\overline{g}(t,\overline{y}_0)| = |\varepsilon\int_\infty^t T(t-s)[h_2(s,\Phi y(s,t,y_0,g)+ \tag{12}$$

$$+g(s,y(s,t,y_0,g))) - h_2(s,\Phi y(s,t,\overline{y}_0,\overline{g}) + \overline{g}(s,y(s,t,\overline{y}_0,\overline{g}))]\,ds| \le$$
$$\le \varepsilon CH_2 \int_{-\infty}^{t} e^{-b(t-s)}[(|\Phi|+\gamma)|y(s)-\overline{y}(s)| + \|g-\overline{g}\|\,ds| \le$$
$$\le \varepsilon CH_2 \int_{-\infty}^{t} e^{-b(t-s)}\,ds\|g-\overline{g}\| + \varepsilon CH_2(|\Phi|+\gamma)\int_{-\infty}^{t} e^{-b(t-s)}|y(s)-\overline{y}(s)|\,ds =$$
$$\varepsilon CH_2 \int_{-\infty}^{t} e^{-b(t-s)}\,ds\|g-\overline{g}\| + \varepsilon C^2H_2(|\Phi|+\gamma)\int_{-\infty}^{t} e^{(-b+a+C\varepsilon H_1)(t-s)}(|y_0-\overline{y}_0|+$$
$$+\frac{C\varepsilon H_1}{a}\|g-\overline{g}\|)\,ds \le \varepsilon CH_2\frac{\|g-\overline{g}\|}{b} + \frac{C^2\varepsilon H_2(|\Phi|+\gamma)}{b-a-C\varepsilon H_1}(|y_0-\overline{y_0}| + \frac{C\varepsilon H_1}{a}\|g-\overline{g}\|).$$

Also,

$$|Ug(t,y_0)| \le \varepsilon CH_2 \int_{-\infty}^{t} e^{-b(t-s)}\,ds = \frac{\varepsilon CH_2}{b} \tag{13}$$

From (12) and (13) it is clear that for sufficiently small ε $U(L(\rho,\gamma)) \subset L(\rho,\gamma)$ and $Ug = g$ has a unique solution. ∎

Remark 2. Dealing with the problem

$$\frac{du}{dt} + Au = Lu_t + f(u_t)$$
$$u_0 = \varphi$$
$$u(0) = x$$

where L, X, Y be as in §2., and $f : Y \to X$ be a Lipschitz continuous bounded function, we can formulate a solution in the form

$$z(t) = (u(t), u_t) = T(t-\sigma)z(\sigma) + \int_{\sigma}^{t} T(t-s)[f(u_s),0]\,ds, \tag{14}$$

where T is defined in §2. Then the semigroup $T(t)$ satisfies all assumptions $H_1 - H_4$ from [**Pu**]. If we define $F(x,z) = (f(z),0)$ a function $Z \to Z$, all assumptions for the existence of the center unstable manifold and foliation (see [**Pu**]) are satisfied and this problem is a special case of the problem studied in [**Pu**]. Of course, the main idea to get this manifold was based on the formula (14) from [**Mi**,2].

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ÚAMVT UK, Mlynská dolina, 842 15 Bratislava, Czechoslovakia

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