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Remarks on Krasnoselskii bifurcation theorem

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Abstract. We give a generalization of the classical bifurcation theorem from eigenvalues of odd multiplicity to the case were the nonlinearity need not be Frechet differentiable.

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Classification: 34B15, 47H15

Introduction.

The following assumptions will be made throughout this paper:

H1) X is a real Banach space with norm denoted $\|\cdot\|$;

H2) $T : D(T) \to X$ is a linear operator with compact resolvent, i.e. $(T-\mu I)^{-1}$ is compact for some (and hence for all) μ not belonging to the spectrum $\sigma(T)$ of T;

H3) $F : X \to X$ is a continuous operator mapping bounded sets onto bounded sets and such that $F(0) = 0$;

H4) μ_0 is an eigenvalue of T of odd algebraic multiplicity.

A classical result due to Krasnoselskii [4] asserts that if F is Frechet differentiable at 0 with $F'(0) = 0$, then μ_0 is a bifurcation point for the problem

$$Tu + F(u) = \mu u \qquad u \in D(T) \tag{1}$$

in other words, given any $\varepsilon > 0$, there exists an eigenvalue–eigenfunction pair (μ, u) of (1) with $|\mu - \mu_0| < \varepsilon$ and $0 < \|u\| \le \varepsilon$.

Notice that, due to the assumption on T, every eigenvalue of T is necessarily isolated and of finite multiplicity, and the whole spectrum $\sigma(T)$ consists of only such points.

It is our aim to give here a generalization of this result to the case where F is not necessarily differentiable, but is merely assumed to be sublinear in a neighborhood of 0. Precisely, for $r > 0$ we set

$$k(F;r) = \sup_{\|u\|\le r} \frac{\|F(u)\|}{\|u\|} \tag{2}$$

assuming that it is finite at least for small values of r. As $k = k(F;r)$ is a nondecreasing function of r, we also set

$$k_0(F) = \lim_{r\to 0} k(F;r) \tag{3}$$

and remark at once that F is differentiable at 0 with $F'(0) = 0$ if and only if $k_0(F) = 0$.

By using coincidence degree [5], we first show (Theorem 1) that bifurcation near μ_0 occurs if there exist $\underline{\mu} < \overline{\mu}$ with $[\underline{\mu}, \overline{\mu}] \cap \sigma(T) = \{\mu_0\}$ and

$$\min\{\|(T - \underline{\mu}I)^{-1}\|^{-1}, \|(T - \overline{\mu}I)^{-1}\|^{-1}\} > k_0(F).$$

Note that when $F'(0)$ exists and is equal to zero, the condition above is trivially satisfied and Theorem 1 reduces to Krasnoselskii's result.

When T is a selfadjoint operator in a Hilbert space, our generalization becomes even more apparent, for we prove that, if μ_0 has *isolation distance* (Kato [3] p. 273)

$$\text{dist}(\mu_0, \sigma(T) \setminus \{\mu_0\}) > 2k_0(F),$$

then bifurcation takes place from the interval $[\mu_0 - k_0(F), \mu_0 + k_0(F)]$, in a sense made precise by the statement of Theorem 3.

Existence and bifurcation theorems without differentiability assumptions on the nonlinear term were proved by other authors (see in particular Schmitt/Smith [6] and the references therein), but the above result seems to be new. We have to mention that, in the line of Rabinowitz [7], these authors also gave *global* bifurcation results, i.e. they proved the existence of continua (maximal connected sets) of nontrivial solutions meeting the line $(\mu, 0)$ of trivial solutions within the bifurcation interval. This extension could be done in our case as well, but we have preferred to bound ourselves to show the main idea in the simplest (i.e. local) setting.

On the other hand, we remark that an explicit dependence on r in $k(F; r)$ has not been considered up to now. That this is not a vacuous assumptions can be seen form the study of a class of integrodifferential equations (on compact intervals of R) that we consider in Section 3 as application of Theorem 3; these problems are not covered in the references cited above. It is clear that the same study, with only minor modifications, can be performed e.g. for the problem (in a bounded open set Ω of R^n)

$$\begin{cases} -\Delta u(x) + u(x) \displaystyle\int_\Omega k(x,y) f(y, u(y))\, dy = \mu u(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with f subquadratic and μ near an eigenvalue of odd multiplicity of $-\Delta$, subject to zero Dirichlet boundary conditions on $\partial\Omega$.

Main results.

For $r > 0$, we let $B_r := \{u \in D(T) : \|u\| \leq r\}; \partial B_r$ will denote the boundary of B_r. Recall that if H1) and H2) hold, then the *coincidence degree* of the couple (T, N) with respect to B_r, denoted $d[(T, N), B_r]$, will be defined for any map $N : X \to X$ satisfying H3), provided $T(u) \neq N(u)$ for $u \in \partial B_r$ (Gaines–Mawhin [5], Chapter iii). We shall set for convenience

$$d(T - N, B_r) := d[(T, N), B_r].$$

Moreover, for $\mu \notin \sigma(T)$ we set

$$c(\mu) = \inf\{\|Tu - \mu u\|, u \in D(T), \|u\| = 1\} = \|(T - \mu I)^{-1}\|^{-1}. \tag{4}$$

Theorem 1. *Assume there exist $\underline{\mu}, \overline{\mu}$ such that:*

i) *$\underline{\mu} < \mu_0 < \overline{\mu}$ and $\sigma(T) \cap [\underline{\mu}, \overline{\mu}] = \{\mu_0\}$;*

ii) *$\min\{c(\underline{\mu}), c(\overline{\mu})\} > k_0(F)$*

where $c(\mu)$ and $k_0(F)$ are as in (4), (3) respectively. Then, for all sufficiently small $r > 0$ (1) has an eigenvalue–eigenfunction pair (μ_r, u_r) with $\|u_r\| = r$ and $\mu_r \in [\underline{\mu}, \overline{\mu}]$.

PROOF : By definition of $k_0(F)$, there exists $r_0 > 0$ such that

$$\min\{c(\underline{\mu}), c(\overline{\mu})\} > k(F; r) \tag{5}$$

for all $r : 0 < r \leq r_0$. We show that the assertion is true for all such r. For, fix $r : 0 < r \leq r_0$ and assume by contradiction that

$$T(u) + F(u) \neq \mu u$$

for all $\mu \in [\underline{\mu}, \overline{\mu}]$ and all $u \in D(T) : \|u\| = r$. Then, by homotopy invariance of the degree

$$d(T + F - \underline{\mu} I, B_r) = d(T + F - \overline{\mu} I, B_r). \tag{6}$$

On the other hand, for all $t \in [0,1]$ and all $u \in D(T) : \|u\| \leq r$,

$$\begin{aligned} \|Tu + tF(u) - \underline{\mu} u\| &\geq \|Tu - \underline{\mu} u\| - t\|F(u)\| \\ &\geq c(\underline{\mu})\|u\| - \|F(u)\| \\ &\geq [c(\underline{\mu}) - k(F; r)]\|u\|. \end{aligned}$$

Therefore, form (5) we get

$$Tu + tF(u) \neq \underline{\mu} u$$

for all $u \in D(T)$ with $0 < \|u\| \leq r$ and all $t \in [0,1]$ so that, again by homotopy invariance,

$$d(T + F - \underline{\mu} I, B_r) = d(T - \underline{\mu} I, B_r).$$

Now observe that, since $\underline{\mu} \notin \sigma(T), Tu - \underline{\mu} u = 0$ iff $u = 0$, and therefore the last degree is the same as the (coincidence) *index* of $T - \underline{\mu} I$ at zero ([5], p.201) that we shall simply denote with $i(\underline{\mu})$.

The argument used for $T + F - \underline{\mu} I$ applies to $T + F - \overline{\mu} I$ as well, so that from (6) we have $i(\underline{\mu}) = i(\overline{\mu})$. On the other hand, due to i) and to the odd multiplicity assumptions on μ_0, the Leray–Schauder formula ([5], p.201) shows that

$$i(\underline{\mu}) \neq i(\overline{\mu}).$$

This contradiction proves the claim. ∎

Remark 2. Under the above assumptions, we can say that $[\underline{\mu}, \overline{\mu}]$ is a *bifurcation interval* for (1). Notice that, if F is Frechet differentiable at 0 with $F'(0) = 0$, then ii) is trivially satisfied, and as $\underline{\mu}, \overline{\mu}$ can be taken arbitrarily close to μ_0, the assertion reduces to saying that given any $r > 0$, there exists $(\mu r, u_r)$ with $\|u_r\| = r$ and $|\mu_r - \mu_0| \leq r$, i.e. μ_0 is a bifurcation point of (1).

We specialize now to a real Hilbert space H and assume further that $T : D(T) \rightarrow H$ is selfadjoint. Then we have:

Theorem 3. *Assume that*

$$\operatorname{dist}(\mu_0, \widehat{\sigma}(T)) > 2k_0(F)$$

where $\widehat{\sigma}(T) = \sigma(T) \setminus \{\mu_0\}$ and $k_0(F)$ is as in (3). Then, for all sufficiently small $r > 0$, (1) has a solution pair (μ_r, u_r) with $\|u_r\| = r$ and

$$\mu_r \in [\mu_0 - k(r), \mu_0 + k(r)]$$

where $k(r) = k(F; r)$ is as in (2).

PROOF : Let $r_0 > 0$ be such that

$$\operatorname{dist}(\mu_0, \widehat{\sigma}(T)) - 2k(r) > 0, \qquad 0 < r \leq r_0. \tag{7}$$

Consider now the intervals

$$\underline{I} =]\mu_0 - (k(r) + \varepsilon), \mu_0 - k(r)[$$
$$\overline{I} =]\mu_0 + k(r), \mu_0 + (k(r) + \varepsilon)[$$

and notice that, if $\mu \in \underline{I} \cup \overline{I}$, then

$$\operatorname{dist}(\mu, \sigma(T)) > k(r). \tag{8}$$

This follows from the inequality

$$\begin{aligned}\operatorname{dist}(\mu, \sigma(T)) &\geq \operatorname{dist}(\mu_0, \widehat{\sigma}(T)) - |\mu - \mu_0| \\ &= 2k(r) + \varepsilon - |\mu - \mu_0|\end{aligned}$$

on remarking that, for all $\mu \in \underline{I} \cup \overline{I}$,

$$k(r) < |\mu - \mu_0| < k(r) + \varepsilon.$$

Now take $\underline{\mu} \in \underline{I}, \overline{\mu} \in \overline{I}$. Since by (7), $[\mu_0 - k(r), \mu_0 + k(r)]$ contains the only point μ_0 of $\sigma(T)$, (8) shows that $\underline{\mu}, \overline{\mu}$ satisfy condition i) of Theorem 1 above as well as the inequality (5) because, due to selfadjointness of $T, c(\mu) = \operatorname{dist}(\mu, \sigma(T))$.

Therefore, arguing as above, we get the existence of a solution pair (μ_r, u_r) of (1) with $\|u_r\| = r$ and $\mu \in [\underline{\mu}, \overline{\mu}]$. On the other hand, (1) has no nontrivial solution if $\|u\| \leq r$ and $\mu \in \underline{I} \cup \overline{I}$,: indeed, if $\|u\| \leq r$,

$$\begin{aligned}\|Tu + F(u) - \mu u\| &\geq c(\mu)\|u\| - k(r)\|u\| \\ &= [\operatorname{dist}(\mu, \sigma(T)) - k(r)]\|u\|\end{aligned}$$

and the assertion follows again by (8). This shows that necessarily $\mu_r \in [\mu_0 - k(r), \mu_0 + k(r)]$. ∎

Remark 4. The above argument also shows that if $\operatorname{dist}(\overline{\mu}, \sigma(T)) > k_0(F) = k_0$, then (1) has no nontrivial solution in some ball $B_r (r > 0)$ for μ near $\overline{\mu}$, so that $\overline{\mu}$ cannot be a bifurcation point. In other words, the set of bifurcation points of (1) is contained in

$$\sigma_{k_0}(T) := \{\mu : \operatorname{dist}(\mu, \sigma(T)) \le k_0\}.$$

This generalizes the well known fact that, when $F'(0) = 0$, bifurcation points necessarily belong to $\sigma(T)$ (see e.g. Rabinowitz [7]).

Remark 5. The existence of solutions u_r of (1) with $\|u_r\| = r$ can be proved as long as $r < \widetilde{r}$ where

$$\widetilde{r} = \sup\{r > 0 : 2k(r) < \operatorname{dist}(\mu_0, \widehat{\sigma}(T))\}.$$

In particular, solutions of any norm exist if $2k_\infty(r) < \operatorname{dist}(\mu_0, \widehat{\sigma}(T))$, where

$$k_\infty(F) = \lim_{r \to +\infty} k(F; r).$$

The cases dealt with in [1] and [2] are of this particular kind, with $k(F; r) = k =$ const.

Example.

We consider the eigenvalue problem

$$\text{(9)} \qquad \begin{cases} Lu(t) + u(t) \int_a^b k(t, s) g(s, u(s))\, ds = \mu u(t) \\ \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(a) + \delta u'(b) = 0 \end{cases}$$

on the compact interval $[a, b]$ of the real line R, where $Lu := -(p(t)u')' + q(t)u$ is a regular Sturm–Liouville operator with real coefficients $p \in C^1([a, b]), p > 0, q \in C([a, b])$; the kernel k belongs to $L^\infty((a, b) \times (a, b))$ and $g : [a, b] \times R \to R$ satisfies $g(t, 0) = 0$ for all $t \in R$ and is ***subquadratic***, i.e.

$$|g(t, u)| \le c + d|u|^2$$

for some $c, d \ge 0$ and all $(t, u) \in [a, b] \times R$. We suppose moreover that $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) > 0$.

Under these assumptions, we prove:

Theorem 6. *Let μ_n^0 denote the eigenvalues of L subject to the above boundary conditions, and let $c_1 = \|k\|_\infty c(b - a)$. Then, for n sufficiently large, the intervals*

$$[\mu_n^0 - c_1, \mu_n^0 + c_1]$$

are bifurcation intervals for (9); more precisely, for all sufficiently small $r > 0$, (9) has an eigenvalue–eigenfunction pair (μ_r, u_r) with $\|u_r\| = r$ and

$$\mu_r \in [\mu_n^0 - (c_1 + c_2 r^2), \mu_n^0 + (c_1 + c_2 r^2)]$$

where $c_2 = \|k\|_\infty \cdot d$.

PROOF : For $u \in L^2(a,b)$, we set

$$V(u(t)) = \int_a^b k(t,s)g(s,u(s))\,ds \qquad (\text{a.a.} t \in [a,b])$$

and

$$F(u) = uV(u)$$

so that (9) becomes

$$Tu + F(u) = \mu u, \qquad u \in D(T) \subset H$$

where $H = L^2(a,b), D(T) = \{u \in H : u', u'' \in H \text{ and } \alpha u(a) + \beta u'(a) = \gamma u(b) + \delta u'(b) = 0\}$, and $Tu = Lu$ for $u \in D(T)$.

About the nonlinear term F, we note that V satisfies, for all $u \in L^2$ and a.a. $t \in [a,b]$,

$$\begin{aligned} |V(u(t))| &\le \int |k(t,s)| \quad |g(s,u(s))|\,ds \\ &\le \|k\|_\infty \int [c + d|u(s)|^2]\,ds \end{aligned}$$

whence

$$|V(u)(t)| \le \|k\|_\infty [c(b-a) + d\|u\|^2] = c_1 + c_2\|u\|^2$$

where $\|u\|^2 = \int_a^b u^2(t)\,dt$.

Therefore $V(u) \in L^\infty(a,b)$ for $u \in L^2$ and

$$\begin{aligned} \|F(u)\| = \|uV(u)\| &\le \|u\| \quad \|V(u)\|_\infty \\ &\le (c_1 + c_2\|u\|^2)\|u\| \end{aligned}$$

whence it follows that

$$k(r) = \sup_{\|u\| \le r} \frac{\|F(u)\|}{\|u\|} \le c_1 + c_2 r^2 \quad (r > 0)$$

and $k_0(F) = \lim_{r \to 0} k(r) \le c_1$.

Now the result follows from Theorem 3 observing that, since $\mu_n^0 - \mu_{n-1}^0 \to \infty$ as $n \to \infty$ (see e.g. [10]), the condition

$$\text{dist}(\mu_n^0, \widehat{\sigma}(T)) > 2c_1$$

is satisfied for sufficiently large n. ∎

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