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Abstract. Methods developed for the study of general closure operators are used to construct various topological hulls for concrete categories. In particular, the notion of concretely cartesian closed topological hull a concrete category over a cartesian closed base is generalized to arbitrary base categories. We clarify why this concept coincides with the one of universally topological hull for concrete categories over the terminal category, i.e., for pre–ordered classes, but not in general. The new notion is characterized in terms of injectivity in a suitable quasi–category.

Keywords: Concrete category, (universally) topological category, final completion closure operator, closure commuting with pullbacks, (concretely) cartesian closed category

Classification: 18D15,18B25

0. Introduction

In the following we will concerned with *concrete categories* over a *base category* $\mathcal{X}$, i.e., pairs $(\mathcal{A},U)$ consisting of a category α and a faithful functor $\mathcal{A} \xrightarrow{U} \mathcal{X}$. For convenience, and to simplify the presentation, we also require U to be *amnestic*, i.e., every $\mathcal{A}$–isomorphism whose U–image is a identity must be an identity. This forces the U–fibres, i.e., the pullbacks of U along the $\mathcal{X}$–objects $1 \xrightarrow{x} \mathcal{X}$, to be partially ordered classes, not just pre–ordered ones. (α,U) is called *fibre–small* if all U–fibres are sets. Without loss generality we assume the hom–sets $(A,B)\underline{\mathcal{A}}$ to be subsets of the hom–sets $(AU,BU)\underline{\mathcal{X}}$ for all $\mathcal{A}$–objects A and B.

Recall that an $\mathcal{A}$–sink (κ,A), i.e., a family κ of $\mathcal{A}$–morphisms with common codomain A, is called *U–final*, if every $\mathcal{X}$–morphism $AU \xrightarrow{f} BU$ is an $\mathcal{A}$–morphism from A to B, provided that $gU \cdot f$ is an $\mathcal{A}$–morphism from the domain of g to B for each $g \in \kappa$. $(\mathcal{A},U)$ is said to be *topological*, if each (possibly large) family of objects of the *comma category* (cf. below) α/U with common codomain has a U–final lift.

For a base category $\mathcal{X}$ let $\mathbf{c}CAT/\mathcal{X}$ denote the full subcategory of the quasi–category $\mathbf{CAT}/\mathcal{X}$, whose objects are the concrete categories over $\mathcal{X}$; morphisms from $(\mathcal{A},U)$ to $(\mathcal{B},V)$ are functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ which satisfy $FV = U$. We say that F is *finally dense*, if each $\mathcal{B}$–object is the codomain of a V–final sink with members that have domains in the image of F. If the functor F is full (and hence equivalent to a full embedding), the $\mathbf{c}CAT/\mathcal{X}$–morphism F is called an *extension*. We say that F is *essential*, provided that a $\mathbf{c}CAT/\mathcal{X}$–morphism $(\mathcal{B},V) \xrightarrow{G} (\mathcal{C},W)$ is an extension iff FG is one.

If $\mathcal{X}$ is cartesian closed, $(\mathcal{A},U)$ is called *concretely cartesian closed*, if U preserves binary products, exponentiantion and evaluation. Consider the non–full subcategory $\mathcal{X}_{fp}$ of $\mathbf{c}CAT/\mathcal{X}$ whose objects are fibre–small and have (domains with) concrete finite products, and whose morphisms are those $\mathbf{C}AT/\mathcal{X}$–morphisms which preserve concrete finite products. A variant of the dual notion of final density plays a role here. A c$CAT/\mathcal{X}$-morphisms $(\mathcal{A},U) \xrightarrow{F} (\mathcal{B},V)$ into a concretely cartesian closed category $(\mathcal{B},V)$ is called *cartesian dense*, if the powers $[AF, A'F], A, A' \in \mathcal{A}$-**O***b*, are finitely dense in $\mathcal{B}$.

0.00 Theorem. *(cf.[**11**])*

(0) *An $\mathcal{X}_{fp}$-objects is injective with respect to extensions iff it is a concretely cartesian closed topological (=CCCT-) category.*
(1) *An extension in $\mathcal{X}_{fp}$ is essential iff it is finally dense and cartesian dense.*
(2) *Injective hulls in $\mathcal{X}_{fp}$ are CCCT-hulls.*

If $\mathcal{X}$ is terminal, i.e., has just one morphisms, c$CAT/\mathcal{X}$ is the quasi-category of partially ordered classes and order-preserving functions, while $\mathcal{X}_{fp}$ is just the category **m***SLat* of meet-semilattices and meet-preserving functions. Theorem 0.00 then specializes to

0.01 Corollary. *(cf.[**03**] and [**09**])*

(0) *A meet-semilattice is injective with respect to extensions iff it is a locale, i.e., a complete Heyting algebra.*
(1) *An extension in* **m***SLat is essential iff it is supremum-dense and cartesian dense.*
(2) *Injective hulls in* **m***SLat are locale-hulls.*

Two facts indicate that it may be possible to relax the assumptions for Theorem 0.00. First, the construction of CCCT-hulls, which dates back to [**02**], explicitly avoids using the cartesian closedness of the base category $\mathcal{X}$, but rather works with the finite products of $\mathcal{X}$. This is reflected by the fact concrete products are the important notion in the definition of the quasi-category $\mathcal{X}_{fp}$. Secondly, for meet-semilattices there is another quite different construction of locale-hulls in [**03**]. Both constructions generalize to arbitrary posets where they still coincide. Can one drop the requirement that $(\mathcal{A},U)$ has concrete finite products and still form something like a CCCT-hull? If so, is Theorem 0.00 still true in a larger subcategory c$CAT/\mathcal{X}$ than $\mathcal{X}_{fp}$? We will answer both of these questions affirmatively.

The second construction of locale-hulls for posets can be viewed as an adaptation of the sheafication construction, known from topos theory, where **Set** is replaced by the symmetric closed monoidal category **2**. A different point of view is present in [**01**]. Among the topological categories over $\mathcal{X}$ those for which final sinks are preserved by pullbacks are called *universally topological.* For a finitely complete base $\mathcal{X}$ consider the non-full subcategory $\mathcal{X}_{ps}$ of $\mathcal{X}_{fp}$ whose objects $\mathcal{A} \xrightarrow{U} \mathcal{X}$ admit initial lifts for regular monomorphisms $X \xrightarrow{m} AU$, and whose morphisms preserve such lifts. In [**01**] the injective objects in $\mathcal{X}_{ps}$ with respect to embeddings are characterized as the universally topological categories. For terminal $\mathcal{X}$ this specializes to parts (0) and (2) of Corollary 0.01. But as in Theorem 0.00, the general poset case is not covered.

Both types of topological hulls, as well as others, can be constructed by the same general method, which is derived from the study of closure operators on the functor category $[\mathcal{A}, \mathbf{Set}^{op}]^{op}$. The connection between such closure operators and final completions is investigated in Section 2, while final completions with special properties are constructed in Section 3.

Section 1 reviews concrete categories from the perspective of functor categories to set the stage for Section 2. Section 4 then indicates how to handle the concept of injectivity (with respect to extensions) in a subcategory $\mathcal{X}_b$ of $\mathbf{c}CAT/\mathcal{X}$ which properly contains $\mathcal{X}_{fp}$.

For functors $\mathcal{X} \xrightarrow{F} \mathcal{Z} \xleftarrow{G} \mathcal{Y}$ the objects of the *comma category* F/G are all triples (X, h, Y) with $X \in \mathcal{X}-\mathbf{O}b, Y \in \mathcal{Y}-\mathbf{O}b$ and $h \in (XF, YG)\underline{\mathcal{Z}}$. A morphism from (X, h, Y) to (X', h', Y') is a pair $(f : g) \in (X, X')\underline{\mathcal{X}} \times (Y, Y')\underline{\mathcal{Y}}$ with $fF \cdot h' = h \cdot gG$. The *domain functor* $F/G \to \mathcal{X}$ and the *codomain functor* $F/G \to \mathcal{Y}$ map $(f : g)$ to f and g, respectively.

$\mathcal{A}_Y$ denotes the functor category $[\mathcal{A}, \mathbf{Set}^{op}]^{op} \simeq [\mathcal{A}^{op}\mathbf{Set}]$. We call the images of $\mathcal{A}$-objects under the Yoneda embedding $\mathcal{A} \xrightarrow{\mathcal{A}_y} \mathcal{A}_Y$ *principal $\mathcal{A}$-ideals* and write A^y for $A\mathcal{A}_y, A \in \mathcal{A}\text{-}\mathbf{O}b$. Notice that monomorphisms in a functor-category over $\mathbf{Set}$ are pointwise such. We use the term *subfunctor* for a monomorphism in $\mathcal{A}_Y$ which is pointwise an inclusion. Subfunctors of principal $\mathcal{A}$-ideals are called *$\mathcal{A}$-sieves*.

The *lower Yoneda representation* $\mathcal{X} \xrightarrow{U_y} \mathcal{A}_Y$ of a functor $\mathcal{A} \xrightarrow{U} \mathcal{X}$ maps $f \in (X, Y)\underline{\mathcal{X}}$ to $UX^y \xrightarrow{Uf^y} UY^y$. Clearly, U_y as the composition of the Yoneda embedding $\mathcal{X}_y$ with $\mathcal{X}_Y \xrightarrow{U_Y} \mathcal{A}_Y$ (i.e., left composition with U) preserves limits.

Of primary importance will be the notion of *orthogonality* (cf. [**13**] and [**16**]). In a category C a morphism $A \xrightarrow{a} A'$ is called *left-orthogonal* to $B \xrightarrow{b} B'$, written $a \perp b$, if for every commutative diagram

(0-00)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[g]{} & B' \end{array}$$

there exists a unique diagonal $A' \xrightarrow{d} B$ making both induced triangles commute. This notion generalizes to sinks a and sources b; of particular interest will be the case when b is empty.

We use the set-theoretical foundations of [**10**], but will talk about "collections" instead of "conglomerates".

1. Concrete categories

For a concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$ define the quasi-category $(\mathcal{A}, U)_s$ of U-*sieves* to be the full subcategory of $\mathcal{A}_Y/U_y$ spanned by all subfunctors of U_y-images of $\mathcal{X}$-objects. U_s denotes the restriction of the faithful and amnestic codomain functor to $(\mathcal{A}, U)_s$. Notice that $(\mathcal{A}, U)_s$ still is *locally small*, i.e., admits a hom-functor into $\mathbf{Set}$. In fact, $((\mathcal{A}, U)_s, U_s)$ is precisely Herrlich's category $(\mathcal{A}^{-2}, U^{-2})$ of [**07**], just viewed from a slightly different perspective, and without the smallness constraint on $\mathcal{A}$.

We will usually suppress the domains of subfunctors and write (m, X) instead of (M, m, X) for $(\mathcal{A}, U)_s$–objects. $(\mathcal{A}, U)_s$–morphisms are already determined by their $\mathcal{X}$–component, so we may reduce $(m, X) \overset{(f:g)}{\rightarrow} (n, Y)$ to $(m, X) \overset{g}{\rightarrow} (n, Y)$. The canonical embedding U_s of $(\mathcal{A}, U)$ into $((\mathcal{A}, U)_s U_s)$ maps $f \in (A, B)\underline{A}$ to $(A^u, AU) \overset{fU}{\rightarrow} (B^u, BU)$, where A^u denotes the pointwise inclusion of A^y into $U(AU)^y$. Moreover, if $(A^u, AU) \overset{f:g}{\rightarrow} (m, X)$ is an $(\mathcal{A}, U)_s$–morphism, $g \in AM$ is the Yoneda image of f. We call U–sieves of the form (A^u, AU) *principal U–ideals*. Notice that $(k \cdot A^u, AU)$ is a U–sieve whenever (k, A) is an $\mathcal{A}$–sieve.

We will mainly be concerned with the left–orthogonality of $(\mathcal{A}, U)_s$–morphisms of the form $(m', X) \overset{(x:X)}{\rightarrow} (m, X)$ with respect to $(\mathcal{A}, U)_s$–objects (n, Y), viewed as empty sources.

If one adds all those morphisms to an $\mathcal{A}$–sink (κ, A), which factor through one of its members, the resulting sink (k, A) is U–final iff the original sink is. But k can be interpreted as an $\mathcal{A}$–sieve, namely as the second factor of the (epi–sink, mono)–factorization of the ($\mathcal{A}_Y$–image of the) original sink κ in $\mathcal{A}_Y$.

1.00 Proposition. *The $\mathcal{A}$-sink (κ, A) is U-final iff the $(\mathcal{A}, U)_s$-morphism $(k : AU)$ is left-orthogonal to all principal U-ideals.*

The dual notion of U–initiality does also have a nice internal interpretation in $\mathcal{A}_Y$: An $\mathcal{A}$–morphism $A \overset{f}{\rightarrow} B$ is U–initial iff $A^y \hookrightarrow U(AU)^y$ is the pullback of $B^y \hookrightarrow U(BU)^y$ along $U(fU)^y$. Sources can be handled by intersections of pullbacks.

We leave it to the reader to express notions like "$(\mathcal{A}, U)$ is topological " or "F is finally dense" in these terms, and to discover how simple the diagrams involved are.

Our considerations so far even apply to concrete quasi–categories over $\mathcal{X}$, as long as they are locally small. However, such objects need not belong to $\mathbf{c}CAT/\mathcal{X}$, hence we will continue to use $(\mathcal{A}, U)$ for an ordinary concrete category over $\mathcal{X}$.

We conclude this section with a technical result.

1.01 Lemma. *If all pullbacks of an $\mathcal{A}_Y$-morphism $K \overset{m}{\rightarrow} L$ along $\mathcal{A}_Y$-morphisms of the form $A^y \overset{a}{\rightarrow} L, A \in \mathcal{A}$–**Ob**, are U-final, then $m \perp B^u$ in $\mathcal{A}_Y$, for every $\mathcal{A}$-object B.*

2. Final completions via closure operators

This section deals with the counterpart of categories of sheaves in the setting of concrete categories.

2.00 Definition. If F is a concrete functor from $(\mathcal{A}, U)$ into a concrete (locally small quasi-) category $(\mathcal{B}, V)$, the pair $(F, (\mathcal{B}, V))$ is called a *final (quasi-) completion*, if F is finally dense and $(\mathcal{B}, V)$ is topological.

In particular, the canonical embedding of $(\mathcal{A}, U)$ into $((\mathcal{A}, U)_S, U_S)$ is a final quasi–completion, in fact is the largest one, cf.[07]. All others can be obtained by *reflective modifications* [08], i.e., as concretely reflective subcategories of $((\mathcal{A}, U)_S, U_S)$ which contain $(\mathcal{A}, U)$.

The U_S–fibres of $(\mathcal{A},U)_S$ are (potentially large) complete lattices. Therefore a concrete reflector can be thought of as a family of closure operators, one for each fibre, which are compatible in the sense that pullbacks of closed U–sieves are closed. (This is a very "topological" concept.) However, closure need not commute with pullbacks. A generalized notion of closure operator on nice collections of monomorphisms has been investigated in [06] and [14]; its roots go as far back as [04], cf. also [05]. In the following we combine both points of view.

2.01 Definition. A *closure operator* on a concrete quasi–category $(\mathcal{B},V)$ is a pair (δ,Γ) consisting of a functor $\mathcal{B} \xrightarrow{\Gamma} \mathcal{B}$ and a natural transformation $V \xrightarrow{\delta} \Gamma$ which satisfy $\Gamma V = V$ and $\delta V = \mathcal{B}$. We write Γ–**Fix** for the collection of fixed points of Γ, also refered to as Γ–*closed $\mathcal{B}$–objects*. (δ,Γ) is called idempotent, if Γ is idempotent.

Closure operators are partially ordered by the pointwise partial order on the fibres. (Beware: this means that *smaller* closure operators have *more* closed objects.) Every closure operator on a topological category $(\mathcal{B},V)$ is dominated by a least idempotent one, its *idempotent hull.*

Let $\mathcal{M}$ be the collection of all $\mathcal{A}_Y$–monomorphisms which are pointwise inclusions, i.e., subfunctors, viewed as a full subcategory of the comma category $\mathcal{A}/\mathcal{A}$. We write ∂_0 and ∂_1 for the restrictions of the domain functor and the codomain functor to $\mathcal{M}$, respectively. U_S turns out to be a pullback of ∂_1 along U_y

$$\begin{array}{ccc} (\mathcal{A},U)_S & \longrightarrow & \mathcal{M} \\ {\scriptstyle U_S}\big\downarrow & & \big\downarrow{\scriptstyle \partial_1} \\ \mathcal{X} & \underset{U_y}{\longrightarrow} & \mathcal{A}_Y. \end{array} \tag{2-00}$$

Now $(\mathcal{M},\partial_1)$ is topological, and closure operators on $\mathcal{M}$ can be pulled back to $(\mathcal{A},U)_S$. The problem with this approach is that we do not know whether every reflective modification of $(\mathcal{A},U)_S$ arises in this way. However, it is convenient to be able to work in $\mathcal{M}$, or $\mathcal{A}_Y$ for that matter.

In the following, (δ,Γ) always will be an idempotent closure operator on $(\mathcal{A},U)_S$. We denote the Γ–image of (m,X) by $(m\Gamma,X)$. Each U–sieve $\bullet \overset{m}{\hookrightarrow} UX^y$ factors as $\bullet \overset{m\delta\partial_0}{\hookrightarrow} \bullet \overset{m\Gamma}{\hookrightarrow} UX^y$. Notice that δ is pointwise mono. While the second factor is Γ–closed, in general it does not make sense to call the first one dense, unless it has a codomain of the form UZ^y for some $\mathcal{X}$ –object Z. However, a notion of relative density can be defined in this setting.

2.02 Definition. An $(\mathcal{A},U)_S$–mono $(m',X) \overset{(x:X)}{\longrightarrow} (m,X)$ is called *relatively Γ–dense*, if $m'\Gamma = m\Gamma$.

2.03 Proposition.

(0) *An $(\mathcal{A},U)_S$–mono $(x : X)$ is relatively Γ–dense iff $(x : X) \perp (n,Y)$ for all Γ–closed U–sieves (n,Y).*

(1) *A U–sieve (n,Y) is Γ–closed iff $(x : X) \perp (n,Y)$ for all relatively Γ–dense $(\mathcal{A},U)_S$–monos $(x : X)$.*

PROOF : Use the fact that $m\delta$ is always relatively Γ–dense, since Γ is idempotent. ∎

The orthogonality relation now induces a Galois correspondence between collections Δ of $(\mathcal{A}, U)_S$–monos, and collections Σ of U–sieves. $\Delta^\perp$ and $\Sigma_\perp$ denote the respective Galois–images.

2.04 Proposition. *For every collection Δ of $(\mathcal{A}, U)_S$–monos $\Delta^\perp$ is the collection of fixed points for a unique idempotent closure operator on $(\mathcal{A}, U)_S$.*

PROOF : Clearly, $\Delta^\perp$ is closed under limits, in particular intersections and pullbacks. For a U–sieve (m, X) we define $(m\Gamma, X)$ to be the smallest U–sieve in $\Delta^\perp$ containing (m, X). This is easily seen to be an idempotent closure operator. The uniqueness follows from Proposition 2.03. ∎

2.05 Example. A concrete category $(\mathcal{A}, U)$ over the terminal category **1** is just a partially ordered class $(A, \sqsubseteq)$. It is topological iff it is a (possibly large) complete lattice. $\mathcal{A}$–sieves can be interpreted as pairs (K, a), where $K \subseteq A$ is a lower segment and $a \in A$ is an upper bound of K. Such a pair is U–final iff a is the supremum of K. Subfunctors, in particular U–sieves, have the terminal object of $\mathcal{A}_Y$ as codomain, hence they may be viewed as lower segments of $(A, \sqsubseteq)$. Moreover, the only U–initial morphisms are the identities. Observe that the notions of principal $\mathcal{A}$–ideal and principal U–ideal coincide. $(\mathcal{A}, U_S)$–monos correspond to pairs (M', M) of lower segments with $M' \subseteq M$, and the relation $(M', M) \perp L$ then means that $M \subseteq L$ iff $M' \subseteq L$. Any collection of such pairs determines a closure operators in the usual sense, i.e., order–preserving, extensive, idempotent functions on $(A, \sqsubseteq)$.

Now we have a bijective correspondence between idempotent closure operators on (i.e., reflective modifications of) $(\mathcal{A}, U)_S$ and Galois–closed collections of $(\mathcal{A}, U)_S$–monos. The idea for constructiong special final completions of $(\mathcal{A}, U)$ will be to find suitable collections Δ which satisfy $\Delta^\perp_\perp = \Delta$.

3. Topological hulls with prescribed properties

We want to develope a concrete analogue of the topos–theoretic idea of sheafification. While our approach is not restricted to *modal* closure operators, i.e., those where closure commutes with pullbacks, we start with what could be considered as the concrete notion of a Grothendieck topos.

Definition. (cf. [00, Definition II.7]) $(\mathcal{A}, U)$ is called *universally topological*, if it is topological, and U–final $\mathcal{A}$–sieves are stable with respect to pullbacks along ($\mathcal{A}_y$–images of) $\mathcal{A}$–morphisms.

(In [01] the term *strongly finally complete* is used instead.) It is easy to see that the second condition is equivalent to requiring that the U–final $\mathcal{A}$–sieves form a Grothendieck topology. In fact, every universally topological final quasi–completion $(\mathcal{B}, V)$ of $(\mathcal{A}, U)$ can be obtained via a suitable Grothendieck topology on $\mathcal{A}$, i.e., an idempotent modal closure operator on $\mathcal{M}$, via a pullback as outlined in the last section, cf. Corollary 3.08.

In order to compare the concepts of universally topological and concretely cartesian closed topological category, we need an internal characterization of CCCT–categories which can be formulated without reference to the cartesian closedness of the base category $\mathcal{X}$. Fortunately such a characterization is well known. Recall that $(\mathcal{A}, U)$ is a CCCT–category over a cartesian closed base, if U preserves binary products, exponentiation and evaluation. As a consequence of Wyler's taut lift theorem, this is equivalent to requiring that $\mathcal{A} \overset{C\times -}{\to} \mathcal{A}$ preserves U–final sinks for every $\mathcal{A}$–object C [00, Theorem I.7]. This characterizations allows us to replace cartesian closed bases by bases with finite products. To eliminate the need for finite products as well, we extend the notion of orthogonality to accommodate certain $\mathcal{X}_Y$–morphisms, instead of just $\mathcal{X}$–morphisms. This allows us to generalize the notion of U–finality to certain $\mathcal{A}_Y$–monos which are not $\mathcal{A}$–sieves.

3.01 Definition. The product $(t, C) \times (k, A)$ of two $\mathcal{A}$–sieves (in $\mathcal{A}_Y$) is called *U–final*, if the product $(t : CU) \times (k : AU)$ is left–orthogonal to all principal U–ideals in the sense that for every $\mathcal{X}_Y$–morphism $(CU)^y \times (AU)^y \overset{g}{\to} (BU)^y$ the composition $(C^u \times A^u) \cdot Ug$ factors through B^u if $(t \times k) \cdot (C^u \times A^u) \cdot Ug$ does.

3.02 Definition. $(\mathcal{A}, U)$ is called a CCCT–*category*, if it is topological, and for every U–final $\mathcal{A}$–sieve (k, A) and each $\mathcal{A}$–object C the product $(C^y, C) \times (k, A)$ is U–final.

Notice that $C^y \times k$ is just the pullback of k along the projection $C^y \times A^y \to A^y$. If the product of C and A does not exist in $\mathcal{A}$, this pullback is not an $\mathcal{A}$–sieve.

3.03 Proposition. *Every universally topological is a CCCT–category.*

PROOF :

If $(\mathcal{A}, U)$ is universally topological, the U–final $\mathcal{A}$–sieves form a Grothendieck topology for which all principal U–ideals are closed. The claim follows, since *any* pullback of dense $\mathcal{A}_Y$–morphisms is again dense. ∎

Definition 3.00 can be weakened by restricting the class of $\mathcal{A}$–morphisms, along which pullbacks have to preserve U–finality. Particularly interesting is the class of U–initial monos. This leads to the notion of *hereditary topological category*, cf. [00, Remarks II.3(iii) and (iv)].

3.04 Example. A complete lattice $(A, \sqsubseteq)$ is universally topological iff for every lower segment K with supremum a every lover bound b of a is the supremum of its lower bounds in K. But every lower bound b of a is of the form $c \sqcap a$ for some c, so this condition is equivalent to $(A, \sqsubseteq)$ being a CCCT–category over $\mathbf{1}$, i.e., finite meets distributing over arbitrary joins. So the reason that both concepts agree for partially ordered classes in that here all morphisms are projections. On the other hand, every complete lattice is hereditary topological.

The final quasi–completions of $(\mathcal{A}, U)$ form a (possibly large) complete lattice. The smallest element is the Dedekind–MacNeille completion, $(\mathcal{A}^{-4}, U^{-4})$ in the terminology of [**08**]. Are there minimal final quasi–completions of the types discussed above? Since all three types are defines by similar conditions involving pullbacks of final sieves, we use a unified approach. For the Dedekind–MacNeille completion no pullbacks have to be considered.

If $(\mathcal{B}, V)$ is obtained from $((\mathcal{A}, U)_S, U_S)$ by means of a closure operator Γ, we need to characterize V–final $\mathcal{B}$–sieves $R \overset{r}{\hookrightarrow} (m, X)^y$. Consider the smallest $\mathcal{A}$–sieve

$(\widetilde{r}, X)$ through which every $\mathcal{A}_Y$–morphism $\bullet \overset{n}{\hookrightarrow} UY^y \overset{Uh^y}{\to} UX^y$ with $(n, Y) \in \mathcal{B}$–Ob and $h \in (n,Y)R$ factors. Clearly, $(\widetilde{r}, X) \overset{X}{\to} (m, X)$. But by construction $R \overset{r}{\hookrightarrow} (m, X)^y \hookrightarrow VX^y$ factors through $(\widetilde{r}\Gamma, X)^y \hookrightarrow VX^y$, hence the V–finality of r implies that (m, X) is the $\mathcal{B}$–reflection of $(\widetilde{r}, X)$. So in order to guarantee that V–finality of r implies that (m, X) is the $\mathcal{B}$–reflection of $(\widetilde{r}, X)$. So in order to guarantee that V–finality is preserved by pullbacks along certain $\mathcal{B}_Y$ morphisms, loosely speaking Γ–closure needs to commute with these pullbacks.

The only other condition we have to meet is that principal U–ideals have to be closed. This lead us to define collections of $(\mathcal{A}, U)_S$–monos as follows.

3.05 Definition. Let $(m, X) \overset{(x:X)}{\to} (m, X)$ be an $(\mathcal{A}, U)_S$–mono.

(0) $(x : X) \in \Delta_D$ iff $(x : X)$ is left–orthogonal to every principal U–ideal.

(1) $(x : X) \in \Delta_U$ iff for every $(\mathcal{A}, U)_S$–morphism $(t, Q) \overset{(p:q)}{\to} (m, X)$ the pullback of $(x : X)$ along $(p : q)$ is left–orthogonal to every principal U–ideal.

(2) $(x : X) \in \Delta_H$ iff for every U_S–initial $(\mathcal{A}, U)_S$–morphism $(t, Q) \overset{(p:q)}{\to} (m, X)$ the pullback of $(x : X)$ along $(p : q)$ is left–orthogonal to every principal U–ideal.

(3) $(x : X) \in \Delta_c$ iff for each U–sieve (t, Q) the product $(t\partial_0 : Q) \times (x : X)$ is left–orthogonal to every principal U–ideal.

We write $\Gamma_D, \Gamma_U, \Gamma_H$ and Γ_C for the idempotent closure operators induced by $\Delta_D, \Delta_U, \Delta_H$ and Δ_C, respectively, cf. Proposition 2.04. While Δ_D is Galois–closed simply by definition, and clearly induces the Dedekin–MacNeille completion, an explicit proof of this fact will be useful to illustrate the other three cases.

Before proceeding with the main result of this section, we simplify the descriptions of Δ_U, Δ_H and Δ_C. For Δ_C, this relates our construction to the one in [**02**], cf. also [**15**].

3.06 Lemma.

(0) $(x : X) \in \Delta_U$ *if every pullback of* $(x : X)$ *along an* $(\mathcal{A}, U_S)$*–morphism of the form* $(C^u, CU) \overset{(c:\widetilde{c})}{\to} (m, X)$ *is left orthogonal to each principal* U*–ideal.*

(1) $(x : X \in \Delta_H$ *if every pullback of* $(x : X)$ *along an* $(\mathcal{A}, U)_S$*–morphism of the form* $(k, CU) \overset{(h:\widetilde{c})}{\to} (m, X)$*, where* $(C^u, CU) \hookrightarrow (k, CU)$ *and* k *is the pullback of* m *along* $U\widetilde{c}^y$*, is left orthogonal to each principal* U*–ideal.*

(2) $(x : X) \in \Delta_c$ *if every product of the form* $(C^y : CU) \times (x : X)$ *is left–orthogonal to each principal* U*–ideal.*

PROOF :

(0) If $(x : X) \notin \Delta_U$, we may assume that $(x : X) \perp (B^u, BU)$ fails for an $\mathcal{A}$–object B and an $(\mathcal{A}, U)_S$–morphism $(m', X) \overset{s}{\to} (B^u, BU)$. Hence for some $C^y \overset{c}{\to} m\partial_0$ the Yoneda–image $CU \overset{\widetilde{c}}{\to} X$ of $c \cdot m$ composed with s does not belong to $(C, B)\underline{\mathcal{A}}$. Therefore the pullback of $(x : X)$ along $(c : \widetilde{c})$ fails to be left–orthogonal to (B^u, BU).

(1) As in (0), just replace $(c, \tilde{c})$ by $(h, \tilde{c})$, where h is the $\mathcal{A}_Y$–pullback of $U\tilde{c}^y$ along m.

(2) If $(x : X) \notin \Delta_C$, there exists a U–sieve (t, Q), an $\mathcal{A}$–object B, and an $\mathcal{X}_Y$–morphism $Q^y \times X^y \xrightarrow{s} (BU)^y$ such that $(t \times m') \cdot Us$ factors through $B^y \hookrightarrow U(BU)^y$, but $(t \times m)$. Us does not. Thus one can find an $\mathcal{A}_Y$–morphism $C^y \xrightarrow{g} (t \times m)\partial_0$, such that the $\mathcal{X}$–morphism $CU \to BU$ induced by $g \cdot (t \times m) \cdot Us$ does not belong to $(C, B)\underline{\mathcal{A}}$. The composition of $g \cdot (t \times m)$ with the projection onto UQ^y has the Yoneda–image $CU \xrightarrow{h} Q$. Now $(CU)^y \times X^y \xrightarrow{(h^y \times X^y) \cdot s} (BU)^y$ witnesses the failure of $(C^y : CU) \times (x : X)$ to be left–orthogonal to (B^u, BU).

■

Recall that in $\mathcal{A}_Y$ the notions of epi–sink and of monomorphism can be handled pointwise. In particular, $\mathcal{A}_Y$ is an (epi–sink, mono)–category, and epi–sinks in $\mathcal{A}_Y$ are pullback–stable and *effective* in the sense that they are colimits of their *kernels*, which are the diagrams generated by forming the pullbacks of all pairs of morphisms in the sink.

3.07 Theorem.

(0) *The full subcategory $((\mathcal{A}, U)_D, U_D)$ of $((\mathcal{A}, U)_S, U_S)$ spanned by the Γ_D–closed U–sieves is the Dedekind–MacNeille completion of $(\mathcal{A}, U)$.*

(1) *The full subcategory $((\mathcal{A}, U)_U, U_U)$ of $((\mathcal{A}, U)_S, U_S)$ spanned by the Γ_U–closed U–sieves is the universally topological hull of $(\mathcal{A}, U)$.*

(2) *The full subcategory $((\mathcal{A}, U)_H, U_H)$ of $((\mathcal{A}, U)_S, U_S)$ spanned by the Γ_H–closed U–sieves is the hereditary topological hull of $(\mathcal{A}, U)$.*

(3) *The full subcategory $((\mathcal{A}, U)_C, U_C)$ of $((\mathcal{A}, U)_S, U_S)$ spanned by the Γ_H–closed U–sieves is the CCCT–hull of $(\mathcal{A}, U)$.*

PROOF :

For $i \in \{D, U, H, C\}$ and a U–sieve (m, X) define $m\Psi_i$ to be the pointwise union of all U–sieves (m', X) with $m = x \cdot m'$ and $(x : X) \in \Delta_i$. Since unions are effective, the induced $(\mathcal{A}, U)_S$–morphism $(m, X) \xrightarrow{(m\varphi_i : X)} (m\Psi_i, X)$ is left–orthogonal to every principal U–ideal.

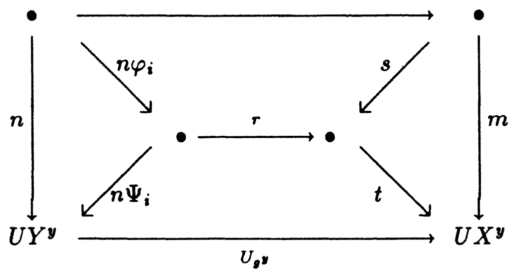
We first show that $(m\varphi_i : X) \in \Delta_i$. For $i = D$ this is trivial. For $i = U$ consider an $(\mathcal{A}, U)_S$–morphism $(C^u, CU) \xrightarrow{(c:\tilde{c})} (m\Psi_U : X)$. Every factorization $m = x \cdot m'$ with $(x : X) \in \Delta_U$ induces a unique factorization $m\varphi_U = x \cdot w$. Form the following pullbacks in $\mathcal{A}_Y$:

(3-00)
$$\begin{array}{ccc} \bullet & \xrightarrow{d} & \bullet \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle w} \\ C^y & \xrightarrow[c]{} & (m\Psi_U)\partial_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle x} \\ \bullet & \xrightarrow[d]{} & \bullet \end{array}$$

Clearly, $(q : CU)$ is the pullback of $(x : X)$ along $(p \cdot C^u, CU) \xrightarrow{(d:\tilde{c})} (m'X)$ and hence belongs to Δ_U. Since the defining effective sink for $m\Psi_U$ in $\mathcal{A}_Y$ pulls back to an effective sink along $C^y \xrightarrow{c} (m\Psi_U)\partial_0$, the pullback $(q \cdot p : CU)$ of $(m\varphi_U : X)$ along $(c : \widetilde{c})$ is left–orthogonal to every principal U–ideal. For $i = H$ the same reasoning works if $(c : \widetilde{c})$ is replaced by $(h : \widetilde{c})$, where h is the pullback of $U\widetilde{c}^y$ along $m\Psi_H$ in $\mathcal{A}_Y$. For $i = C$ the argument is similar.

In order to show that Ψ_i is an endo–functor on $(\mathcal{A}, U)_S$, consider an $(\mathcal{A}, U)_S$–morphism $(n < Y) \overset{(f:g)}{\to} (m, X)$.Let $((r, s), t)$ be the (epi–sink, subfunctor)–factorization of $(n\Psi_i \cdot Ug^y, m)$

(3-01)



We have to show that $(s : X) \in \Delta_i$. For $i = D$ this follows since $(n\varphi_D : Y) \in \Delta_D$ and (r, s) is an effective epi–sink. For $i = U$ consider an $(\mathcal{A}, U)_S$–morphism $(C^u, CU) \overset{(c:\widetilde{c})}{\to} (t, X)$ and form the following pullbacks in $\mathcal{A}_Y$:

(3-02)
$$\begin{array}{ccc} L & \xrightarrow{c'} & UY^y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle Ug^y} \\ U(CU)^y & \xrightarrow[U\widetilde{c}^y]{} & UX^y \end{array} \quad \text{and} \quad \begin{array}{ccc} K & \xrightarrow{\widehat{c}} & (n\Psi_U)\partial_0 \\ {\scriptstyle \widehat{r}}\downarrow & & \downarrow{\scriptstyle r} \\ C^y & \xrightarrow[c]{} & t\partial_0. \end{array}$$

These induce a unique $K \xrightarrow{q} L$ with $q \cdot g' = \widehat{r} \cdot C^u$ and $q \cdot c' = \widehat{c} \cdot n\varphi_U$. Let $\widehat{s}$ be the pullback of s along c, and let be the pullback of $n\varphi_U$ along $\widehat{c}$. Clearly, $\widetilde{p}, \widehat{r}$ factors through $\widehat{s}$. Any $\mathcal{A}_Y$–morphism $A^y \xrightarrow{a} K$ now induces a unique $U(AU)^y \xrightarrow{\widetilde{a}} L$ with $A^u \cdot \widetilde{a} = a \cdot q$. Let $AU \xrightarrow{b} Y$ and $AU \xrightarrow{d} CU$ be the Yoneda–images of $\widetilde{a} \cdot g'$ respectively. Now the pullback $(v : AU)$ of $(n\varphi_U : Y) \in \Delta_U$ along $(a \cdot \widehat{c} : b)$ is left–orthogonal to every principal U–ideal, and $(a \cdot \widehat{r} : d)$ is an $(\mathcal{A}, U)_S$–morphism from (A^u, AU) to (C^u, CU).Since the sink $(\widehat{r}, \widehat{s})$ is left–orthogonal to all principal U–ideals as well. Hence $(s : X) \in \Delta_U$, as desired. For $i = H$ we replace $(c, \widetilde{c})$ by $(k, CU) \overset{(h:\widetilde{c})}{\to} (t, X)$, where k is the pullback of t along $U\widetilde{c}^y$. Moreover, we need the fact that the sink consisting of the pullbacks $\check{a}$ of $U(AU)^y \xrightarrow{\widetilde{a}} L$ along $K \xrightarrow{q} L$, where $A^y \xrightarrow{a} K$, is an epi–sink and hence effective. The case $i = C$ requires another lengthy but similar diagram chase.

By construction it is clear that $\Delta_i^{\perp} = \Psi_i\text{–}\mathbf{Fix}$, i.e., $\Psi_i = \Gamma_i$, that pullbacks of U_i–final $(\mathcal{A}, U)_i$–sieves along the appropriate $((\mathcal{A}, U)_i)_Y$–morphism are again U_i–final, and that Γ_i is the largest closure operator (in the pointwise ordering) with these properties. Therefore among those it has the smallest collection of fixed points, which constitute the desired topological hull. ■

There always exits a largest Grothendieck topology J_U on $\mathcal{A}$ such that all principal U–ideals are J_U–closed. J_U consists of all those $\mathcal{A}$–sieves (j, A) with the property that for each $\mathcal{A}$–morphism $C \xrightarrow{f} A$ the pullback of j along f^y is U–final, cf. [14, Theorem 3.08]. Let Γ_G be the restriction to $(\mathcal{A}, U)_S$ of the idempotent modal closure operator on $\mathcal{M}$ generated by J_U. In view of Lemma 1.01 we get

3.08 Corollary. $\Gamma_U = \Gamma_G$.

This result justifies our earlier assertion that universally topological categories in the context of concrete categories play the role Grothendieck topoi play for ordinary categories.

4. Injectivity and CCCT–hulls

It is easy to verify Theorem 0.00 when $\mathcal{X}$ just has binary products without being cartesian closed; the proof in [**11**] still works. On the other hand, it has been shown in [**01**] that the analogous result does not hold for universally topological categories, if one looks at pullback–preserving concrete functors. All finite limits need to be taken into account, as well as embeddings. Our interest here is to generalize Theorem 0.00 in such a way, that for $\mathcal{X} = 1$ it covers the poset case, i.e., we want to eliminate the need for concrete binary products to exist in the objects of $\mathcal{X}_{fp}$. This develops an idea explored to some extent in [**13**]. However, we cannot do without binary products in $\mathcal{X}$.

For a concrete functor $(\mathcal{A}, U) \xrightarrow{F} (\mathcal{B}, V)$ the *inverse image functor* $((\mathcal{B}, V)_S, V_S) \xrightarrow{F_S} ((\mathcal{A}, U)_S U_S)$ maps an $(\mathcal{B}, V)$–morphism $(m, X) \xrightarrow{g} (n, Y)$ to $(Fm, X) \xrightarrow{g} (Fn, Y)$. A concrete right adjoint F_R to F_S can be defined as follows: Whenever (m, X) is a U–sieve, let $(m', X) = (m, X)F_R$ be the smallest V–sieve with codomain X such that m functors through $m'F$. (Similarly one can define a concrete right adjoint F_T of F_S).

We write U_{ds} for the concrete embedding of the Dedekind–MacNeille completion of $(\mathcal{A}, U)$ into $((\mathcal{A}, U)_S, U_S)$, and U_{sd} for the concrete left adjoint induced by Γ_D.

4.00 Definition. The *Dedekind lift* F_D of a concrete functor $(\mathcal{A}, U) \xrightarrow{F} (\mathcal{B}, V)$ is defined by the following composition

$$\begin{array}{ccc} ((\mathcal{A}, U)_D, U_D) & \xrightarrow{F_D} & ((\mathcal{B}, V)_D, V_D) \\ {\scriptstyle U_{ds}} \downarrow & & \downarrow {\scriptstyle V_{sd}} \\ ((\mathcal{A}, U)_S, U_S) & \xrightarrow[F_R]{} & ((\mathcal{B}, V)_S, V_S). \end{array} \tag{4-00}$$

Notice that the notion of Dedekind lift is *not* functorial, as the following simple example in **Pos** shows [13, Example 3.15]:

4.01 Example. Consider the embedding F of the discretely ordered set $A = \{x, y\}$ into the poset $(B, \sqsubseteq)$ given by

$$x \longrightarrow z \longleftarrow y,$$

and define the **Pos**–morphism $(B, \sqsubseteq) \xrightarrow{G} 2^{op}$ by $xG = 1$ and $zG = 0$. Then A is a Dedekind cut in $(A, =)$, but not in $(B, \sqsubseteq)$. Hence $F_D G_D$ maps A to $\{0, 1\}$, while $(FG)_D$ maps A to $\{1\}$.

We now use Dedekind lifts to define a suitable extension of the quasi–category $\mathcal{X}_{fp}$, which contains objects without binary products. The problem is to find the right morphisms. Essentially we want the Dedekind lifts to preserve binary products, but care has to be taken to insure that the morphisms defined in this way compose properly.

4.02 Definition.

(0) For any concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$ define $(\mathcal{A}, U)_B$ to be the smallest full subcategory of $(\mathcal{A}, U)_D$ which contains all binary products of principal U–ideals. We write U_B for the restriction of U_D to $(\mathcal{A}, U)_B$, and U_b for the concrete embedding of $(\mathcal{A}, U)$ into $(\mathcal{A}, U)_B$.

(1) Let $\mathcal{X}_b$ be the (non–full) subcategory of **c**$CAT/\mathcal{X}$, whose objects are fibre-small concrete categories, and whose morphisms form $(\mathcal{A}, U)$ to $(\mathcal{B}, V)$ are those concrete functors, whose Dedekind lift restricts tot a concrete functor from $((\mathcal{A}, U)_B, U_B)$ to $((\mathcal{B}, V)_B, V_B)$.

Clearly $(\mathcal{A}, U)_B$ is small–fibred if $(\mathcal{A}, U)$ is. Moreover, $(\mathcal{A}, U)_B$ has the same final quasi–completions as $(\mathcal{A}, U)$. It is important to notice that for two composable $\mathcal{X}_b$–morphisms F and G we have $(FG)_B = F_B G_B$. Obviously, $\mathcal{X}_{fp}$ is a full subcategory of $\mathcal{X}_b$.

The framework outlined above will allow us to extend Theorem 0.00, if we can rephrase the notion of cartesian denseness without using powers.

4.03 Definition. A concrete functor $(\mathcal{A}, U) \xrightarrow{F} (\mathcal{B}, V)$ into a concrete quasi-category is called *cartesian dense*, provided that an $\mathcal{X}$–morphism $BU \xrightarrow{f} CU$ lifts to a $\mathcal{B}$–morphism from B to C iff for any two $\mathcal{A}$-objects A and D and every $(\mathcal{B}, V)_B$–morphism $AF \times C \xrightarrow{g} DF$ the composition $(AU \times f) \cdot gV$ lifts to a $(\mathcal{B}, V)_B$–morphism from $AF \times B$ to DF.

If $(\mathcal{B}, V)$ is an ordinary concrete category, this means principal V–sieves of the form (A^v, AFV) and (D^v, DFV) suffice to define Δ_C, i.e., $(x : X) \in \Delta_C$ iff for all $\mathcal{A}$–objects A the product $((AF)^y : AFV) \times (x : X)$ is left–orthogonal to every principal V–ideal $(D^v, DFV), D \in \mathcal{A}$–**O**$b$. If the base is cartesian closed this translates into assertion that the powers $[AFV, DFV]$ are initially dense in the CCCT–hull of $(\mathcal{B}, V)$. By default, $(\mathcal{B}, V)$ is cartesian dense in its CCCT–hull.

References

[00] Adámek J., Herrlich H., *Cartesian closed categories, quasitopoi and topological universes,* Comment.Math.Univ.Carolinae **27** (1986), 235-257.

[01] Adámek J., Herrlich H., *A characterization of concrete quasitopoi by injectivity,* preprint.

[02] Adámek J., Strecker G.E., *Construction of cartesian closed topological hulls*, Comment. Math.Univ.Carolinae **22** (1981), 235-254.

[03] Bruns G., Lakser H., *Injective hulls of semilattices*, Canadian Bull.Math. **13** (1970), 115-118.

[04] Ehrbar H., Wyler O., "On subobjects and images in categories," Technical Report 68-34, Dept.Math., Carnegie–Mellon Univ.,1968;and Preprint, 1969.

[05] Ehrbar H., Wyler O., *Images in categories as reflections*, Cahiers Top. Geom.Diff.Cat. **28** (1987), 143-159.

[06] Dikranjan D., Giuli E., *Closure operators I,*, Topoiogy and its Applications **27** (1987), 129-143.

[07] Herrlich H., *Initial completions*, Math.Z. **150** (1976), 101-110.

[08] Herrlich H., *Universal topology*, Categorical Topology, Proc.Int.Conf.Categorical Topology, Toledo, OH, 1983, Heldermann Verlag Berlin, 1984, 223-281.

[09] Horn A., Kimura N., *The category of semilattices*, Alg.Univ **1** (1971), 26-38.

[10] Herrlich H., Strecker G.E., "Category Theory, 2nd ed.," Heldermann Verlag, Berlin, 1979.

[11] Herrlich H., Strecker G.E., *Cartesian closed topological hulls as injective hulls*, Quaestiones Mathematicae **9,1-4** (1986), 263-280.

[12] Johnstone P.T., "Topos Theory," London Math.Soc.Monographs No. 10, Academic Press, London–New York–San Francisco, 1977.

[13] Koslowski J., "Dedekind cuts and Frink ideals for categories," Thesis, Kansas State University, Manhattan,KS, 1986.

[14] Koslowski J., *Closure operators with prescribed properties*, Springer Lecture Notes in Mathematics **1348** (1988), 208-220.

[15] Strecker G.E., *On cartesian closed topological hulls*, Categorical Topology, Proc.Int.Conf. Categorical Topology, Toledo, OH, 1983, Heldermann Verlag Berlin, 1984, 523-539.

[16] Tholen W., *Factorizations, localizations, and the orthogonal subcategory problem*, Math. Nachr. **114** (1983), 63-85.

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