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Abstract. In this paper we study interval π–equivalences, that is we want to study Sd–functions from the class of rational numbers Q to Q by means of these π–equivalences. A theorem is proved which says that to each interval π–equivalence there exists an Sd^*–function to which the π–equivalence corresponds.

Keywords: Alternative Set Theory, interval π–equivalence, function.

Classification: 03E70, 54C30

Introduction.

A classical real function $\mathcal{F}$ (i.e. a closed figure in Q^2) can be represented by an Sd-function $F : Q \to Q$ such that $\mathcal{F} = \mathrm{Fig}(F)$. We want to study $\mathcal{F}$ by means of that Sd-function F and the Sd-function by means of an interval π–equivalence R_F on the class of all rational numbers Q which is in a canonical way assigned to F.

Throughout the paper we use usual notations and principles of the Alternative Set Theory (see [V]). In the first section, basic propositions concerning interval π–symmetries are proved, discrete basis theorem is also proved. Then the structure of Q and the π–symmetries are studied in a connection with automorphisms. Finally there is proved an important theorem stating that to each interval π–equivalence R there exists an Sd^*-function F such that $R = R_F$.

First section, basic notions and motivations of this paper are due to P.Vopěnka. I also thank K.Čuda for many valuable remarks to the studied matter.

1. Interval π–symmetries (equivalences).

Let the letters x, y, z (event. with indices) be variables for rational numbers from Q.

Definition. A symmetry R is called to be an interval if

$$(\forall x, y, z)(\langle x, z\rangle \in R \;\&\; x \leq y \leq z \to \langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in R).$$

Obviously if $\mathcal{M} \neq 0$ is a class of interval symmetries (equivalences) then $\cap\mathcal{M}$ is an interval symmetry (equivalence). If R is a symmetry then we denote

$$\overline{R} = \{\langle x, y\rangle; (\exists x_1, y_1)(\langle x_1, y_1\rangle \in R \;\&\; x_1 \leq x, y \leq y_1)\}.$$

Obviously $\overline{R}$ is an interval symmetry. If R is an equivalence then $\overline{R}$ is an interval equivalence. If R is a π–class then $\overline{R}$ is also a π–class.

Definition. Let R be an interval symmetry. We say that X is an R–cut if

1) $X \subseteq Q$ & $\emptyset \neq X \neq Q$,
2) $(\forall x, y)(x \in X$ & $y \leq x \rightarrow y \in X)$,
3) $R''X = X$.

We say that x is its inner or outer R–head if $X = \{y; y \leq x\} \cup R''\{x\}$ or $Q - X = \{y; x \leq y\} \cup R''\{x\}$ respectively.

Proposition 1. *Let $S \subseteq R$ be two interval symmetries. Let X be an R-cut. Then X is an S-cut. If x is moreover an inner (outer) S-head of the cut X, then x is an inner (outer) R-head of the cut X.*

PROOF: $X \subseteq S''X \subseteq R''X = X$. Let $X = \{y; y \leq x\} \cup S''\{x\}$ then $X \subseteq \{y; y \leq x\} \cup R''\{x\} \subseteq R''X = X$. ∎

We say that a property $\varphi(n)$ holds for almost all $n \in FN$ if there exists an $m \in FN$ such that $\varphi(n)$ holds for all $n \geq m$.

Proposition 2. *Let $\{R_n; n \in FN\}$ be a sequence of interval π-symmetric such that $R_{n+1} \subseteq R_n$ for all n. Put $R = \bigcap\{R_n; n \in FN\}$. Let $X \subseteq Q$ be such that the classes $X, Q - X$ are revealed. The following holds:*

(a) *$R''X = X$ iff $R_n''X = X$ for almost all $n \in FN$.*
(b) *X is an R-cut iff X is an R_n-cut for almost all n. Moreover X has an inner (outer) R-head iff X has an inner (outer) R_n-head for almost all n.*

PROOF: (a) The case of $X = \emptyset$ or $X = Q$ is trivial, hence let $\emptyset \neq X \neq Q$. Let $R_m''X = X$, then $X \subseteq R''X \subseteq R_m''X = X$. On the other hand let $X = R''X$. Let us suppose that $X \neq R_n''X$ for all $n \in FN$, hence $R_n \cap (X \times (Q - X)) \neq \emptyset$ for all n. Then $R \cap (X \times (Q - X)) \neq \emptyset$, thus $X \neq R''X$ - a contradiction. By that we have proved that there exists an m such that $X = R_m''X$. Let $n \geq m$ then $X \subseteq R_n''X \subseteq R_m''X = X$.

(b) From (a) it follows that X is an R–cut iff there exists an m such that X is an R_n–cut for all $n \geq m$. Let x be an inner (outer) R–head of the cut X. Then (by the proposition 1) x is an inner (outer) R_n–head of X for almost all n. Let conversely x_n be an inner R_n–head of the cut X for all $n \geq m$. Let $x \in X$ be such that $x_n \leq x$ for all $n \geq m$. We prove that x is an inner R–head of the cut X. Let $x \leq y$. If $y \in X$ then $\langle x_n, y\rangle \in R_n$ for all $n \geq m$, thus $y \in R''\{x\}$. Hence $X \subseteq \{y; y \leq x\} \cup R''\{x\} \subseteq R''X = X$, which means that x is an inner R–head of X. The case of the outer head is similar. ∎

Proposition 3. *Let R be an interval π-symmetry. Let X be an Sd-class such that $X \subseteq Q, \emptyset \neq X \neq Q, R''X = X$. Then there exists a set-theoretically definable R-cut Y.*

PROOF: Obviously $Q - X$ is an Sd–class and $R''(Q - X) = Q - X$. Let us assume that $Q - X$ is not an R–cut. Then there exist $x_0 \in X, y_0 \in (Q - X), x_0 < y_0$, thus $x_0 \in X, y_0 \notin X$. Put $Y = \{x; (\exists y \in X)(x \leq y < y_0)\}$. Obviously Y is an Sd–class which satisfies the first two conditions from the definition of R–cut. Let us prove that it satisfies the third condition. By contradiction let us assume that there exist

$x_1 \in Y, z \notin Y$ such that $\langle x_1, z\rangle \in R$. Let $x_2 \in X$ be from the definition of Y such that $x_1 \le x_2 < y_0$. Obviously $x_2 < z$ because otherwise it would be $z \in Y$. It implies $\langle x_2, z\rangle \in R$, thus $z \in R''X = X$. If $y_0 \le z$ then we would have $\langle x_2, y_0\rangle \in R$, thus $y_0 \in R''X = X$, but $y_0 \notin X$. Consequently $z < y_0$. Since $z \in X$, $z \in Y$, and this is the desired contradiction. ∎

Proposition 4. *Let R be an interval π-symmetry. Then there exists its generating sequence $\{R_n; n \in FN\}$ such that R_n is an interval Sd-symmetry for all n. Moreover if R is an equivalence then $R_{n+1} \circ R_{n+1} \subseteq R_n$ can be assumed for all n.*

PROOF: Let $\{S_n; n \in FN\}$ be a generating sequence of the π–symmetry R. Obviously $\overline{S}_n$ is an interval Sd–symmetry, $\overline{S}_{n+1} \subseteq \overline{S}_n, S_n \subseteq \overline{S}_n$ for all $n \in FN$. From this $R = \bigcap\{S_n; n \in FN\} \subseteq \bigcap\{\overline{S}_n; n \in FN\}$. Let $\langle x, y\rangle \in \bigcap\{\overline{S}_n; n \in FN\}$. We want to prove $\langle x, y\rangle \in R$. There exists a sequence $\{\langle x_n, y_n\rangle; n \in FN\}$ such that $\langle x_n, y_n\rangle \in S_n, x_n \le x, y \le y_n$ for all n. Let $\{\langle x_\alpha, y_\alpha\rangle; \alpha \in \gamma\}$ be a prolongation of this sequence such that $x_\alpha \le x, y \le y_\alpha$ for $\alpha \in \gamma$ and $\langle x_\alpha y_\alpha\rangle \in S_n$ for $n \in FN$, $\alpha \ge n$. Hence $\langle x_\alpha, y_\alpha\rangle \in R$ if $\alpha \in \gamma - FN$ and since R is an interval symmetry, we see $\langle x, y\rangle \in R$. We have proved that $\{\overline{S}_n; n \in FN\}$ is a generating system of the π–symmetry R with the desired properties. If R is moreover an equivalence then by the theorem III.1.1[V] it is possible to select from this sequence a generating subsequence $\{R_n; n \in FN\}$ such that $R_{n+1} \circ R_{n+1} \subseteq R_n$ for all n. ∎

Definition. We say that a class $D \subseteq Q$ is a discrete basis of a π–symmetry R if

1) $(\forall x)(\exists y \in D)(\langle x, y\rangle \in R)$,
2) $(\forall \gamma \in N)$ Set $\{x; x \in D \;\&\; -\gamma \le x \le \gamma\}$.

We say that $x, y \in D$ are neighbouring if $x \ne y$ and

$$(\forall z)(\min\{x, y\} < z < \max\{x, y\} \rightarrow z \notin D).$$

Theorem. *Let R be an interval π-symmetry. Then the following conditions are equivalent:*

(a) *There exists a discrete basis of the π-symmetry R.*
(b) *Each set-theoretically definable R-cut has an inner and an outer R-head.*

PROOF: (a)→(b). Let D be a discrete basis of the π–symmetry R. Let X be a set–theoretically definable R–cut. Let $x_0 \in X, y_0 \notin X$ and $x_1, y_1 \in D$ be such that $\langle x_0, x_1\rangle \in R, \langle y_0, y_1\rangle \in R$. Obviously $x_1 \in X, y_1 \notin X$. Let $\gamma \in N$ be such that $-\gamma \le x_1 < y_1 \le \gamma$. Put $u = X \cap \{x; x \in D \;\&\; -\gamma \le x \le \gamma\}$. We see $x_1 \in u, y_1 \notin u$. Let x_2 be the greatest element in the set u in the natural ordering of $Q, y_2 \in D, y_2 > x_2$ its neighbour in D. Obviously $y_2 \notin X$. If $x_2 \le z \le y_2$ then either $\langle x_2, z\rangle \in R$ or $\langle z, y_2\rangle \in R$ and these two cases exclude one another because $\langle x_2, z\rangle \in R$ implies $z \in R''X = X$ and $\langle z, y_2\rangle \in R$ implies $z \in R''(Q - X) = Q - X$. From this it follows $X = \{y; y \le x_2\} \cup R''\{x_2\}$, $Q - X = \{y; y \ge y_2\} \cup R''\{y_2\}$. Consequently x_2 is an inner and y_2 and y_2 an outer R–head of the R–cut X.

(b)→(a). Let $\{R_n; n \in FN\}$ be a generating sequence of the π–symmetry R such that R_n is an interval Sd–symmetry for all n (see Proposition 4). There exist (by the theorem III.1.3[V]) Sd–classes D_n such that D_n is a maximal R_n–net. If $m \leq n$ then $R_n \subseteq R_m$ and hence

(1') $(\forall n)(m \leq n \rightarrow (\forall x)(\exists y \in D_n)(\langle x,y\rangle \in R_m))$.

We will prove that the following holds:

(2') $(\forall n)(\forall \gamma \in N)\,\mathrm{Set}\{x, x \in D_n \ \&\ -\gamma \leq x \leq \gamma\}$.

Choose an $n \in FN$ and let $\gamma \in N$ be such that (2') does not hold. Put $Y = \{y; \mathrm{Set}\{x; x \in D_n \ \&\ -\gamma \leq x \leq \gamma \ \&\ x \leq y\}\}$. Obviously Y is an Sd–class satisfying the first two conditions from the definition of the R–cut. We prove $Y = R_n''Y$. Let $x_0 \in Y, y_0 \notin Y, \langle x_0, y_0\rangle \in R_n$, obviously $x_0 < y_0$. If $z_1, z_2 \in D_n$ would be such that $x_0 \leq z_1 < z_2 \leq y_0$ then $\langle z_1, z_2\rangle \in R_n$, which is impossible because D_n is an R_n–net. So between x_0, y_0 there lies at most one element of the class D_n and so $y_0 \in Y$ – a contradiction. By that we have proved that Y is an R_n–cut and thus also an R–cut. Let x_1 be its inner and y_1 outer R–head and thus also R_n–head (see Proposition 1). Let $z_1, z_2, z_3 \in D_n$ be such that $x_1 \leq z_1 < z_2 < z_3 \leq y_1$. Then either $\langle x, z_2\rangle \in R_n$ and so $\langle z_1, z_2\rangle \in R_n$ or $\langle z_2, y_1\rangle \in R_n$ and so $\langle z_2, z_3\rangle \in R_N$. Between x_1, y_1 thus there can lie at most two elements of the class D_n. This implies $y_1 \in Y$ – a contradiction.

Let $\{D_\alpha; \alpha \in \delta\}$ be an Sd^*–prolongation of the sequence $\{D_n; n \in FN\}$ such that for all $\alpha \in \delta$ the following holds:

$$(\forall \gamma \in N)\,\mathrm{Set}\{x; x \in D_\alpha \ \&\ -\gamma \leq x \leq \gamma\}.$$

Let $\delta_m \in (\delta - FN)$ be such for all $\alpha \in N, m \leq \alpha \leq \delta_m$, it holds

$$(\forall x)(\exists y \in D_\alpha)(\langle x,y\rangle \in R_m).$$

Choose $\alpha \in \delta - FN$ so that $\alpha \leq \delta_m$ for all m. Then

$$(\forall m)(\forall x)(\exists y \in D_\alpha)(\langle x,y\rangle \in R_m),$$

thus $R_m''\{x\} \cap D_\alpha \neq \emptyset$ for every m, x. Since D_α is an Sd^*–class, D_α is revealed and so $R''\{x\} \cap D_\alpha = \bigcap\{R_m''\{x\}; m \in FN\} \neq \emptyset$. But it means that

$$(\forall x)(\exists y \in D_\alpha)(\langle x,y\rangle \in R).$$

We have proved that D_α is a discrete basis of the π–symmetry R. ∎

Proposition 5. *Let R be an interval π–symmetry which has a discrete basis D. Let X be an R–cut, X a sharp class, i.e.* $(\forall u)\,\mathrm{Set}(X \cap u)$. *Then X is an Sd–class.*

PROOF: Let $\gamma \in N$ be such that $-\gamma \in X, \gamma \notin X$. Put $d = \{y; -\gamma \leq y \leq \gamma \ \&\ y \in D\}$. Then $X = \{y; y \leq \gamma\} \cup R''(d \cap X), Q - X = \{y; \gamma \leq y\} \cup R''(d - X)$, thus X and $Q - X$ are π–classes and so Sd–classes. ∎

Proposition 6. *Let R be an interval compact π–symmetry. Then R has a discrete basis which is a set and the class $\mathcal{M}$ of all set–theoretically definable R–cuts is at most countable.*

PROOF: Let $\{R_n; n \in FN\}$ be a generating sequence of the π–symmetry R. Let d be a set such that $(\forall x)(\exists y \in d)(\langle x,y\rangle \in R)$ (see Theorem III.1.5[V]). Obviously d is a discrete basis of the π–symmetry R. If $x \in d$ then $\bar{x}$ denote its right neighbour in the set d. For $X \in \mathcal{M}$ let c_X denote the greatest element of the set $X \cap d$. Obviously $\langle c_X, \bar{c}_X\rangle \notin R$ and $c_X \neq c_Y$ for $X, Y \in \mathcal{M}, X \neq Y$. Put $A_n = \{c_X; X \in \mathcal{M}$ & $\langle c_X, \bar{c}_X\rangle \notin R_n\}$. Obviously

$$\{c_X; X \in \mathcal{M}\} = \cup\{A_n; n \in FN\}.$$

Thus it suffices to prove that each A_n is a finite class. If $x, y \in A_n, x < y$ then $\bar{x} \leq y$ from the definition of $\bar{x}$, hence $\langle x,y\rangle \notin R_n$. But this means that A_n is an R_n–net and so by the theorem III.1.3[V] A_n is finite. ∎

More generally as a consequence of some deeper results of [Č] it can be proved that the class of all clopen figures in a compact π–symmetry is countable.

2. Interval π–symmetries and automorphisms.

Let $\stackrel{\circ}{=}$ mean the basic equivalence on the universe V (see [ČK] or the section V.1[V]).

Proposition 1. *Let $X \subseteq Q$ be an Sd–cut. Then* $\mathrm{Fig}_{\stackrel{\circ}{=}}(X)$ *is also a cut.*

PROOF: If $F : V \to V$ is an automorphism then obviously $F''X$ is also an Sd–cut. Since $\mathrm{Fig}_{\stackrel{\circ}{=}}(X) = \cup\{F''X; F \text{ is an automorphism}\}, \mathrm{Fig}_{\stackrel{\circ}{=}}(X)$ is a cut. ∎

Proposition 2. *Let F be an automorphism, $X \subseteq Q$ an Sd–cut which is not $Sd_\spadesuit$. Then $F''X \neq \mathrm{Fig}_{\stackrel{\circ}{=}}(X)$.*

PROOF: Let us suppose that $F''X = \mathrm{Fig}_{\stackrel{\circ}{=}}(X)$. It implies that $\mathrm{Fig}_{\stackrel{\circ}{=}}(X)$ is an Sd–class. It is also a $\stackrel{\circ}{=}$–figure, it is proved in the section V.1[V] that then it is an $Sd_\spadesuit$–class. Consequently X is an $Sd_\spadesuit$–class – a contradiction. ∎

Sd–cuts represent classical real numbers in the sense of Dedenkind's cuts. The following proposition says that these Sd–cuts are being moved by automorphisms in the limits given by $Sd_\spadesuit$–cuts which are firm with respect to the automorphisms.

Define an interval π–equivalence

$$R_0 = \cap\{Z^2 \cup (Q - Z)^2; Z \text{ is an } Sd_\spadesuit\text{–cut}\}$$

Proposition 3. *Let X be a cut, then* $\mathrm{Fig}_{\stackrel{\circ}{=}}(X) = \mathrm{Fig}_{R_0}(X)$.

PROOF: Obviously $\mathrm{Fig}_{\stackrel{\circ}{=}}(X) \subseteq \mathrm{Fig}_{R_0}(X)$. From the definition of $\stackrel{\circ}{=}$

$$\mathrm{Fig}_{\stackrel{\circ}{=}}(X) = \cap\{A; A \text{ is } Sd_\spadesuit \ \& \ X \subseteq A\}.$$

For an $Sd_\spadesuit$–class $A \supseteq X$ put $Z_A = \{y \in A; (\forall z)(z \leq y \to z \in A)\}$, it is an $Sd_\spadesuit$–cut. Since $\mathrm{Fig}_{\stackrel{\circ}{=}}(X) = \cap\{Z_A; A \text{ is } Sd_\spadesuit \ \& \ X \subseteq A\}, \mathrm{Fig}_{R_0}(X) \subseteq \mathrm{Fig}_{\stackrel{\circ}{=}}(X)$. ∎

$Sd_\emptyset$–cuts occupy a special place among all rational cuts. Thus let us define a class of concrete real numbers:

$$CR = \{X; X \text{ is an } Sd_\emptyset\text{–cut } \ \& \ X \text{ has not a last element}\}.$$

All finite rational and algebraic numbers, π, e etc. belong in the classical sense to CR. This class is countable and is closed under algebraic operations and under the operation of supremum over $Sd_\emptyset$–subclasses. The nonexistence of an infinitesimally small concrete real number is equivalent to the axiom of elementary equivalence.

Now let R be an interval π–symmetry and X an Sd–R–cut. We say that X is limit if X has not its inner or outer head. We will give a sufficient condition on R to have a limit Sd–cut.

In the rest of this section we suppose that the axiom of elementary equivalence holds (i.e. Def $= FV$).

Proposition 4. *Let S be an interval $Sd_\emptyset$-symmetry. If there exists an Sd-cut X of S such that $X \cap BQ \neq \emptyset, BQ - X \neq \emptyset$ and $X \notin Sd_\emptyset$ then S has a limit Sd-cut.*

PROOF: Let X be an Sd–cut, $X \notin Sd_\emptyset, S''X = X$. Let us suppose that $0 \in X$, $1 \notin X$. Let A be a maximal $Sd_\emptyset$–S–net on [0,1]. If A would be a finite class then X could not be limit. If Set(A) then card(A) $\in$ Def but card(A) $\notin FN$. Thus A is a proper uncountable Sd–class. From the theorem of the preceding section it follows that there has to exist an Sd–cut Y of S which is limit. ∎

Corollary. *Let R be an interval $\pi_\emptyset$-equivalence. If R has an Sd-cut X such that $X \cap BQ \neq \emptyset, BQ - X \neq \emptyset$ and $X \notin Sd_\emptyset$ then R has a limit Sd-cut.*

PROOF: See Proposition 1.2. ∎

The converse implication does not hold – the interval $\pi_\emptyset$–equivalence

$$R_+ = \{\langle x, y\rangle; x = y = 0 \text{ or } x \neq 0 \ \& \ y \neq 0 \ \& \ (\forall n)(|\frac{x}{y} - 1| \leq \frac{1}{n})\}$$

has just two Sd–cuts $\{x; x < 0\}$ and $\{x; x \leq 0\}$ which are both $Sd_\emptyset$ and limit.

3. Correspondence between interval π–equivalences and rational Sd–functions.

Definition. Let $F : Q \to Q$ be a function. We define the relation

$$R_F = \{\langle x, y\rangle; (\forall z)(\min\{x, y\} \leq z \leq \max\{x, y\} \to F(x) \doteq F(z) \doteq F(y))\},$$

where $\doteq$ is the standard compact indiscernible equivalence on Q.

Proposition 1. *If $F : Q \to Q$ is an Sd-function then R_F is an interval π-equivalence.*

PROOF: Obviously R_F is an interval equivalence. Let $\{S_n; n \in FN\}$ be a generating system of $\doteq$. Put

$$R_{F,n} = \{\langle x, y\rangle; (\forall z)(\min\{x, y\} \leq z \leq \max\{x, y\} \to \\ \to \langle F(x), F(z)\rangle \in S_n \ \& \ \langle F(z), F(y)\rangle \in S_n)\},$$

then $R_{F,n}$ is an Sd–class and $R_F = \bigcap\{R_{F,n}; n \in FN\}$. ∎

Examples. (i) Let $x \in Q$, then $\alpha \leq x < \alpha + 1$ for an $\alpha \in N \subseteq Q$. Define $F(x) = (x - \alpha)(-1)^{\alpha} + (\alpha + 1 - x)(-1)^{\alpha+1}$ (see fig. 1). Then $R_F = \{\langle x, y\rangle; |x - y| \doteq Q\}$ is a noncompact π–equivalence on Q.

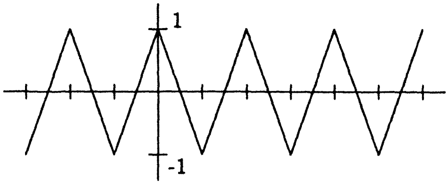

Fig.1

(ii) Let $x \in Q$, then $x = \alpha/\beta$ where $\alpha, \beta \in N$ are relatively prime. Put $F(x) = (-1)^{\alpha}$. Then $R_F = \{\langle x, y\rangle; x = y\}$ is a discrete equivalence on Q.

We want to investigate Sd–functions from Q to Q through interval π–equivalences R_F. Results on interval π–equivalences can be applied on Sd–functions.

In this section we will show that to each interval π–equivalence R there exists an Sd^*–function F such that $R = R_F$.

Definition. Let R be a symmetry on Q, we define the relation of connectedness of R as usually

$$Cntd_R(u) \equiv (\forall v \subseteq u)(\emptyset \neq v \neq u \rightarrow (\exists z_1 \in v)(\exists z_2 \in u - v)(\langle z_1, z_2\rangle \in R)),$$
$$S = \{\langle x, y\rangle; (\exists u)(x, y \in u \ \ \& \ \ Cntd_R(u))\}.$$

Proposition 2. *Let R be an interval symmetry, S the relation of connectedness of R. Then*

(a) *$R \subseteq S$ and S is an interval equivalence.*
(b) *An Sd–class X is R–cut iff it is S–cut.*
(c) *If R is an interval π–symmetry then S is an interval π–equivalence.*

PROOF: (a) It is obvious from the definition that $R \subseteq S$. Since $u_1 \cap u_2 \neq \emptyset$ & $Cntd_R(u_1)$ & $Cntd_R(u_2)$ implies $Cntd_R(u_1 \cup u_2)$, we see that S is an equivalence. Finally let $x < z < y$ and $\langle x, y\rangle \in S$, then there is a $u \subseteq Q$ such that $Cntd_R(u)$ and $x, y \in u$. Put $v = \{x_1 \in u; x_1 \leq z\}$, $z_1 = \max(v), z_2 = \min(u - v)$. Then necessarily $\langle z_1, z_2\rangle \in R$ and so $\langle z_1, z\rangle \in R, \langle z, z_2\rangle \in R$. Consequently $Cntd_R(v \cup \{z\})$, $Cntd_R((u - v) \cup \{z\})$ and $\langle x, z\rangle \in S, \langle z, y\rangle \in S$. We have proved that S is an interval equivalence.

(b) If an Sd–class X is an S–cut, it is also an R–cut because $R \subseteq S$. Let an Sd–class X be an R–cut and $x \in X, y \notin X$ be such that $\langle x, y\rangle \in S$. It means that there is a $u \subseteq Q$ such that $x, y \in u$ and $Cntd_R(u)$. Put $v = u \cap X$, there has to exist $z_1 \in v, z_2 \in u - v$ such that $\langle z_1, z_2\rangle \in R$, but $z_1 \in X, z_2 \notin X$ – a contradiction.

(c) If R is an Sd–class then it is obvious from the definition that S is also Sd. Let R be an interval π–symmetry, $R = \cap\{R_n; n \in FN\}$ where R_n are interval

Sd–symmetries (see Proposition 1.4). Let S_n be the relations of connectedness of R_n, hence S_n are Sd–classes. It holds that (see theorem III.3.1[V])

$$(1) \qquad Cntd_R(u) \Leftrightarrow (\forall n)(Cntd_{R_n}(u)).$$

It is obvious $S \subseteq \cap\{S_n; n \in FN\}$. Let $\langle x,y\rangle \in \cap\{S_n; n \in FN\}$. Then there are $u_n \subseteq Q$ such that $x,y \in u_n$ & $Cntd_{R_n}(u_n)$. Take a prolongation $\{u_\delta; \delta \in \gamma\}$ of the sequence $\{u_n; n \in FN\}$ such that $\delta \in \gamma - FN$ and $n \in FN$ implies $x,y \in u_\delta$ and $Cntd_{R_n}(u_\delta)$. Take a $\delta \in \gamma - FN$, then $x,y \in u_\delta$ and $Cntd_R(u_\delta)$ (see (1)), thus $\langle x,y\rangle \in S$. ∎

Proposition 3. *Let S be an interval Sd–equivalence. Then there exists an Sd–function $G : Q \to \{-1,1\}$ such that $S = R_G$.*

PROOF: Put $S_0 = \{\langle x,y\rangle; (\forall z)(\min\{x,y\} \le z \le \max\{x,y\} \to \langle x,z\rangle \in S$ or $\langle z,y\rangle \in S)\}$, obviously S_0 is an interval Sd–symmetry. Let S_1 be the equivalence of connectedness of S_0. Obviously $S \subseteq S_1$. Let A_1 be a maximal set–theoretically definable S_1–net and $A \supseteq A_1$ a maximal set–theoretically definable S–net. By induction we construct a function G on A_1. Let $P : N \to A_1$ be an Sd–numbering of the class A_1 (if A_1 is a set, we consider $P : \alpha \to A_1$).

I. $G(P(0)) = 1$.
II. $G(P(\alpha)) = -G(x_0)$ where $x_0 = \max\{x \in P''\alpha; x < P(\alpha)\}$ if $\{x \in P''\alpha; x < P(\alpha)\} \ne 0$.
$G(P(\alpha)) = 1$ otherwise.

By this an Sd–function G on A_1 is defined. Now let $x \in A$. Then there exists just one $x_0 \in A_1$ such that $\langle x_0, x\rangle \in S_1$. Thus there is a $u \subseteq Q$ such that $x, x_0 \in u$ and $Cntd_{S_0}(u)$. Put $Z = \{z \in A; \min\{x_0,x\} \le z \le \max\{x_0,x\}\}$. Let $z_1 \in Z$, then there exists a $z_2 \in u$ such that $\langle z_1,z_2\rangle \in S$. Indeed, let $z_1 \notin u$, otherwise it should hold with $z_2 = z_1$. Put $v = \{z \in u; z < z_1\}$, there are $z_2 \in v$, $z_3 \in (u - v)$ such that $\langle z_2,z_3\rangle \in S$. Since $z_2 \le z_1 \le z_3$, also $\langle z_1,z_2\rangle \in S$. Since Z is an S–net and S is an Sd–equivalence, there is a one–one Sd–function from Z into u. Thus Set(Z) and we can put $\alpha = \text{card}(Z)$ and $G(x) = (-1)^{\alpha-1}G(x_0)$.

Finally let $x \in Q$. Then put $G(x) = G(x_0)$ where $x_0 \in A$ is such that $\langle x_0,x\rangle \in S$. We have defined an Sd–function $G : Q \to \{-1,1\}$. It remains to prove that $S = R_G$. It is obvious that $S \subseteq R_G$. Let $\langle x,y\rangle \notin S, x < y$, we can suppose $x,y \in A$. We shall use the common notation $[x,y] = \{z, x \le z \le y\}$ and $(x,y) = \{z; x < z < y\}$. If there exist $z_1,z_2 \in A \cap [x,y], z_1 < z_2$ such that $\langle z_1,z_2\rangle \in S_1$, then from the definition of G it is obvious that $\langle x,y\rangle \notin R_G$. Let us suppose the contrary. Then $(A - A_1) \cap (x,y) = 0$ and $\langle x,y\rangle \notin S_1$. If Set($A \cap [x,y]$) then $u = A \cap [x,y]$ would be S_0 connected –a contradiction. Thus $A \cap [x,y]$ and also $A_1 \cap [x,y]$ is an uncountable proper Sd–class. Let $x = P(\alpha)$, necessarily there exist a $\beta > \alpha$ such that $P(\beta) \in A_1 \cap [x,y]$. Let β_0 be the first such β. Then $G(P(\alpha)) \ne G(P(\beta_0))$, hence again $\langle x,y\rangle \notin R_G$. ∎

Theorem. *Let R be an interval π–equivalence. Then there exists an Sd^*–function F such that $R = R_F$.*

PROOF: Let $R = \bigcap\{R_n; n \in FN\}$ where $\{R_n; n \in FN\}$ is a generating system consisting of interval Sd-symmetries such that $R_{n+1} \circ R_{n+1} \subseteq R_n$, $\{R_\alpha; \alpha \in \gamma\}$ be its Sd^*-prolongation and S_α the relations of connectedness of R_α. We will construct a sequence $\{\langle F_n, A_n\rangle; n \in FN\}$ of Sd-functions $F_n : Q \to Q$ and set-theoretically definable maximal R_n-nets A_n.

We say that an $x \in Q$ lies between connected neighbours $x_1, x_2 \in A_n$ if $x_1 \leq x \leq x_2$, x_1 and x_2 are neighbours in A_n and $\langle x_1, x_2\rangle \in S_n$. We say that x lies on the edge $x_0 \in A_n$ if $x \geq x_0, x \in R''_n\{x_0\}$ and x_0 is maximal in $S''_n\{x_0\} \cap A_n$ or $x \leq x_0, x \in R_n\{x_0\}$ and x_0 is minimal in $S''_n\{x_0\} \cap A_n$. We want to satisfy the following conditions (for $\alpha \in FN$):

(a_α) If $m < \alpha$ then $A_\alpha \supseteq A_m$ and $(\forall z \in A_m)(F_\alpha(z) = F_m(z))$.

(b_α) If $x < y \in A_\alpha$ then

$$(\exists z_1, z_2 \in [x, y] \cap A_\alpha)(|F_\alpha(z_1) - F_\alpha(z_2)| \geq 1/4^\alpha).$$

If moreover $x, y \in A_\alpha$ are connected neighbours then

$$|F_\alpha(x) - F_\alpha(y)| \leq 1/2^\alpha.$$

(c_α) If $m \leq \alpha$ and x lies between connected neighbours $x_1, x_2 \in A_m$, then
(1) $|F_\alpha(x) - F_\alpha(x_i)| \leq 1/2^m + (1/4^{m+1} + \cdots + 1/4^\alpha) \leq 1/2^m + 1/(3.4^m)$ $(i = 1, 2)$.
If x lies on the edge $x_0 \in A_m$, then
(2) $|F_\alpha(x) - F_\alpha(x_0)| \leq 1/4^{m+1} + \cdots + 1/4^\alpha \leq 1/(3.4^m)$. ∎

Lemma. *Let $\{\langle F_k, A_k\rangle; k \leq n\}$ satisfy the conditions $(a_k), (b_k)$ and (c_k) $(k = 0, \ldots, n)$. Then there exists an Sd-function $F_{n+1} : Q \to Q$ and a set-theoretically defined maximal R_{n+1}-net A_{n+1} satisfying again the conditions $(a_{n+1}), (b_{n+1})$ and (c_{n+1}).*

PROOF: Let $A_{n+1} \supseteq A_n$ be a maximal set-theoretically defined R_{n+1}-net. Proposition 3 says that there is an Sd-function G such that $S_{n+1} = R_G$. Let us define F_{n+1} firstly in the points of A_{n+1}. For $z \in A_n$ put $F_{n+1}(z) = F_n(z)$. For $z \in A_{n+1} - A_n$ we distinguish two cases:

A. z lies between two connected neighbours $x, y \in A_n$.

If $\langle x, y\rangle \in S_{n+1}$, then in all points $z \in A_{n+1} \cap (x, y)$ define $F_{n+1}(z)$ so that

(a) $F_{n+1}(z)$ lies between the values $F_n(x)$ and $F_n(y)$,

(b) if z_1, z_2 are neighbouring in A_{n+1}, then

$$|F_{n+1}(z_1) - F_{n+1}(z_2)| \in [1/4^{n+1}, 1/2^{n+1}].$$

There is a $z \in (A_{n+1} \cap (x, y))$ because $R_{n+1} \circ R_{n+1} \circ R_{n+1} \subseteq R_n$. Let us suppose the contrary, it means $[x, y] \subseteq R''_{n+1}\{x, y\}$. Since $\langle x, y\rangle \in S_{n+1}$ and $\langle x, y\rangle \notin R_{n+1}$ there are $z_1 \in R''_{n+1}\{x\}, z_2 \in R''_{n+1}\{y\}, x \leq z_1 \leq z_2 \leq y$ such that $\langle z_1, z_2\rangle \in R_{n+1}$. It implies $\langle x, y\rangle \in R_n$ – a contradiction. Moreover by the induction hypothesis

$|F_n x - F_n y| \in [1/4^n, 1/2^n]$. It is thus possible to satisfy these two conditions (see fig.2).

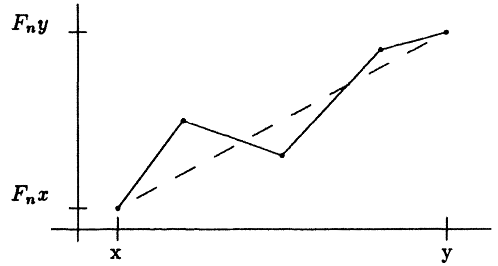

Fig. 2

Let $\langle x, y\rangle \notin S_{n+1}$. Put $d = F_n(y) - F_n(x)$. In all points $z \in A_{n+1} \cap (x, y)$ define $F_{n+1}(z)$ so that

(a) $F_{n+1}(z) \in \{F_n(x), F_n(x) + d/4\}$ if $G(x) = G(z)$,
(b) $F_{n+1}(z) \in \{F_n(y) - d/4, F_n(y)\}$ if $G(x) \neq G(z)$ or $z \in S''_{n+1}\{y\}$,
(c) if $z_1, z_2 \in A_{n+1} \cap [x, y]$ are neighbouring, then

$$|F_{n+1}(z_1) - F_{n+1}(z_2)| \geq d/4.$$

It is obvious (see fig.3) that these three conditions can be satisfied.

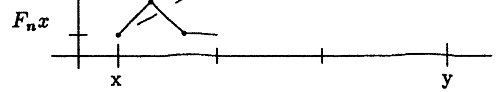

Fig. 3

B. z lies on the edge $x \in A_n$. Let $z \in S_n\{x\} \cap [x, \infty)$ where $x = \max(S''_n\{x\} \cap A_n)$, the second case is similar. Define F_{n+1} in all points $z \in S''_n\{x\} \cap [x, \infty) \cap A_{n+1}$ so that

(a) $F_{n+1}(z) \in \{F_n(x) - 1/4^{n+1}, F_n(x)\}$ if $G(x) = G(z)$,
(b) $F_{n+1}(z) \in \{F_n(x), F_n(x) + 1/4^{n+1}\}$ if $G(x) \neq G(z)$
(c) if z_1, z_2 are neighbouring in A_{n+1}, then

$$|F_{n+1}(z-1) - F_{n+1}(z-2)| \geq 1/4^{n+1}.$$

Again it is possible to satisfy these conditions (see fig. 4).

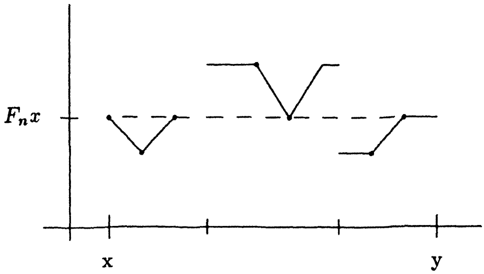

Fig. 4

On the rest of Q define F_{n+1} so that it is linear on $[x,y]$ where $x,y \in A_{n+1}$ are connected neighbours or constant on $S''_{n+1}\{x\} \cap [x,\infty)$ or $S''_{n+1}\{x\} \cap \langle(-\infty,x]$ where x is maximal or minimal in $S''_{n+1}\{x\} \cap A_{n+1}$ respectively.

The conditions (a_{n+1}) and (b_{n+1}) are obvious from the construction of F_{n+1}. Let us prove (c_{n+1}).

Firstly let $m = n+1$. If x lies between connected neighbours $x_1, x_2 \in A_m$, then $|F_m(x) - F_m(x_i)| \leq 1/2^m$ $(i = 1,2)$. If x lies on the edge $x_0 \in A_m$, then $|F_m(x) - F_m(x_0)| = 0$.

Secondly, let $m < n+1$, then the induction hypothesis states that (c_n) with this m holds. Let x lie between connected neighbours $z_1, z_2 \in A_n$. Then $F_{n+1}(x)$ lies between $F_n(z_1), F_n(z_2)$ and since in the first case of $(c_{n+1}) z_1, z_2$ lie between $x_1, x_2 \in A_m$, in the second case z_1, z_2 lie on the edge $z_0 \in A_m$, (1) or (2) of (c_{n+1}) holds. Let x lie on the edge $z_0 \in A_n$, then $|F_{n+1}(x) - F_n(z_0)| \leq 1/4^{n+1}$ as it follows from the construction and since in the first case z_0 lies between $x_1, x_2 \in A_m$, in the second one on the edge $x_0 \in A_m$, we see that (1) or (2) of (c_{n+1}) again holds. ∎

We can suppose that $R_0 = Q^2$, then put $F_0 = 0, A_0 = \{0\}$. From the lemma it follows that there exists a sequence $\{\langle F_n, A_n\rangle; n \in FN\}$ with the desired properties. Let $\{\langle F_\alpha, A_\alpha\rangle; \alpha \in \gamma\}$ be an Sd^*–prolongation consisting of Sd^*–functions $F_\alpha : Q \to Q$ and Sd^*–maximal R_α–nets A_α such that $(a_\alpha), (b_\alpha), (c_\alpha)$ for $\alpha \in \gamma$ hold.

Take an $\alpha \in \gamma - FN$ and put $F \equiv F_\alpha$. It remains to prove that $R = R_F$. Firstly observe that F is bounded, indeed $|F(x)| \leq 1/3$ for $x \in Q$ as follows from (c_α) with $m = 0$.

Let $\langle x,y\rangle \in R$ and $x_0, y_0 \in A_n$ be such that $x \in R''_n\{x_0\}, y \in R''_n\{y_0\}$ and $x_0 = y_0$ or x_0, y_0 are connected neighbours in A_n. Necessarily there are such x_0, y_0. From $(a_\alpha), (b_n)$ and (c_α) it follows

$$|F(x) - F(y)| \leq |F_\alpha(x) - F_\alpha(x_0)| + |F_n(x_0) - F_n(y_0)| + |F_\alpha(y) - F_\alpha(y_0)| \leq$$
$$\leq 1/2^n + 2(1/2^n + 1/(3 \cdot 4^n)).$$

Since it holds for each $n \in FN$, $F(x) \doteq F(y)$. We have proved generally $(\forall x,y)$ $(\langle x,y\rangle \in R \to F(x) \doteq F(y))$. Thus $\langle x,y\rangle \in R$ implies $\langle x,y\rangle \in R_F$.

On the other hand let $\langle x,y\rangle \notin R, x<y$. If there are $n\in FN$ and $x_0,y_0\in A_n$ such that $x\le x_0<y_0\le y$, then (a_α) and (b_α) imply that $\langle x,y\rangle\notin R_F$. Let $\mathrm{card}(A_n\cap[x,y])\le 1$ for all n. Firstly let us suppose that $A_n\cap(x,y)=0$ for all n. Necessarily $\langle x,y\rangle\notin S$. Let n be such that $\langle x,y\rangle\notin S_n$ and $x_0,y_0\in A_n$ such that $x\in R_n''\{x_0\}, y\in R_n''\{y_0\}$. Then $x_0\le x<y\le y_0$ and x lies on the edge $x_0\in A_n, y$ lies on the edge $y_0\in A_n$. Since $x_0,y_0\in A_n$ are neighbouring, (b_n) and (a_α) imply

$$|F(x_0)-F(y_0)|\ge 1/4^n.$$

Finally from (2) of (c_α) it follows

$$|F(x)-F(y)|\ge|F(x_0)-F(y_0)|-|F(x_0)-F(x)|-|F(y)-F(y_0)|\ge$$
$$\ge 1/4^n-2/(3\cdot 4^n)=1/(3\cdot 4^n).$$

Thus $\langle x,y\rangle\notin R_F$. If $A_n\cap(x,y)=\{x_0\}$ for an $n\in FN$ then $A_m\cap(x,y)=\{x_0\}$ for all $m\ge n$. Obviously $\langle x,x_0\rangle\notin R$ or $\langle x_0,y\rangle\notin R$. Since $\mathrm{card}(A_n\cap(x,x_0))=\mathrm{card}(A_n\cap(x_0,y))=0$ for all $n\in FN$, it holds $\langle x,x_0\rangle\notin R_F$ or $\langle x_0,y\rangle\notin R_F$. This implies $\langle x,y\rangle\notin R_F$. ■

Corollary. *Let R be an interval π-equivalence. Then there exists a nondecreasing Sd^*-function F such that $R=R_F$ iff R is compact.*

PROOF: It is obvious that if F is nondecreasing, then R_F is compact. Let R be compact. By the preceding theorem there exists an Sd^*-function G such that $R=R_G$. It would suffice to construct a "variation" of the function G. But we know that even a classically continuous function has not to have a variation. Nevertheless, in this case it suffices to prove the following ■

Lemma. *Let G be a compact rational Sd-function (it means that R_G is compact). Then there exists its generalized variation, i.e. a nondecreasing Sd-function F such that $R_G=R_F$.*

PROOF: Put $\doteq_\alpha=\{\langle x,y\rangle; |x-y|<1/\alpha \text{ or } x,y>\alpha \text{ or } x,y<-\alpha\}$,

$$R_\alpha=\{\langle x,y\rangle;(\forall z \text{ between } x,y)(\langle G(x),G(z)\rangle\in\doteq_\alpha \;\&\; \langle G(z),G(y)\rangle\in\doteq_\alpha)\}.$$

Let $\gamma>FN$ be such that for each $\alpha\in\gamma,\alpha\ge 1$ there exists a maximal R_α-net u_α. If $G(x)\doteq c$ for all $x\in Q$ then put $F(x)\equiv c$. If there are x,y such that $G(x)\not\doteq G(y)$ then we can suppose that there are x,y such that $G(x)=0,G(y)=1$. Thus we can suppose that there are $x_0<y_0$ such that

$$(\forall\alpha\in\gamma)(x_0=\min(u_\alpha \;\&\; y_0=\max(u_\alpha)) \text{ and } u_\alpha=\{x_0=x_0^\alpha<\dots<x_{\omega_\alpha}^\alpha=y_0\}.$$

Put

$$F_\alpha(x_\beta^\alpha)=\sum_{\delta=1}^{\beta}|G(x_\delta^\alpha)-G(x_{\delta-1}^\alpha)|.$$

If $\langle x^\alpha_\beta, x^\alpha_{\beta+1}\rangle \in S_\alpha$, then let F_α be linear on $[x^\alpha_\beta, x^\alpha_{\beta+1}]$. If x^α_β is an edge of S_α, let F_α be constant on $S''_\alpha\{x^\alpha_\beta\} \cap [x^\alpha_\beta, \infty)$ or $S''_\alpha\{x^\alpha_\beta\} \cap (-\infty, x^\alpha_\beta]$. Finally put

$$F(x) = \sum_{\alpha\in\gamma,\alpha\geq 1} F_\alpha(x)/(2^\alpha F_\alpha(y_0)) \text{ for } x \in Q.$$

F is rational nondecreasing Sd-function such that $0 \leq F \leq 2$. We want to prove $R_G = R_F$.

If $\langle x, y\rangle \in R_G$, then $F_\alpha(x) \doteq F_\alpha(y)$ for all α and so $F(x) \doteq F(y)$, i.e. $\langle x, y\rangle \in R_F$.

Let $\langle x, y\rangle \notin R_G$ and let $F(x) \doteq F(y)$. Let there be $x_n, y_n \in u_n$ such that $x \leq x_n < y_n \leq y$, then $F(x_n) \doteq F(y_n)$. From the construction of F it follows that $G(x_n) \doteq G(z) \doteq G(y_n)$ for all $z \in (u_m \cap [x_n, y_n]), m \geq n$ and thus $G(x_n) \doteq G(z) \doteq G(y_n)$ for all $z \in [x_n, y_n]$. It means $\langle x_n, y_n\rangle \in R_G$ – a contradiction. Thus $\text{card}([x, y] \cap u_n) \leq 1$ for all $n \in FN$. This implies $\langle x, y\rangle \notin S$ where S is the relation of connectedness of R_G. Thus we have $G(x) \not\doteq G(y)$ and $G(x) \doteq G(z)$ for $z \in S''\{x\} \cap [x, y]$ and $G(z) \doteq G(y)$ for $z \in S''\{y\} \cap [x, y]$. Consequently $F_n(x) \not\doteq F_n(y)$ for an $n \in FN$ and so $\langle x, y\rangle \notin R_F$. ∎

References

[V] Vopěnka P., "Mathematics in the Alternative Set Theory," Teubner–Texte, Leipzig, 1979.

[Č] Čuda K., *A contribution to topology in AST: Compactness*, Comment.Math.Univ.Carolinae **28,1** (1987), 43-61.

[ČK] Čuda K., Kussová B., *Basic equivalences in the alternative set theory*, Comment.Math. Univ. Carolinae **23** (1982), 629-644.

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