Simple motions

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Abstract. We single out a special kind of motion, called here "simple", and study its topological features. The main result (which is the nonstandard version of a theorem of Eilenberg) is that a set u is the range of a simple motion if the space $E = u \times u - \mathrm{Fig}(\Delta)$ can be partitioned into two disjoint open figures ($\Delta = \{\langle x, x\rangle; x \in u\}$).

Keywords: Alternative set theory, Π–equivalence, motion, simple motion, connected set, simply connected set

Classification: 03E70 , 54J05

"Motion" in AST is a function the values of which at near points are indiscernible. This makes it a useful device for describing and simulating real processes. Besides it offers an intuitive approach to topological connectedness.

Here we shall be concerned with the latter, that is some relations between motions and connected spaces.

0.Preliminaries. By a topology we shall understand a π–equivalence $\doteq$ and by a space a pair $\langle w, \doteq\rangle$ where w is a set.

A homeomorphism between the spaces $\langle w_1, \doteq_1\rangle$, $\langle w_2, \doteq\rangle$ is a 1-1, onto function $f : w_1 \to w_2$ such that $(\forall xy)(x \doteq_1 y \leftrightarrow f(x) \doteq_2 f(y))$.

Recall the following definitions and facts from [V]. A figure X (with respect to $\doteq$) is closed iff $X = \mathrm{Fig}(u)$ for some set u.

A set u is said to be connected if there is no partition of u into sets u_1, u_2 such that $\mathrm{Fig}(u_1) \cap \mathrm{Fig}(u_2) = \emptyset$.

A class X is said to be connected if for any two points $x, y \in X$, there is a connected set $u \subseteq X$ such that $x, y \in u$.

Of course we have besides the classical notion of connectedness, according to which the space cannot be decomposed into two disjoint open (hence clopen) sets.

This classical version is stronger than that given in terms of $\doteq$. They coincide, however for closed figures. Namely, the following holds.

Lemma 0.1.

a) *The set u is connected iff Fig(u) is connected.*

b) *Fig(u) is connected iff it cannot be decomposed into two closed disjoint figures.*

[1]This paper was partly written during the author's two–month staying in Prague. Thanks are due to P.Vopěnka, A.Sochor and other people of the Prague Seminar for helpful discussions and suggestions.

PROOF:

a) [V, Ch.III, §3]

b) Immediate from a) and the definitions. ■

As for the open figures the following implication holds:

Lemma 0.2. *If X is an open figure and X is decomposed into two open disjoint figures, then X is disconnected.*

PROOF: Let X_1, X_2 be open figures such that $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$, $X_1, X_2 \neq \emptyset$. Let $x \in X_1$, $y \in X_2$. If $u \subseteq X$ is a set containing x, y and put $u_1 = u \cap X_1$, $u_2 = u \cap X_2$, then $u_1 \cup u_2 = u$, $u_1 \cap u_2 = \emptyset$ and u_1, u_2 are Σ–classes (because X_1, X_2 are such). Hence u_1, u_2 are sets such that $\mathrm{Fig}(u_1) \cap \mathrm{Fig}(u_2) = \emptyset$. Thus, u is disconnected. ■

A function f is said to be a motion if $\mathrm{dom}(f) = \vartheta \in N$ and $f(\alpha) \doteq f(\alpha+1)$ for all $\alpha + 1 \in \vartheta$.

The following is an important characterization of connected sets and classes.

Lemma 0.3.

a) *A set u is connected iff it is the range of a motion.*

b) *A class X is connected iff for any two points $x, y \in X$ there is a motion f such that $f(0) = x$, $f(\vartheta - 1) = y$ and $rng(f) \subseteq X$.*

PROOF:

a) [V, Ch.IV.]

b) Immediate from a) and the definitions. ■

1. Interval topologies. Every motion f with respect to $\doteq$ with domain ϑ induces a topology $\underset{f}{\doteq}$ on ϑ defined as follows:

$$\alpha \underset{f}{\doteq} \beta \text{ iff } (\forall \gamma)(\gamma \text{ between } \alpha, \beta \rightarrow f(\gamma) \doteq f(\alpha)).$$

Clearly we have then

$$(\forall \alpha\beta\gamma)(\alpha \underset{f}{\doteq} \gamma \ \ \& \ \ \alpha < \beta < \gamma \rightarrow \alpha \underset{f}{\doteq} \beta). \tag{1}$$

Because of (1) the equivalence classes of $\underset{f}{\doteq}$ are π–intervals of the segment ϑ, that is convex π-classes. Moreover, no monad of $\underset{f}{\doteq}$ is a set[2]. This leads to the following definition.

Definition. A topology $\doteq$ on a segment ϑ of N is called interval topology (int.top.) iff all of its monads are convex classes.

The int. top. $\doteq$ is said to be non degenerate (n.d.) if all of its monads are proper classes.

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[2] Except of the trivial equivalence $\doteq \; = V^2$.

Examples.

(1) Let I be a π–cut closed under addition. Then the equivalence

$$x \doteq_I y \text{ iff } |x - y| \in I$$

is a n.d. inter. top.

(2) If $\doteq$ is an int. top. on ϑ, then the equivalence $\doteq^{\bullet}$ such that

$$x \doteq^{\bullet} y \text{ iff } \vartheta - 1 - x \doteq \vartheta - 1 - y$$

is an int. top. too, called the inverse of $\doteq$. Clearly, if $\doteq$ is n.d., then so is $\doteq^{\bullet}$.

(3) (This is due to P.Zlatoš). Define on N the equivalence

$$\alpha \sim \beta \text{ iff } (\forall n)(|1 - \frac{\alpha}{\beta}| < \frac{1}{n}) \;\; \& \;\; \alpha, \beta \neq 0,$$

that is the rational $\frac{\alpha}{\beta}$ is near to 1.

This is an int. top. but is degenerate, because for $n \in FN$ $\text{Mon}(n) = \{n\}$. However for every $\alpha > FN$, $\text{Mon}(\alpha)$ is a proper class, so if we choose some $\alpha > FN$ and put

$$x \sim_\alpha y \text{ iff } x + \alpha \sim y + \alpha,$$

then $\sim_\alpha$ is a n.d. int. top.

Interval topologies raise the interesting problem of their classification. In particular n.d. inter. top. which appear in close relation to motions. We shall not deal with this problem here.

We shall cite only the following result:

Proposition 1.1.

a) *The intersection of countably many n.d. int. top. is a n.d. int. top.*

b) *For any n.d. int. top. $\doteq$ there is a cut I (π and closed under addition) such that $\doteq_I \subseteq \doteq$.*

PROOF: a) Let $\doteq_n$, $n \in FN$, be n.d. int. top. Clearly $\doteq = \bigcap_n \doteq_n$ is an int. top. For every x the monads $\text{Mon}_n(x)$, with respect to $\doteq_n$, are all proper π–intervals. If $\bigcap_n \text{Mon}_n(x) = [\gamma, \delta]$ a set–interval, then we easily deduce from the properties of π–classes that there is a finite number of monads whose intersection is $[\gamma, \delta]$. But then, clearly, some of them is equal to $[\gamma, \delta]$, a contradiction.

b) Let $\text{o}(n, x)$, $n \in FN$ be for every x the sequence of sets such that $\text{Mon}(x) = \bigcap\{\text{o}(n, x) : n \in FN\}$. Clearly, $\text{o}(n, x)$ can be all intervals containing x as an interior point, and let $\text{o}(n, x) = [x_n^0, x_n^1]$. Choose $\gamma > FN$ such that $[0, \gamma] \subseteq \text{Mon}(0)$ and $[\vartheta - 1 - \gamma, \vartheta - 1] \subseteq \text{Mon}(\vartheta - 1)$. Put

$$\alpha_n = \min(\{|x - x_n^0|, |x - x_n^1| : x \in [\gamma, \vartheta - 1 - \gamma]\} \cup \{\gamma\}).$$

Since $\doteq$ is n.d. $\alpha_n > FN$ for every n. Choose some β such that $FN < \beta < \alpha_n$ for every $n \in FN$ and put $I = \bigcap\{\beta/_{n} \colon n \in FN\}$.

Clearly I is a π–cut closed under addition and if $|x - y| < I$, then $|x - y| < \alpha_n$, hence $y \in \text{o}(n, x)$ for every n, thus $x \doteq y$.

2. Simple motions. It is evident that for every motion f

(2)
$$(\forall\alpha\beta)(\alpha \underset{f}{\doteq} \beta \rightarrow f(\alpha) \doteq f(\beta)).$$

The converse of (2) would mean that the motion f enters at most once each monad and this is the simplest kind of motion we can imagine.

So we give the following.

Definition. A motion f is simple if

i) f is $1-1$
ii) $(\forall\alpha\beta)(\alpha \underset{f}{\doteq} \beta \leftrightarrow f(\alpha) \doteq f(\beta)$.

The set u is said to be simply connected if it is the range of a simple motion. (This term has a different meaning in classical topology).

There is a close connection between simple motions and n.d. int. top. because of the following straightforward result.

Proposition 2.1..

a) *The motion f is simple iff f is a homeomorphism between the spaces $\langle dom(f), \underset{f}{\doteq}\rangle$ and $\langle rng(f), \doteq\rangle$.*
b) *u is simply connected iff it is the homeomorphic image of a space $\langle\vartheta, \doteq\rangle$ where $\doteq$ is a n.d. int. top.*

PROOF: Both claims are immediate from the definitions. ■

From b) above we see that simply connected sets posses an inherent linear ordering transferred to them by the homeomorphism and the natural ordering of ϑ.

We shall see in the sequel that this ordering is almost unique.

Lemma 2.2. *Let $\doteq_1$, $\doteq_2$ be two n.d. int. top. on ϑ and let $f : \langle\vartheta, \doteq_1\rangle \rightarrow \langle\vartheta, \doteq_2\rangle$ be a homeomorphism. Let $Mon_1(x)$, $Mon_2(x)$ be the monads of x with respect to $\doteq_1$, $\doteq_2$. Then, either*

$$(\forall xy)(x < y \rightarrow Mon_2(f(x)) \leq Mon_2(f(y))$$

or

$$(\forall xy)(x < y \rightarrow Mon_2(f(y)) \leq Mon_2(f(x)).$$

(Let us call f almost increasing and almost decreasing in the two cases respectively).

PROOF: We first show that one of the following holds:

a) $f(0) \doteq 0$ and $f(\vartheta - 1) \doteq \vartheta - 1$
b) $f(0) \doteq \vartheta - 1$ and $f(\vartheta - 1) \doteq 0$.

Clearly it suffices to see that $f(0) \doteq 0$ or $f(0) \doteq \vartheta - 1$. The case for $\vartheta - 1$ is similar and the combination of them yields a) or b).

Suppose that $f(0) = x \not\doteq 0, \vartheta - 1$. Then either $\mathrm{rng}(f) \subseteq [0, \mathrm{Mon}_2(x)]$ or $\mathrm{rng}(f) \subseteq [\mathrm{Mon}_2(x), \vartheta - 1]$.

(Both of them are impossible because f is onto).

Indeed let $y < z$ be such that $f(y) <$ Mon$_2(x) < f(z)$ or $f(z) <$ Mon$_2(x) < f(y)$. Let $\beta = \max\{\alpha : f(\alpha) < x\}$, $\gamma = \max\{\alpha : f(\alpha) > x\}$. Then, some of them, e.g. β, is less then $\vartheta - 1$. But $f(\beta + 1) < x$ too, since $f(\beta + 1) \doteq f(\beta)$. This contradicts the definition of β.

Therefore a) or b) holds.

We claim that if a) is true then f is almost increasing and if b) is true then f is almost decreasing. Let us verify the first claim the second being similar.

Assume a) is true and for some pair $x < y$ Mon$_2(f(y)) <$ Mon$_2(f(x))$. Then inductively we see that $f(y+1), f(y+2), \ldots, f(\vartheta - 1) < f(x)$. But $f(\vartheta - 1) \doteq \vartheta - 1$ a contradiction. ■

Lemma 2.3. *Let ϑ, $\doteq_1$, $\doteq_2$, f be as in the previous lemma. Then, either*

$$(\forall x)(f(x) \doteq_2 x) \text{ or } (\forall x)(f(x) \doteq_2 \vartheta - 1 - x).$$

PROOF: By the preceding lemma f is either almost increasing or almost decreasing. Assume the first. It suffices to show $f''m = m$ for every monad m of $\doteq_1$. Suppose $f''m = \mu \neq m$. Then the cuts $I_1 = [0, m)$ and $I_2 = [0, \mu)$ are different. However, since f is almost increasing $f''I_1 = I_2$. By overspill and the fact that f is $1-1$ this is easily proved to be a contradiction.

Suppose f is almost decreasing. Then we have to show that $f''m = \vartheta - m = \{\vartheta - 1 - x : x \in m\}$. This is proved as before. ■

Theorem 2.4. *Let f, g be simple motions (with respect to the same $\doteq$) with same domain ϑ and same range u. Then, either*

$$(\forall x)(f(x) \doteq g(x)) \text{ or } (\forall x)(f(x) \doteq g(\vartheta - 1 - x)).$$

PROOF: Put $h = g^{-1} \circ f$. Then

$$x \underset{f}{\doteq} y \leftrightarrow f(x) \doteq f(y) \leftrightarrow g^{-1}f(x) \underset{g}{\doteq} g^{-1}f(y).$$

Therefore h is a homeomorphism between the spaces $\langle \vartheta, \underset{f}{\doteq} \rangle$, $\langle \vartheta, \underset{g}{\doteq} \rangle$ where $\underset{f}{\doteq}$, $\underset{g}{\doteq}$ are n.d. int. top. According to the preceding lemma, $(\forall x)(h(x) \underset{g}{\doteq} x)$ or $(\forall x)(h(x) \underset{g}{\doteq} \vartheta - 1 - x)$ from whence the conclusion follows. ■

Given a simply connected set u, "enumerated" by the simple motion f, we can say that to every monad Mon$(f(\alpha))$ there corresponds the "symmetric" monad Mon$(f(\vartheta - \alpha))$. The preceding theorem says that this pairing of monads is independent of the particular motion f used to enumerate u. Moreover the following holds.

Theorem 2.5. *Let u be simply connected and let $f : u \to u$ be a homeomorphism. Then for every pair of symmetric monads m, m' of u, either $f''m = m$ or $f''m = m'$.*

PROOF: Take some simple motion g such that rng$(g) = u$. Then the function $h = f \circ g$ is a simple motion too with same domain and range u. By the previous theorem we have either $(\forall x)(h(x) \doteq g(x))$ or $(\forall x)(h(x) = g(\vartheta - 1 - x))$.

In the first case $f \circ g(x) \doteq g(x)$ hence $f''m = m$ while in the second $f \circ g(x) \doteq g(\vartheta - 1 - x)$ hence $f''m = m'$ since the monads of $g(x)$ and $g(\vartheta - 1 - x)$ are symmetric. ∎

Corollary 2.6. *If u is a simply connected set with $|u| = \vartheta$ we have essentially two ways to range over its elements by a simple motion. And these are inverse to each other that is, if one goes from 0 to $\vartheta - 1$ the other traces the same steps back from $\vartheta - 1$ to 0. "Essentially" means that we ignore disturbances in the interior of monads which are described by the relations $f(x) \doteq g(x)$, $f(x) \doteq g(\vartheta - 1 - x)$.*

3.Simply connected sets. In this section we shall give a necessary and sufficient condition in order that a connected set be simply connected. This result (Theorem 3.6) in its classical version concerns orderability of topological spaces and is due to S. Eilenberg (see [E]). The modifications of the version adapted here concern the treatment of monads.

Lemma 3.1. *Let u be a connected set. Then u is simply connected iff there is a set-ordering $<$ on u such that*

$$(\forall xy)(x < y \ \& \ x \not\doteq y \rightarrow Mon(x) < Mon(y)) \tag{3}$$

(where $Mon(x) < Mon(y)$ has the obvious meaning).

PROOF: Since u is connected all monads of u are proper classes. Suppose u is simply connected and $u = \text{rng}(f)$ with $\text{dom}(f) = \vartheta$ where f is simple. If we put

$$f(\alpha) < f(\beta) \text{ iff } \alpha < \beta,$$

then clearly $<$ has the property required. Conversely suppose $<$ has the property (3) and let $|u| = \vartheta$. Obviously there is a function $f : \vartheta \rightarrow u$, $1-1$, onto, such that $\alpha < \beta \leftrightarrow f(\alpha) < f(\beta)$. Since the monads of u are not sets $f(\alpha) \doteq f(\alpha+1)$, otherwise there is some x such that $\text{Mon}(f(\alpha)) < x < \text{Mon}(f(\alpha+1))$, which contradicts the fact that f preserves the ordering. Hence f is a motion and, by (3), it is a simple motion. ∎

In fact we can weaken considerably the assumptions of the last lemma.

Given a set-relation $<$ on a space $\langle u, \doteq\rangle$, consider the following properties:

(A) $(\forall xy)$ (exactly one of the relations $x < y$, $x = y$, $y < x$ holds)
(B) $(\forall xy)(x < y \ \& \ x \not\doteq y \rightarrow \text{Mon}(x) < \text{Mon}(y))$
(C) $(\forall xyz)(x, y, z$ not all in the same monad and $x < y$ and $y < z$ then $x < z)$.

Clearly (A), (B), (C) together are weaker than the assumptions of Lemma 3.1, because (C) is weaker than full transitivity of $<$.

We are going to prove, first, that (A)–(C) suffice for a connected set u to be simply connected and, second that, in a connected u (A) and (B) imply (C). Therefore, finally, a connected u satisfying (A) and (B) is simply connected.

For the first goal, clearly, we have to show that (A), (B), (C) imply transitivity not necessarily for $<$, but for a relation $<^*$ satisfying (A), (B) and agreeing with $<$ on pairs $\langle x, y\rangle$ such that $x \not\doteq y$.

Definition. A cycle (with respect to $<$) is a set-sequence

$$\{\langle x_0, x_1\rangle, \langle x_1, x_2\rangle \dots, \langle x_{\alpha-1}, x_\alpha\rangle, \langle x_\alpha, x_0\rangle\} \subseteq < .$$

Lemma 3.2.

(1) *Let u, $\doteq$, $<$ satisfy (A), (B), (C) and let $\{x_0, \dots, x_\alpha\}$ be a set such that $Mon(x_0) \leq Mon(x_1) \leq \dots \leq Mon(x_\alpha)$. Then $Mon(x_0) < Mon(x_\alpha)$.*

(2) *If for some β in the above sequence $Mon(x_\beta) < Mon(x_{\beta+1})$, then $Mon(x_0) < Mon(x_\alpha)$.*

(3) *If $x < y$ and $x \not\doteq y$, then $\langle x, y\rangle$ does not belong to any cycle of $<$.*

PROOF: 1) Observe first that the following is true: For any x, y

$$\text{Mon}(x) < \text{Mon}(y) \leftrightarrow (\exists n \in FN)(\mathbf{o}(n, x) < \mathbf{o}(n, y)) \tag{4}$$

where $\mathbf{o}(n, x)$ are the sets producing the monad of x.

One direction of (4) is trivial and the other is a simple consequence of the prolongation axiom.

Given the set $\{x_0, \dots, x_\alpha\}$ such that $\text{Mon}(x_0) \leq \dots \leq \text{Mon}(x_\alpha)$, consider the function $f : \alpha \to N$ defined by

$$f(\gamma) = \min\{\beta : \mathbf{o}(\beta, x_\gamma) < \mathbf{o}(\beta, x_{\gamma+1})\} \text{ for } \gamma < \alpha$$

(Put $f(\gamma) = 0$ if no such β exists).

It follows from (4) that $f''\alpha \subseteq FN$.

Let $m \in FN$ be a bound for the set $f''\alpha$. Then it is clear that

$$\text{Mon}(x_\gamma) \leq \text{Mon}(x_{\gamma+1}) \leftrightarrow \mathbf{o}(m, x_\gamma) \leq \mathbf{o}(m, x_{\gamma+1}) \text{ for all } \gamma < \alpha \tag{5}$$

Put $P = \{\mathbf{o}(m, x_\gamma) : \gamma < \alpha\}$. P is a set and we claim that the relation $<$ on P is an ordering. It suffices of course to see that $<$ is transitive on P. Let $\mathbf{o}(m, x)$, $\mathbf{o}(m, y)$, $\mathbf{o}(m, z)$ be elements of P such that $\mathbf{o}(m, x) < \mathbf{o}(m, y)$ and $\mathbf{o}(m, y) < \mathbf{o}(m, z)$. Let x_1, y_1, z_1 be elements belonging to the above three sets respectively. We have to show that $x < z$. Since by assumption $x < y$, $y < z$ it suffices (in view of (C)) to show that x, y, z are not all in the same monad. Assume the contrary. Then $x_1 < \text{Mon}(y) < z_1$. By (B), $\text{Mon}(x_1) < \text{Mon}(y) < \text{Mon}(z_1)$ which is a contradiction since $\text{Mon}(x_1) = \text{Mon}(z_1)$.

Therefore $<$ is an ordering on P. By the hypothesis and (5) we have $\mathbf{o}(m, x_0) \leq \dots \leq \mathbf{o}(m, x_\alpha)$, whence $\mathbf{o}(m, x_0) \leq \mathbf{o}(m, x_\alpha)$. By (4) again, $\text{Mon}(x_0) \leq \text{Mon}(x_\alpha)$.
2) This is immediate from the equivalence (5) and the fact that if for some β, $\mathbf{o}(m, x_\beta) < \mathbf{o}(m, x_{\beta+1})$, then $\mathbf{o}(m, x_0) < \mathbf{o}(m, x_\alpha)$, since $<$ is an ordering on P.

3) Suppose

$$z < \dots < x < y < \dots < z$$

is a cycle containing the pair $x < y$, $x \not\doteq y$. Then by (B),

$$\text{Mon}(z) \leq \dots \leq \text{Mon}(x) < \text{Mon}(y) \leq \dots \text{Mon}(z).$$

It follows from 2) above that $\text{Mon}(z) < \text{Mon}(z)$, a contradiction. This completes the proof. ■

Lemma 3.3. *If $<$ is a relation satisfying (A), (B), (C), then there is an ordering $<^*$ on u satisfying (A), (B) and such that*

$$x < y \ \& \ x \not\approx y \to x <^* y$$

PROOF: Let $R = < -\{\langle x,y\rangle : \langle x,y\rangle$ belongs to some cycle of $<\}$. It is easy to see that R is transitive, hence a partial ordering.

Indeed, let $\langle x,y\rangle$, $\langle y,z\rangle \in R$. This means that $\langle x,y\rangle$, $\langle y,z\rangle$ do not belong to any cycle.

Suppose $\langle x,z\rangle \notin R$. Then, either $\langle z,x\rangle \in <$, hence $z < x < y < z$, which is a cycle containing $\langle x,y\rangle$, or $\langle x,z\rangle \in <$ and $\langle x,z\rangle$ is contained in a cycle $w < \cdots < x < z < \cdots < w$. But then, clearly $\langle x,y\rangle$, $\langle y,z\rangle$ are contained in a cycle too. In both cases we get a contradiction, therefore, $\langle x,z\rangle \in R$.

Now, it is well known that every partial set–ordering of a set u can be extended to a total set–ordering on u. Extend R to such a total ordering $<^*$. According to Lemma 3.2. 3), every pair $x < y$ such that $x \not\approx y$ does not belong to a cycle, hence $\langle x,y\rangle \in R$. Consequently $x <^* y$. This completes the proof ∎

Corollary 3.4. *If u is connected and there is a relation $<$ satisfying (A), (B), (C), then u is simply connected.*

PROOF: Modify, by Lemma 3.3, $<$ to $<^*$ having the properties of Lemma 3.1. Hence u is simply connected. ∎

Let us come now to the second reduction. This is essentially due to Eilenberg.

Lemma 3.5. *If u is connected, then (A) and (B) imply (C). Thus, if u is connected and satisfies (A), (B), then it is simply connected.*

PROOF: For $x \in u$, put

$$x^- = \{y : y \not\approx x \ \& \ y < x\}$$
$$x^+ = \{y : y \not\approx x \ \& \ y < x\}$$

Claim 1. The conjunction of properties (A), (B) is equivalent to the conjunction of the properties

(A') $\qquad (\forall x)(u = x^- \cup \mathrm{Mon}(x) \cup x^+)$

(B') $\qquad$ For every x, the classes x^-, x^+ are open figures.

PROOF: Suppose (A), (B) hold. Then, clearly, (A') holds. Since

$$x^- = \{y : x < y\} - \mathrm{Mon}(x),$$

x^-, x^+ are Σ–classes. From (B) also follows that they are figures, hence open figures. The converse is similar. ∎

Claim 2. If (A'), (B') hold, then the classes $x^- \cup \mathrm{Mon}(x), x^+ \cup \mathrm{Mon}(x)$ are connected closed figures.

PROOF: First observe that $x^- \cup \mathrm{Mon}(x) = \mathrm{cl}(x^-)$ and $x^+ \cup \mathrm{Mon}(x) = \mathrm{cl}(x^+)$. Indeed, since x^-, x^+ are open and disjoint, $\mathrm{cl}(x^-) \subseteq x^- \cup \mathrm{Mon}(x)$ from whence $\mathrm{cl}(x^-) = x^- \cup \mathrm{Mon}(x)$ since x^- is a figure.

Suppose $\mathrm{cl}(x^-)$ is not connected. Since it is a closed figure, it can be decomposed (see Lemma 0.1) into two disjoint figures $\mathrm{Fig}(v_1)$, $\mathrm{Fig}(v_2)$. Thus, $u = \mathrm{Fig}(v_1) \cup \mathrm{Fig}(v_2) \cup x^+$. If $\mathrm{Mon}(x) \subseteq \mathrm{Fig}(v_2)$, then, obviously, $\mathrm{Fig}(v_2) \cup x^+ = \mathrm{Fig}(v_2) \cup \mathrm{cl}(x^+) = \mathrm{Fig}(w)$ and $\mathrm{Fig}(w)$ is disjoint from $\mathrm{Fig}(v_1)$. Hence u is decomposed into two disjoint closed figures, a contradiction. ∎

Let us come now to prove the assertion of the Lemma. Suppose $x < y$, $y < z$ and either $x \not\doteq y$ or $y \not\doteq z$. We shall show that $x < z$.

Assume $x \not\doteq y$ (the other case is similar). Then $\mathrm{Mon}(x) \subseteq y^-$ or, equivalently, $u - y^- \subseteq u - \mathrm{Mon}(x)$, hence according to (A'), $\mathrm{Mon}(y) \cup y^+ \subseteq x^- \cup x^+$.

By claim 2, $\mathrm{Mon}(y) \cup y^+$ is connected, while $x^- \cup x^+$ is not (see Lemma 0.2 and claim 1). Thus $y^+ \cup \mathrm{Mon}(x)$ must be included either in x^- or in x^+. Since by hypothesis $y \in x^+$, it follows that $y^+ \subseteq x^+$. By hypothesis, $z \in y^+$, hence $z \in x^+$, that is $x < z$. ∎

To come to the main theorem, consider on $u \times u$ the product topology, denoted again by $\doteq$, that is

$$\langle x, y\rangle \doteq \langle x_1, y_1\rangle \overset{\mathrm{df}}{\leftrightarrow} x \doteq x_1 \;\;\&\;\; y \doteq y_1$$

Obviously $\mathrm{Mon}(\langle x, y\rangle) = \mathrm{Mon}(x) \times \mathrm{Mon}(y)$.

Let $\Delta = \{\langle x, x\rangle : x \in u\}$

Theorem 3.6. *Let u be a connected set. u is simply connected iff the space $E = u \times u - Fig(\Delta)$ can be decomposed into two disjoint open figures.*

PROOF: Direction "$\to$" is easy. Suppose u is simply connected. By Lemma 3.1. there is an ordering $<$ of u satisfying (3). Put

$$E_1 = \{\langle x, y\rangle : x < y\} - \mathrm{Fig}(\Delta), E_2 = \{\langle x, y\rangle : y < x\} - \mathrm{Fig}(\Delta).$$

E_i are Σ–classes and if $\langle x, y\rangle \in E_1$, then $x < y$ and $x \not\doteq y$, hence by (3) $\mathrm{Mon}(x, y) \subseteq E_1$. Thus E_i are open figures such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$.

To prove the converse implication we need the following.

Lemma 3.7. *Suppose u is connected and $E_1, E_2 \neq \emptyset$ is a partition of E into open figures.*
Then $(\forall xy)(\langle x, y\rangle \in E_1 \leftrightarrow \langle y, x\rangle \in E_2)$.

PROOF: Let $s : u \times u \to u \times u$ be the function $s(\langle x, y\rangle) = \langle y, x\rangle$. s is a homeomorphism. Suppose E_1, E_2 do not satisfy the conclusion of the lemma. Then, $s''E_1 \cap E_1 \neq \emptyset$. Put

$$D_1 = s''E_1 \cap E_1, D_2 = E_2 \cup s''E_2.$$

Clearly, D_1, D_2 are open figures such that $D_1 \cup D_2 = E$, $D_1 \cap D_2 = \emptyset$, $D_1, D_2 \neq \emptyset$ and $s''D_i = D_i$ for $i = 1,2$.

Put for every x,

$$D_1(x) = \{y : \langle x, y\rangle \in D_1\}, D_2(x) = \{y : \langle x, y\rangle \in D_2\}.$$

Since D_1, D_2 are open figures we easily see that $D_i(x)$ are open figures too and

(5) $$x \doteq x' \to D_i(x) = D_i(x') \text{ for } i = 1,2.$$

Moreover, for every x

$$u = D_1(x) \cup \mathrm{Mon}(x) \cup D_2(x).$$

By the same reasoning used to prove the claim 2 of Lemma 3.5., we see that

$$\mathrm{cl}(D_i(x)) = D_i(x) \cup \mathrm{Mon}(x)$$

and that $\mathrm{cl}(D_i(x))$ is connected.

Suppose, now, that for some x, $D_1(x) \neq \emptyset$ and $D_2(x) \neq \emptyset$. Let $y \in D_1(x)$, $z \in D_2(x)$. ∎

Claim. $\mathrm{cl}(D_1(x)) \times \mathrm{Mon}(z) \subseteq D_1 \cup D_2$ and $\mathrm{cl}(D_2(x)) \times \mathrm{Mon}(y) \subseteq D_1 \cup D_2$.

PROOF: We show the first; the other is similar. Let $w_1 \in \mathrm{cl}(D_1(x))$, $w_2 \in \mathrm{Mon}(z)$. We have to verify that $w_1 \not\doteq w_2$ (since $D_1 \cup D_2 = E_1 \cup E_2 = E$).Since $\mathrm{cl}(D_1(x)) = D_1(x) \cup \mathrm{Mon}(x)$ either $w_1 \in D_1(x)$ or $w_1 \in \mathrm{Mon}(x)$.

i) $w_1 \in D_1(x) \leftrightarrow \langle x, w_1\rangle \in D_1$. Now $w_2 \in \mathrm{Mon}(z)$ and $z \in D_2(x)$, thus $\langle x, z\rangle \in D_2$, hence $\langle x, w_2\rangle \in D_2$. Suppose $w_1 \doteq w_2$. Then $D_1 \cap D_2 \neq \emptyset$ since they are figures, a contradiction.

ii) $w_1 \in \mathrm{Mon}(x) \to x \doteq w_1$ and again $\langle x, w_2\rangle \in D_2$. Hence $\langle w_1, w_2\rangle \in D_2$. Therefore $w_1 \not\doteq w_2$. The claim is proved.

Now, $\mathrm{cl}(D_1(x) \times \mathrm{Mon}(z))$ is connected and $D_1 \cup D_2$ is not. Since $\langle x, z\rangle \in D_2$, it follows $\mathrm{cl}(D_1(x)) \times \mathrm{Mon}(z) \subseteq D_2$. But $y \in D_1(x)$, therefore $\langle y, z\rangle \in D_2$.

Similarly from the inclusion $\mathrm{cl}(D_2(x)) \times \mathrm{Mon}(y) \subseteq D_1 \cup D_2$ we get $\langle z, y\rangle \in D_1$. The relation $\langle y, z\rangle \in D_2$ & $\langle z, y\rangle \in D_1$ contradict the fact that $s''D_i = D_i$.

Therefore for every x, some of the sets $D_1(x)$, $D_2(x)$ is necessarily empty.

Assume for some x, $D_1(x) = \emptyset$. Then $u = D_1(x) \cup \mathrm{Mon}(x)$. If $x \doteq x'$, (5) implies $D_1(x') = D_1(x) \neq \emptyset$. If $y \in D_1(x)$, then $x \in D_1(y)$ thus, $D_1(y) \neq \emptyset$. We infer that $D_2(x) \neq \emptyset$ for every x, consequently, $D_2 = \emptyset$, a contradiction. This contradiction completes the proof of the lemma. ∎

Proof of the direction "$\rightarrow$" of the Theorem 3.6.

Suppose E_1, E_2 form an open–figure decomposition of E. It suffices, by Lemma 3.5., to show that there is a relation $<$ on u with properties (A), (B).

E_1, E_2 are disjoint Σ–semisets and, by the foregoing lemma, $s''E_1 = E_2$. We can extend by the prolongation axiom E_1 to a set u_1 such that $s''u_1 \cap u_1 = \emptyset$. Also u_1 can be taken to be total. (This is easily done by a "completion" construction).

Define the relation $<$ on u as follows:

$$x < y \text{ iff } \langle x, y\rangle \in u_1.$$

Clearly $<$ satisfies (A) since it is total and $s''u_1 \cap u_1 = \emptyset$ and satisfies (B) because contains E_1 and E_1 is a figure. This finishes the proof of the theorem. ∎

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(*Received* May 5,1988, *revised* September 1,1988)