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Combinatoric properties of classes in AST

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Abstract. We say that a class $Q \subseteq [[V]^k]^m$ is $\langle k,m\rangle$–complete on Z iff $(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)([[v]^k]^m \subseteq Q)$ where $[Z]^\infty = \{u \subseteq X; u \text{ is an infinite set}\}$. We discover some theorems on an existence of complete classes, namely those which are figures in an equivalence $\overset{\circ}{\underset{\{P\}}{=}}$ with P fully revealed (and exact). Note that Ramsay theorem is a special case of such theorems on completeness.

Keywords: Completeness, condesation, equivalence $\overset{\circ}{\underset{\{P\}}{=}}$, figure, Ramsay theorem

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Introduction

Having an equivalence E on a class $[Z]^k$ we call, as usual, a subclass $U \subseteq Z$ homogenous for E iff $(\exists x \in Z)([U]^k \subseteq E''\{x\})$, i.e. iff $[[U]^k]^2 \subseteq \dot{E}$, where $\dot{E} = \{\{x,y\} \in [V]^2; \langle x,y\rangle \in E\}$. Assuming that E, Z are set–definable and there is only a finite number of factor–classes of E, we conclude by using the Ramsay theorem (see §1) that

$$(1) \qquad (\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)([[v]^k]^2 \subseteq \dot{E}).$$

(We put $[Z]^\infty = \{u \subseteq Z; u \text{ is an infinite set}\}$.) Let us agree on calling $\dot{E}$ submitted to (1) $\langle k,2\rangle$–complete on Z. Note that the condition that $[Z]^k/E$ is finite is equivalent to

$$(2) \qquad (\forall u \in [[Z]^k]^\infty)([u]^2 \cap \dot{E} \neq \emptyset)$$

Such an $\dot{E}$, satisfying (2), is called 2–condensating on $[Z]^k$. Thus the following "completeness theorem" holds: Let E be an equivalence on $[Z]^k$ and suppose that $k \geq 1$, E, Z are set–definable. If $\dot{E}$ is 2–condensating on $[Z]^k$ then $\dot{E}$ is $\langle k,2\rangle$–complete on Z.

We naturally generalize the notion of completeness: writing m instead of 2 and Q instead of $\dot{E}$ in (1) we obtain the definition of $\langle k,m\rangle$–completeness of Q on Z. Similarly can be gained the notion that Q is m–condesating on Z. Now, completeness theorems are those which conclude from condensation to completeness; we discover four such theorems. The point is in finding a type of classes such that a completeness theorem holds for Q, contained in the system of classes of such a type. We find two such systems of classes: fully revealed classes and so called m–K–symmetric classes. To justify the introduction of the second one, let us observe that 2–$\dot{E}$–symmetric classes, where E is an equivalence on V, are just figures in E. Moreover, we describe an operation $\sqcup(K)$ such that, roughly speaking, $\sqcup(K)$ is a lower bound for m–K–symmetric k–condensating classes. We introduce an operation $\nabla(K)$ which is in an important special case of K inverse to $\sqcup$. By this special case we mean that $K = \dot{E}\{P\}$, where P is fully revealed and exact (see preliminaries and §4). Note that $E\{P\}$ denotes an equivalence, usually known as $\underset{\{P\}}{\overset{\circ}{=}}$. We obtain, as a conclusion, the least element among k–condensating figures Q in $E\{P\}$, which are k–transitive, i.e. they satisfy $\nabla(Q) \subseteq Q$. This element is, moreover, $\langle l, k\rangle$–complete for each $l \in FN$.

The results of Alena Vencovská (see[Č]) which are identical with those obtained here when specifying $k = 2 = m$ and P is a set, have been a source of inspiration for the problems of this paper.

PRELIMINARIES

We shall use the obvious notation of the Alternative set theory; recall that i, j, k, l, m, n range over finite natural numbers.

We define $[X]^0 = X$, $[X]^n = \{t \subseteq X;\ t \hat{\approx} n\}$ if $n \geq 1$ and $[X]^{\langle m,n\rangle} = [[X]^m]^n$; note that $[X]^{\langle 0,n\rangle} = [X]^n$. Assume that $\emptyset \neq \tau \subseteq FN^2$. We put $[X]^\tau = \bigcup\{[X]^p; p \in \tau\}$.

Having $\emptyset \neq T \subseteq FN$ we write $(T, m\rangle$ instead of $T \times \{m\}$ and $\langle k, m\rangle$ instead of $\{\langle k, m\rangle\}$.

Let us introduce the following symbols:

$$[X]^\infty = \{u \subseteq X; u \text{ is an infinite set }\},\ [X]^f = P(X) - [X]^\infty.$$

We put, for an equivalence E,

$$\dot{E} = \{\{x, y\} \in [V]^2; \langle x, y\rangle \in E\}.$$

Let P be a class. We define

$$E\{P\} = \{\langle x, y\rangle; \varphi(x, P) \leftrightarrow \varphi(y, P) \text{ holds for every normal formula } \varphi(v, P) \in FL\}$$

and

$$\mathrm{Def}\{P\} = \{x;\ \text{there exists a normal formula } \varphi(v, P) \in FL \text{ such that } (\exists! v)\varphi(v, P) \wedge \varphi(x, P) \text{ holds}\}.$$

Let P be fully revealed. Then $E\{P\}$ is a compact equivalence. Each monad of such an equivalence is either infinite or one–element set $\{x\}$ with $x \in \mathrm{Def}\{P\}$. Assume that $\langle x, y\rangle \in E\{P\}$. Then there exists an automorphism F such that $F(x) = y$ and $F''P = P$ hold.

Definition. We denote by $Nd\{P\}$ the system of all classes $\{x; \varphi(x, P)\}$, where $\varphi(v, Z)$ is a normal formula of the language FL. Writing $X \in Nd\{P\}$ we mean that X is a class from $Nd\{P\}$.

Definition. We say that a class P is exact iff $X \in Nd\{P\} \rightarrow X \cap \mathrm{Def}\{P\} \neq 0$ holds.

Let P be fully revealed, $X \in Nd\{P\}$ and let $\varphi(v, Z) \in FL$ be a normal formula. Then $\{x; \varphi(x, P)\} \in Nd\{P\}$ holds, too. However, not yet that each $X \in Nd\{P\}$ is a figure in the equivalence $E\{P\}$.

§1 Ramsay theorem

Our aim is to prove the following

Ramsay theorem. *Let $l \geq 0$, $m \geq 1$ and let $\{P_i; i < m\}$ be a cover of a class $[Z]^l$. Assume that all classes $-Z$, P_i, $-P_i$, $i < m$, are revealed. Then $(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)(\exists i < m)([v]^l \subseteq P_i)$ holds.*

To justify the name of this theorem, let us introduce the following consequence.

Corollary. $(\forall l, k, m \geq 1)(\exists n)(\forall p)(p$ *is a cover of* $[n]^l \wedge p \approx m \rightarrow (\exists u \subseteq n)(\exists p_i \in p)([u]^l \subseteq p_i \wedge u \approx k)$.

Indeed, we deduce from the previous theorem that, for each $\gamma \notin FN$, holds: $m \geq 1 \rightarrow (\forall p)(p$ is a cover of $[\gamma]^l \wedge p \approx m \rightarrow (\exists u \subseteq n)(\exists p_i \in p)([u]^l \subseteq p_i \wedge u \approx k)$. We conclude from this, using the overspill principle, that the corollary is true.

Remark. *We shall use in the following familiar properties of revealed classes, as, for example: if X, Y are revealed, then $X \cup Y$, $X \cap Y$, are revealed, if $-X$ is revealed then $-[X]^m$ is revealed and X is revealed if $[X]^m$ is revealed.*

The proof of the Ramsay theorem will be given in a sequence of lemmas. We use the following notation: let Rams_l denote the sentence

$$(\forall Q \subseteq [V]^l)((Q \text{ is revealed } \wedge -Q \text{ is revealed }) \rightarrow (\forall u \in [V]^\infty)$$
$$(\exists v \in [u]^\infty)([v]^l \subseteq Q \vee [v]^l \subseteq [V]^l - Q)).$$

Lemma 1. *Rams_l holds for each $l \in FN$.*

This is a key lemma of our proof. Before we give its proof, let us prove the Ramsay theorem from Lemma 1.

Lemma 2. *Let $[Z]^l = P_1 \cup P_2, P_1 \cap P_2 = \emptyset$. Suppose that $-Z$, P_1, P_2 are all revealed. Then $(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)([v]^l \subseteq P_1 \vee [v]^l \subseteq P_2)$.*

Proof: Put $Q = P_1$. Then $-Q = P_2 \cup (-[Z]^l)$ and, consequently, $-Q$ is revealed. (See Remark above.) Now, the lemma 2 follows from the lemma 1 immediately. ■

We can easily prove, by induction on m, the following

Lemma 3. *Let $[Z]^l = P_0 \cup \cdots \cup P_{m-1}, m \geq 1$ and let $\{P_i; i < m\}$ be a partition of $[Z]^l$. Suppose that $-Z, P_0, \ldots, P_{m-1}$ are revealed. Then $(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)(\exists i \in m)([v]^l \subseteq P_i)$.*

Now, let us prove the Ramsay theorem from lemma 3. Put $\overline{P}_0 = P_0, \overline{P}_i = P_i - \bigcup_{j<i} P_j = \bigcap_{j<i}(P_i - P_j)$. Each class $P_i - P_j$ is revealed and, consequently, $\overline{P}_i$ is revealed for all $i < m$. We have, for $i < m$, $\overline{P}_i \subseteq P_i$. Thus the partition $\{\overline{P}_i; i < m\}$ of $[Z]^l$ satisfies the assumptions of the lemma 3 and our theorem is proved.

Proof: of the lemma 1 We shall prove it by induction on l.

(1) Rams_0 and Rams_1 hold trivially.

We shall prove $\text{Rams}_l \to \text{Rams}_{l+1}$. Let us denote by $\rho_{l+1}(X, Q)$ the formula

$$Q \subseteq [V]^{l+1} \wedge (\exists x \in X)(\exists v \in [X]^{\infty})([v]^l \subseteq Q_x),$$

where $Q_x \approx \{t \in [V]^l; t \cup \{x\} \in Q\}$. We prove

$$\text{(2)} \quad \text{Rams}_l \wedge Q \subseteq [V]^{l+1} \wedge Q, -Q \text{ are revealed } \to \to (\forall u \in [V]^{\infty})(\rho_{l+1}(u, Q) \vee \rho_{l+1}(u, [V]^{l+1} - Q)).$$

Let $u \in [V]^{\infty}$, $x \in V$. Then Q_x and $-Q_x$ are revealed, $Q_x \subseteq [V]^l$. We deduce from Rams_l that $(\exists v \in [u]^{\infty})([v]^l \subseteq Q_x \vee [v]^l \subseteq [V]^l - Q_x)$. Consequently, $(\exists x \in u)(\exists v \in [u]^{\infty})([v]^l \subseteq Q_x) \vee (\exists x \in u)(\exists v \in [u]^{\infty})([v]^l \subseteq [V]^l - Q_x)$ holds, too. We have $[V]^l - Q_x = ([V]^{l+1} - Q)_x$ and (2) is proved.

Put

$$Y = \{u \in [V]^{\infty}; \rho_{l+1}(u, Q)\}$$
$$\text{and} \qquad Y' = \{u \in [V]^{\infty}; \rho_{l+1}(u, [V]^{l+1} - Q)\}.$$

Thus, we have $(\forall u \in [V]^{\infty})(u \in Y \vee u \in Y')$, i.e. $Y \cup Y' = [V]^{l+1}$. We have, in addition, $u \in Y \wedge v \supseteq u \to v \in Y$. Thus $Y \subseteq [V]^{\infty}$ is an upper–class in the ordering $\langle [V]^{\infty}, \subseteq \rangle$. We deduce from this that $(\forall u \in [V]^{\infty})(\exists w \in [V]^{\infty})(w \subseteq u \wedge \hat{w} \subseteq Y \vee \hat{w} \cap Y = \emptyset)$, where $\hat{w} = \{v \in [V]^{\infty}; v \subseteq w\}$. We have $(\forall u \in [V]^{\infty})(\exists w \in [u]^{\infty})([w]^{\infty} \subseteq Y \vee [w]^{\infty} \subseteq Y')$ and, consequently,

$$\text{(3)} \quad (\forall u \in [V]^{\infty})(\exists w \in [u]^{\infty})[(\forall v \in [w]^{\infty})\rho_{l+1}(v, Q) \vee (\forall v \in [w]^{\infty})\rho_{l+1}(v, [V]^{l+1} - Q)].$$

Now we prove

$$\text{(4)} \qquad (\forall w \in [V]^{\infty})((\forall v \in [w]^{\infty})\rho_{l+1}(v, Q) \to (\exists v \in [w]^{\infty})([v]^{l+1} \subseteq Q)).$$

Let $a_1 \in w$, $v_1 \in [w]^{\infty}$ be such that $[v_1]^l \subseteq Q_{a_1}$. Suppose that we have, for $i = 0, 1, 2, \ldots$, $a_{i+1} \in v_i - \{a_1, \ldots, a_i\}$ and $v_{i+1} \in [v_i - \{a_1, \ldots, a_i\}]^{\infty}$ such that $[v_{i+1}]^l \subseteq Q_{a_{i+1}}$.

Let $\{a_{i_0}, \ldots, a_{i_l}\} \subseteq [\{a_1, \ldots, a_n\}]^{l+1}$ be such that $1 \leq i_0 \leq i_1 \leq \cdots \leq i_l \leq n$. We have $\{a_{i_1}, \ldots, a_{i_l}\} \in [v_{i_0}]^l$ and, consequently, $\{a_{i_1}, \ldots, a_{i_l}\} \in Q_{a_{i_0}}$, i.e. $\{a_{i_0}, \ldots, a_{i_l}\} \in Q$. Thus, we have the following:

$$(\forall n > l)(\exists z \in [w]^n)([z]^{l+1} \subseteq Q).$$

Choose, for $n > l$, $z_n \in [w]^n$ such that $[z_n]^{l+1} \subseteq Q$. Q is revealed. Thus, there exists u such that $\bigcup_{l<n} [z_n]^{l+1} \subseteq u \subseteq Q$, and we have $(\forall n > l)(\exists z \in [w]^n)([z]^{l+1} \subseteq u)$. We have, consequently, a $\gamma \notin FN$ such that $(\exists z \in [w]^{\gamma})([z]^{l+1} \subseteq u)$; such a z satisfies $z \in [w]^{\infty} \wedge [z]^{l+1} \subseteq Q$ and (4) is proved.

Now, $\text{Rams}_l \to \text{Rams}_{l+1}$ is an easy consequence of (3) and (4). ∎

§2 Completeness

Definition. Let $0 \neq \tau \subseteq FN^2$. Q is called τ–complete on Z iff

$$(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)([v]^\tau \subseteq Q).$$

Proposition. *Let, for every $i \in FN$, Q_i be τ–complete on Z_i. Then $\cap Q_i$ is τ–complete on $\cap Z_i$.*

Proof: Let $u \in [\cap Z_i]^\infty$; choose $v_0 \in [u]^\infty$ such that $[v_0]^\tau \subseteq Q_0$. Let $v_{i+1} \in [v_i]^\infty$ and $[v_{i+1}]^\tau \subseteq Q_{i+1}$, $i \in FN$. Then there exist $v \in [\cap v_i]^\infty$. We have $(\forall i)([v]^\tau \subseteq Q)$. ■

Proposition. *The system of τ–complete classes on Z is closed under countable intersections. If a subclass of a class X is τ–complete on Z, then X is τ–complete on Z, too. Q is τ–complete on Z iff $Q \cap [Z]^\tau$ is τ–complete on Z. Q is τ–complete on $Z \leftrightarrow (\forall p \in \tau)(Q$ is p–complete on$Z)$.*

Proof: Only the last proposition is not quite trivial. But it follows from the fact that $\tau \preceq FN$. ■

Definition. Q is k–condesating on Z iff $(\forall u \in [Z]^\infty)([u]^k \cap Q \neq \emptyset)$.

Proposition. *If Q is k–condesating on Z, $Z' \subseteq Z$ and $Q' \supseteq Q$, then Q' is k–condesating on Z'.*

Example. Let P be fully revealed. Then $dotE\{P\}$ is 2–condesating on V.

Theorem (1.theorem on $\langle 0, k\rangle$–completeness).
Let $Q \subseteq [V]^k$ be k–condesating on Z and let $Q, -Q, Z, -Z$, be revealed. Then Q is $\langle 0, k\rangle$–complete on Z.

Proof: Let $u \in [Z]^\infty$ and put $P_1 = Q \cap [Z]^k$, $P_2 = -Q \cap [Z]^k$. We deduce, by using the Ramsay theorem, that there exists a set $v \in [u]^\infty$, which is homogenous for the partition $\{P_1, P_2\}$ of $[Z]^k$. We can see that $[v]^k \subseteq Q$. ■

Definition. We put, for $k \geq 2$ and $s \in [V]^{k-1}$, $Q_k[s] = \{x; \{x\} \cup s \in Q \cap [V]^k\}$. We omit the index k if there is no danger of confusion.

Theorem (on compactness). *Let Q be k–condensating on W, $k \geq 2$. Assume that $-Q$, W are revealed. Then there exists $w \in [W]^f$ (i.e. finite $w \subseteq W$) such that*

$$W - w \subseteq \bigcup\{Q[s]; s \in [w]^{k-1}\}.$$

Proof: Let us prove, firstly, that

$$(\exists n)(\forall v \subseteq W)(n \hat{\preceq} v \rightarrow [v]^k \cap Q \neq \emptyset).$$

Suppose that there exists, for every $n \in FN$, $v_n \in [W]^f$ such that $n \hat{\preceq} v_n \wedge [v_n]^k \cap Q = \emptyset$. The class $\bigcup v_n \subseteq W$ is countable; let $\overline{u}$ such a set that $\bigcup v_n \subseteq \overline{u} \subseteq W$ holds. We have $\{v_n\}_n \subseteq P(\overline{u}) \subseteq P(W)$. Put $u = P(\overline{u})$. We have $\bigcup\{[v_n]^k\}_{FN} \subseteq -Q$. Thus there exists a set q such that $\{[v_n]^k\}_{FN} \subseteq q \subseteq -Q$. We have $(\forall n)(\exists v \in$

$u)(n\widehat{\prec}v \wedge [v]^k \subseteq q)$. There exists a $\gamma \notin FN$ and a set $v \in u$ (i.e. $V \subseteq W$) such that $\gamma\widehat{\prec}v$ (i.e. $V \in [W]^\infty$) and $[v]^k \subseteq q$ (and, consequently, $[v]^k \cap Q = \emptyset$), which is a contradiction with k-condensation of Q on W.

Put

$$D = \{w \subseteq W; [w]^k \cap Q = \emptyset \wedge (\forall x \in W - w)[\{x\} \cup w]^k \cap Q \neq \emptyset\}$$

Let $w \in D$. Then $w\widehat{\preceq}n$ and, consequently, w has the required properties. ∎

Corollary. *Let Q be k-condesating on W, $k \geq 2$. Assume that Q, W are from $Nd\{P\}$, where P is fully revealed and exact. Then there exists $w \in [W]^f$ such that*

$$W - w \subseteq \bigcup\{Q[s]; s \in [w]^{k-1}\} \text{ and } w \in \ Def\,\{P\}.$$

PROOF: Let D be as above. Then $D \in Nd\{P\}$ and $D \cap \mathrm{Def}\{P\} \neq \emptyset$. ∎

Definition. Let $k \geq 2$, $m \geq 0$. We put

$$\nabla_k^m(W,Q) = \{t \in [W]^m; (\exists s \in [W]^{k-1})(t \in [Q[s]]^m)\}$$
$$\nabla_k^m(Q) = \nabla_k^m(V,Q)$$

Note that $W' \subseteq W \wedge Q' \subseteq Q \rightarrow \nabla_k^m(W',Q') \subseteq \nabla_k^m(W,Q)$.

Proposition. *Let E be an equivalence. Then*

(1) $$\nabla_2^n(\dot{E}) = \bigcup\{[E''\{x\}]^n; E''\{x\}\widehat{\succeq}n+1\}, \text{ whenever } n > 0.$$

(2) $$\nabla_2^2(\dot{E}) \subseteq \dot{E}$$

The proof is easy.

Definition. A class Q is called m-transitive, for $m \geq 1$, iff $\nabla_m^m(Q) \subseteq Q$ holds.

Theorem (first theorem on $\langle l,m\rangle$-completeness).
Let Q be k-condensating on $[Z]^l$, $k \geq 2$, $l \geq 0$, $m \geq 1$. Assume that Q, $-Q$, Z, $-Z$ are revealed. Then $\nabla_k^m([Z]^l,Q)$ is $\langle l,m\rangle$-complete on Z.
Suppose, moreover, that Q is k-transitive. Then Q is $\langle l,k\rangle$-complete on Z.

PROOF: Put $W = [Z]^l$; then Q, W satisfy the assumptions of the previous theorem. Let $w \in [W]^f$ be such that $[Z]^l \subseteq \bigcup\{Q[s]; s \in [w]^{k-1}\} \cup \{w\}$. We obtain, by using the Ramsay theorem, that $(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)(\exists s \in [w]^{k-1})([v]^l \subseteq Q[s])$. We have, for such v, s, $[v]^l \subseteq [Z]^l$ and $s \subseteq [Z]^l$. We conclude from this that

$$(\forall u \in [Z]^\infty)(\exists v \in [u]^\infty)([[v]^l]^m \subseteq \nabla_k^m([Z]^l,Q)).$$

We have $\nabla_k^m([Z]^l,Q) \subseteq \nabla_k^m(Q)$. Thus $m = k \geq 2 \rightarrow \nabla_m^m(Q)$ is $\langle l,m\rangle$-complete on Z, which implies the last assertion of the theorem in question. ∎

Example.

(1) Let E be equivalence on $[Z]^l$, $l \geq 0$. Assume that $\dot{E}$ is 2–condensating on $[Z]^l$, E, $-E$, Z, $-Z$ revealed. Then $\dot{E}$ is $\langle l,2\rangle$–complete on Z.

(2) Let P be fully revealed. Then $\dot{E}\{P\}$ is $\langle l,2\rangle$–complete on V for every $l \in FN$.

PROOF:

(1) $\nabla_2^2\dot{E}$ is $\langle l,2\rangle$–complete and $\nabla_2^2(\dot{E}) \subseteq \dot{E}$.

(2) $\dot{E}\{P\}$is 2–condesating on$[V]^l$. Let $E_i \in Nd\{P\}$ be such equivalences that $E\{P\} = \cap E_i$. Every E_i is $\langle l,2\rangle$–complete on V by (1). Thus $\dot{\mathrm{E}}\{P\}$ is $\langle l,2\rangle$–complete on V, too. ■

Proposition. *Assume that $k \geq 2$, $m \geq 2$ and let K be $\langle 0,m\rangle$–complete on V. Then $\nabla_m^k(K)$ is $\langle 0,k\rangle$–complete on V.*

PROOF: Let $u \in [V]^\infty$; then there exists $v \in [u]^\infty$ such that $[v]^m \subseteq K$. Assume that $t \in [v]^k$. We have an $s \in [v-t]^{m-1}$ and, consequently, $x \in t \rightarrow \{x\} \cup s \in K$ holds. Thus $t \in \nabla_m^k(K)$. ■

Proposition. *Let, for $i \in FN$, Q_i be revealed and $Q_{i+1} \subseteq Q_i$; let W be revealed, too. Then $\nabla_k^n(W, \cap Q_i) = \bigcap \nabla_k^n(W, Q_i)$.*

PROOF: Prove, firstly, that if Q is revealed and $t \in [V]^n$ then $Q^t = \{s; (\forall x \in t)(\{x\} \cup s \in Q)\}$ is revealed.

Indeed, let $C = \{s_n\}_{FN} \subseteq Q^t$ be countable. Put $q_n = \{\{x\} \cup s; x \in t\}$; then $q_n \in [Q]^f$. $\bigcup_n q_n \subseteq Q$ is countable, thus, there exists u such that $\bigcup_n q_n \subseteq u \subseteq Q$. Then $C \subseteq \{s; (\forall x \in t)(\{x\} \cup s \in u)\} \subseteq Q^t$.

The inclusion $\subseteq$ of our proposition is easy. Assume $t \in \bigcap \nabla_k^n(W, Q_i)$. $Q_i^t \cap [W]^{k-1}$ is revealed, thus there exists $s \in \bigcap_i (Q_i^t \cap [W]^{k-1})$. We have $(\forall i)(\forall x \in t)(\{x\} \cup s \in Q_i)$, which implies that $t \in [(\bigcap_i Q_i)[s]]^n$. We have $s \in [W]^{k-1}$ and $t \in [W]^n$, which finishes our proof. ■

§3 K–SYMMETRIC CLASSES

Definition. Let $m \geq 2$. A class Q is called m–K–symmetric on W iff

$$(\forall s \in [W]^m \cap K)(s \cap Q \neq 0 \rightarrow s \subseteq Q)$$

Q is m–K–symmetric iff Q is m–K–symmetric on V

Remark.

(1) Q is m–K–symmetric on $W \leftrightarrow Q \cap W$ is m–K–symmetric on W.

(2) If Q is m–K–symmetric on W, $K' \subseteq K$ and $W' \subseteq W$, then Q is m–K'–symmetric on W', too.

Proposition.

Let E be an equivalence. A class $X \subseteq W$ is 2–$\dot{E}$–symmetric on $W \leftrightarrow X$ is a figure in $E \cap W^2$.

The proof is easy similarly as that of the

Proposition.

The system of m–K–symmetric classes on W is closed under intersections and unions of subsystems and under the complement.

Definition.

Let $k \geq 1$, $0 \neq \tau \subseteq FN^2$. Let K, Z be classes. We define

$$\bigsqcup_{\tau}^{k(Z,K)} = \{t \in [Z]^k ; (\exists u \in [Z]^\infty)([u]^\tau \subseteq K \wedge t \subseteq u)\}.$$

$$\bigsqcup_{\tau}^{k(K)} = \bigsqcup_{\tau}^{k(V,K)} .$$

Proposition.

Let $k \geq 1$, $0 \neq \tau \subseteq FN^2$. Assume that K is τ–complete on Z. Then $\bigsqcup_\tau^k(Z,K)$ is $\langle 0,k\rangle$–complete on Z.

PROOF: If $u \in [Z]^\infty$, then there exists a $v \in [u]^\infty$ such that $[v]^\tau \subseteq K$. We have $[v]^k \subseteq \bigsqcup_\tau^k(Z,K)$. ∎

Example. Let P be fully revealed, $k \geq 1$. Then $\bigsqcup^k_{\langle FN,2\rangle}(\dot{\mathrm{E}}\{P\})$ is $\langle 0,k\rangle$–complete on V.

Theorem(on lower bound).

Let Q be m–K–symmetric on $[V]^k$ and k–condesating on Z; let $m \geq 2$, $k \geq 1$. Then $\bigsqcup^k_{\langle k,m\rangle}(Z,K) \subseteq Q$.

PROOF: At first, we can see that

$$(\forall u)(m \widehat{\leq} [u]^k) \rightarrow ([[u]^k]^m \subseteq K \wedge [u]^k \cap Q \neq \emptyset \rightarrow [u]^k \subseteq Q).$$

Let us prove the inclusion in question. Assume that $t \in \bigsqcup^k_{\langle k,m\rangle}(Z,K)$. Then there exists $u \in [Z]^\infty$ such that $[[u]^k]^m \subseteq K$ and $t \in [u]^k$ holds. We have $[u]^k \cap Q \neq \emptyset$. By using the formula above we obtain $[u]^k \subseteq Q$ and, consequently, $t \in Q$ holds.

Theorem (Second theorem on $\langle 0,k\rangle$–completeness).

Let K be $\langle k,m\rangle$–complete on Z. Assume that Q is m–K–symmetric on $[V]^k$ and k–condesating on Z. Let $m \geq 2$, $k \geq 1$. Then Q is $\langle 0,k\rangle$–complete on Z.

PROOF: $\bigsqcup^k_{\langle k,m\rangle}(Z,K)$ is $\langle 0,k\rangle$–complete on Z. We deduce from the previous theorem that $\bigsqcup^k_{\langle k,m\rangle}(Z,K) \subseteq Q$. Thus, Q is $\langle 0,k\rangle$–complete on Z, too.

Let us clear some properties of K–symmetric classes.

Proposition.

Let Z be m–K–symmetric on W, $k \geq 1$, $m \geq 2$. Then

$$Z \textit{ is } k\text{-} \bigsqcup_{\langle 0,m\rangle}^{k(W,K)} -- \textit{symmetric on } W.$$

PROOF: Suppose that $t \in \bigsqcup^k_{\langle 0,m\rangle}(W,K) \wedge t \cap Z \neq \emptyset$. Then there exists $u \in [W]^\infty$ such that $[u]^m \subseteq K \wedge t \in [u]^k$. Let us choose $a \in t \cap Z$. Then $a \in u$ and we have $s \in [u-\{a\}]^{m-1} \to \{a\} \cup s \in K$. Thus $s \in [u-\{a\}]^{m-1} \to \{a\} \cup s \subseteq Z$ holds. We deduce from this that $u \subseteq Z$ and, especially, $t \subseteq Z$ is true. ■

Proposition. *(on restriction)*

(1) *Let Z be m-K- symmetric, $0 \in T \subseteq FN$. Then* $\bigsqcup^k_{\langle T,m\rangle}(K) \cap [Z]^k \subseteq \bigsqcup^k_{\langle T,m\rangle}(Z,K)$.

(2) *Let Z be k-K-symmetric and assume that $K \subseteq [V]^k$, $n \geq 1$. Then* $\nabla^n_k(K \cap [Z]^k) = \nabla^n_k(K) \cap [Z]^n$.

(3) *Let Z be m-K-symmetric, $k \geq 2$, $n \geq 1$, $0 \in T \subseteq FN$. Then* $\nabla^n_k(\bigsqcup^k_{\langle T,m\rangle}(K) \cap [Z]^k) = \nabla^n_k(\bigsqcup^k_{\langle T,m\rangle}(K)) \cap [Z]^n$.

PROOF: (1) Let $t \in \bigsqcup^k_{\langle T,m\rangle}(K) \cap [Z]^k$ Then there exists $u \in [V]^\infty$ such that $[u]^{\langle T,m\rangle} \subseteq K$ and $t \in [u]^k$. Thus there is an $x \in u \cap Z$. Let $y \in u$ be arbitrary. Then there exists $s \in [u]^m$ with $\{x,y\} \subseteq s$. We have $[[u]^0]^m = [u]^m \subseteq K$. Thus $s \in [V]^m \cap K$ and $s \cap Z \neq 0$. We deduce from this that $s \subseteq Z$ and, consequently, $y \in Z$, $u \subseteq Z$. Thus $t \in \bigsqcup^k_{\langle T,m\rangle}(Z,K)$ holds.

(2) We have:

$$\begin{aligned}\nabla^n_k(K \cap [Z]^k) &= \{t \in [V]^n; (\exists s \in [V]^{k-1})(\forall x \in t)(\{x\} \cup s \in K \cap [Z]^k)\} = \\ &= \{t \in [Z]^n; (\exists s \in [V]^{k-1})(\forall x \in t)(\{x\} \cup s \in K)\} = \nabla^n_k(K) \cap [Z]^n.\end{aligned}$$

Note that in the last but one equality, we have used the implication $\{x\} \cup s \in K \wedge x \in Z \to \{x\} \cup s \in [Z]^k$, which is quaranteed by our assumptions.

(3) We deduce from the previous proposition that Z is k-$\bigsqcup^k_{T,m\rangle}(K)$-symmetric; (3) follows immediately from this and from (2). ■

Corollary. *Let Q be m-K-symmetric on $[V]^k$, k-condensating on Z, $k \geq 1$, $m \geq 2$. Let Z be m-K-symmetric and $\{0,k\} \subseteq T \subseteq FN$. Then*

$$\bigsqcup^k_{\langle T,m\rangle}(K) \cap [Z]^k \subseteq Q.$$

PROOF: We deduce from the theorem on the lower bound that $\bigsqcup^{k(Z,K)}_{\langle T,m\rangle} \subseteq Q$. The assertion follows from this and by using the item (1) of the previous proposition. ■

Now we discuss "inclusive properties" of the operations ∇, $\bigsqcup$, i.e. relations of the form $\nabla(\bigsqcup(K)) \subseteq \nabla(K)$, $\nabla(\bigsqcup(K)) \subseteq K$.

Theorem (on inclusion). *Let $k \geq m \geq 1$, $n \geq 1$. Then*

$$\nabla^n_k(W, \bigsqcup^k_{\langle 0,m\rangle}(K)) \subseteq \nabla^n_m(W,K).$$

PROOF: Assume that $t \in \nabla^n_k(W, \bigsqcup^k_{\langle 0,m\rangle}(K))$, $t = \{x_1, \ldots, x_n\} \in [W]^n$. Then there exists $s \in [W]^{k-1}$ such that $\{x_i\} \cup s \in \bigsqcup^k_{\langle 0,m\rangle}(K)$ holds for $i = 1,2,\ldots,n$.

Having $\widehat{s} \in [s]^{m-1}$, we deduce that, for $i = 1, 2, \ldots, n$, $\{x_i\} \cup \widehat{s} \in K$ is true. Thus $t \subseteq K[s]$ and $t \in [K[s]]^n$, i.e. $t \in \nabla^n_m(W, K)$ holds. ■

Note that $\bigsqcup^k_{\langle T,m\rangle}(K) \subseteq \bigsqcup^k_{\langle 0,m\rangle}(K)$ is true whenever we have $0 \in T \subseteq FN$. Thus $\nabla^n_k(W, \bigsqcup^k_{\langle T,m\rangle}(K)) \subseteq \nabla^n_m(W, K)$ holds under assumptions that $k \geq m \geq 1$, $n \geq 1$ and $0 \in T \subseteq FN$.

Corollary. *Let $k \geq m \geq 2$ and suppose that K is m–transitive. Then*

(1) $\nabla^m_k(\bigsqcup^k_{\langle 0,m\rangle}(K)) \subseteq K$.

(2) *Assume that* $0 \in T \subseteq FN$. *Then*

$$\nabla^m_k(\bigsqcup^k_{\langle T,m\rangle}(Z, K)) \subseteq K.$$

(3) *Assume that* $0 \in T \subseteq FN$ *and let* Z *be* m–K–*symmetric. Then*

$$\nabla^m_k(\bigsqcup^k_{\langle T,m\rangle}(K)) \cap [Z]^m \subseteq K \cap [Z]^m.$$

PROOF: follows directly from the previous theorem and note. We use yet in (3) the item (1) of the proposition on restriction. ■

Proposition. *Let $k \geq 2$, $0 \neq \tau \subseteq FN^2$. Then*

$$\bigsqcup^k_\tau(W, K) \subseteq \nabla^k_k(W, \bigsqcup^k_\tau(W, K)).$$

PROOF: Let $t \in \bigsqcup^k_\tau(W, K)$. Then there exists $u \in [W]^\infty$ such that $t \in [u]^k \wedge [u]^\tau \subseteq K$. Choose $s \in [u - t]^{k-1}$. We have, for each $x \in t$, $\{x\} \cup s \in \bigsqcup^k_\tau(W, K)$; we deduce from this that $t \in \nabla^k_k(W, \bigsqcup^k_\tau(W, K))$. ■

Proposition. *Let $k \geq 2$. Then*

$$\bigsqcup^k_{\langle 0,k\rangle}(W, K) \subseteq \nabla^k_k(W, K).$$

PROOF: We have, by using the theorem on inclusion, that

$$\nabla^k_k(W, \bigsqcup^k_{\langle 0,k\rangle}(W, K) \subseteq \nabla^k_k(W, K).$$

Now, the previous proposition gives the relation in question. ■

§4 Combinatoric properties of $\dot{E}\{P\}$

4.1. Throughout this paragraph, let P be a fully revealed class.

Note that $\dot{E}\{P\}$ is $\langle k, 2\rangle$–complete on V, thus the next proposition follows from the second theorem on $\langle 0, k\rangle$–completeness:

Proposition. *Let $k \geq 1$. Then each figure in $E\{P\}$ which is k-condensating on V, is $\langle 0, k\rangle$-complete on V.*

Definition. Let $n \geq 2$. We put

$$U^{(n)}\{P\} = \bigsqcup^{n}_{\langle n,2\rangle}(\dot{E}\{P\}).$$

Proposition.

(1) $U^{(n)}\{P\} = \bigsqcup^{n}_{\langle FN,2\rangle}(\dot{E}\{P\})$.

(2) $U^{(n)}\{P\}$ *is a figure in $E\{P\}$, i.e. it is 2-$\dot{E}\{P\}$-symmetric on $[V]^n$.*

PROOF: Let, for $i \in FN$, $E_i \in Nd\{P\}$ be such an equivalence that $E_{i+1} \subseteq E_i$ and $\bigcap_i E_i = E\{P\}$ hold. Put $U_i = \{t \in [V]^n; (\exists u)(i \hat{\leq} u \wedge [u]^{\langle i+1,2\rangle} \subseteq \dot{E}_i)\}$. We have $\bigsqcup^{n}_{\langle FN,2\rangle}(\dot{E}\{P\} = \bigcap_i U_i$ (by using the fact that each U_i is revealed) and $U_{i+1} \subseteq U_i$. Thus, $\bigsqcup^{n}_{\langle FN,2\rangle}(\dot{E}\{P\})$ is a figure in $E\{P\}$. But it is, moreover, $\langle 0, n\rangle$–complete on V. We deduce from this, by using the theorem on lower bound, that $U^{(n)}\{P\} \subseteq \bigsqcup^{n}_{\langle FN,2\rangle}(\dot{E}\{P\})$ holds. Finally, the converse inclusion is easy. ■

Now, we obtain immediately the following

Theorem (on least element). *Assume $n \geq 2$. Then $U^{(n)}\{P\}$ is the least among figures in $E\{P\}$ which are n-condensating on V.*

More generally: Let Z be a figure in $E\{P\}$. Then $U^{(n)}\{P\} \cap [Z]^n$ is the least among subclasses of $[V]^n$, which are figures in $E\{P\}$ and n-condensating on Z.

Proposition. *Let $n \geq 2$. Then*

$$U^{(n)}\{P\} \subseteq \nabla^2_2(\dot{E}\{P\}).$$

PROOF: $\nabla^2_2(\dot{E}\{P\}$ is a figure in $E\{P\}$, i.e. it is 2–$\dot{E}\{P\}$-symmetric on $[V]^n$. It is, moreover, n–condensating on V and the relation in question follows from the previous theorem. ■

Definition. We put, for $n \geq 2$,

$$D^{(n)}\{P\} = \nabla^n_2(\dot{E}\{P\}).$$

Now, we have for $n \geq 2$:

$$U^{(n)}\{P\} \subseteq D^{(n)}\{P\}.$$

Remark.

(1) $D^{(n)}\{P\} = \bigcup\{[E\{P\}''\{x\}]^n; x \in V - \mathrm{Def}\{P\}\}$.

(2) $D^{(2)}\{P\} = \dot{E}\{P\}$.

Proposition. $D^{(n)}\{P\}$ *is n-transitive.*

PROOF: Let $t \in \nabla^n_n(D^{(n)}\{P\})$. Then there is $s \in [V]^{n-1}$ such that $x \in t \rightarrow \{x\} \cup s \in D^{(n)}\{P\}$. We have, for each $y \in s$, $D^{(n)}\{P\}[s] \subseteq E\{P\}''\{y\}$. Thus $t \in [E\{P\}''\{y\}]^n$, i.e. $t \in D^{(n)}\{P\}$ holds. ■

Proposition. $U^{(n)}\{P\} \not\subseteq D^{(n)}\{P\}$.

PROOF: Assume that $\{a_1, a_2\} \in \dot{E}\{P\}$. Let F be an automorphism such that $F(a_1) = a_2$ and $F''P = P$. Put $a_3 = F(a_2)$ and $x_1 = \{a_1, a_2\}$, $x_2 = \{a_2, a_3\}$. At first, $\langle x_1, x_2\rangle \in E\{P\}$ holds. Suppose that $y = \{y_1, y_2\}$ satisfies: $[[\{x_1, x_2, y\}]^2]^2 \subseteq \dot{E}\{P\}$. Then, especially, $x_1 \cap y \hat{\approx} 1 \wedge x_2 \wedge y \hat{\approx} 1 \wedge y_1 \neq y_2$ holds. We can easily see that $y \in [\{a_1, a_2, a_3\}]^2$. Assume that $\{x_1, x_2\} \subseteq t \in [V]^n$. Then there is no $u \supseteq t$ such that $u \hat{\geq} 4$ and $[[u]^2]^2 \subseteq \dot{E}\{P\}$ hold. Assuming $t \in U^{(n)}\{P\}$, we see that there exists an infinite $u \supseteq t$ such that $[[u]^2]^2 \subseteq \dot{E}\{P\}$ (see the second proposition of this paragraph). Thus $t \notin U^{(n)}\{P\}$. We can choose t such that $\{x_1, x_2\} \subseteq t \in [V]^n$ and $t \subseteq E\{P\}''\{x_1\}$. Then $t \in D^{(n)}\{P\} - U^{(n)}\{P\}$. ■

Theorem. *Every $D^{(n)}\{P\}$ is (FN, n)-complete on V.*

PROOF: Let E_i be as in the first proof of this section. Thus, $\dot{E}_i$, $\dot{-}E_i$ are revealed and 2-condensating on every $[V]^l$. We conclude, by using the theorem on $\langle l, m\rangle$-completeness that $\nabla_2^n(\dot{E}_i)$ is $\langle l, n\rangle$-complete on V. We have $D^{(n)}\{P\} = \nabla_2^n(\dot{E}\{P\}) = \nabla_2^n(\bigcap_i \dot{E}_i) = \bigcap_i \nabla_2^n(\dot{E}_i)$ and the last class is $\langle l, n\rangle$-complete on V. (We have used the last proposition of §2 and the second one.) ■

4.2. We assume in this paragraph that P is fully revealed and exact.

Proposition. *Let $k \geq 2$, $n \geq 1$, $Q \in Nd\{P\}$, $W \in Nd\{P\}$. Let Q be k-condensating on W. Then*

$$\nabla_2^n(\dot{E}\{P\} \cap [W]^2) \subseteq \nabla_k^n(Q \cap [W]^k).$$

PROOF: Let $w \in [W]^f \cap \mathrm{Def}\{P\}$ be as in the corollary of the theorem on compactness and let $t = \{x_1, \ldots, x_n\}$ satisfy $t \in \nabla_2^n(\dot{E}\{P\} \cap [W]^2)$; we have $t \subseteq W$. There exists $y \in W$ such that $\{x_i, y\} \in \dot{E}\{P\}$ holds for $i = 1, 2, \ldots, n$. Especially, $E\{P\}''\{x_i\} \neq \{x_i\}$ and $t \cap \mathrm{Def}\{P\} = \emptyset$. Let $s \in [w]^{k-1}$ be such that $\{x_1\} \cup s \in Q$. We have, for $i = 1, 2, \ldots, n, \langle x_1, x_i\rangle \in E\{P\}$. Choose $i \in \{1, \ldots, n\}$. Then there exists an automorphism F such that $F(x_1) = x_i$ and $F''P = P$. Thus $F(s) = s$ and $F''Q = Q$, i.e. $\{x_i\} \cap s \in Q$. We have, of course, $t \in [(Q \cap [W]^k)[s]]^n$ and the proposition is proved. ■

Theorem (on exclusion). *Let $k \geq 2$, $n \geq 1$, $\emptyset \neq T \subseteq FN$. Then*

$$\nabla_2^n(\dot{E}\{P\}) \subseteq \nabla_k^n(\bigsqcup_{(T,2)}^k (\dot{E}\{P\})).$$

PROOF: It suffices to prove the relation in question for $T = FN$ only. Let us use the notation of the first proof in 4.1. Every U_i is k-condensating on V. We deduce, by using the previous proposition, that $\nabla_2^n(\dot{E}\{P\}) \subseteq \nabla_k^n(U_i)$ holds for each $i \in FN$. We have

$$\nabla_2^n(\dot{E}\{P\}) = \bigcap_i \nabla_k^n(U_i) = \nabla_k^n(\bigcap_i U_i)$$

(see the last proposition in §2), which finishes our proof. ■

We obtain as a consequence of this theorem and of the theorem on inclusion the following

Theorem (on equality). *Let $k \geq 2$, $n \geq 1$. Then*

$$\nabla_2^n(\dot{E}\{P\}) = \nabla_k^n(\bigsqcup\nolimits_{\langle FN,2\rangle}^{k}(\dot{E}\{P\})).$$

Thus, we have for $k \geq 2$, $n \geq 2$:

$$D^{(n)}\{P\} = \nabla_k^n(U^{(k)}\{P\}).$$

Especially:

$$\dot{E}\{P\} = \nabla_2^2(U^{(2)}\{P\}).$$

Proposition. *No $U^{(n)}\{P\}$ is n-transitive.*

PROOF: Assume that $U^{(n)}\{P\}$ is n–transitive. Then

$$D^{(n)}\{P\} \subseteq \nabla_n^n(U^{(n)}\{P\}) \subsetneq U^{(n)}\{P\} \subseteq D^{(n)}\{P\},$$

which is a contradiction. (See the last proposition in 4.1.) ■

Theorem. *Let $k \geq 2$, $n \geq 2$. Suppose that $Q \subseteq [V]^k$ and Z are two figures in $E\{P\}$ and let Q be k-condensating on Z. Then*

(1) $D^{(n)}\{P\} \cap [Z]^n \subseteq \nabla_k^n(Q)$.

(2) *Assume, in addition, that Q is k- transitive. Then $D^{(k)}\{P\} \cap [Z]^k \subseteq Q$.*

A proof follows immediately from the theorems on least element, on equality and on restriction.

Theorem (second theorem on $\langle l,n\rangle$–completeness). *Let $Q \subseteq [V]^k$ be k-condensating on $[W]^l$, $k \geq 2$, $l \geq 0$, $n \geq 2$. Assume that Q and W are two figures in $E\{P\}$. Then $\nabla_k^n(Q)$ is $\langle l,n\rangle$-complete on W. Suppose, moreover, that Q is k–transitive. Then Q is $\langle l,k\rangle$-complete on W.*

PROOF: Put $Z = [W]^l$. Then Z is a figure in $E\{P\}$. $D^{(n)}\{P\} \cap [Z]^n$ is $\langle l,n\rangle$–complete on W (see the end of 4.1.). We deduce from this by using the previous theorem that the assertions in question hold. ■

We give one application to the problems of indiscernibles. We say that X is a class of $\{P\}$–indiscernibles iff $[X]^{\langle FN,2\rangle} \subseteq \dot{E}\{P\}$ holds.

Proposition. *Let $n \geq 2$. Then $D^{(n)}\{P\} = \{t \in [V]^n; (\exists u \in [V]^\infty)(t \cap u = \emptyset \wedge (\forall x \in t)(\{x\} \cup u$ is a set of $\{P\}$-indiscernibles).*

PROOF: The relation $\supseteq$ is clear; let us prove the converse one. We have $D^{(n)}\{P\} = \bigcap_{k\geq 2} \nabla_k^n(U^{(k)}\{P\}$. Let $t \in D^{(n)}\{P\}$. Then there exists, for each $k \geq 2$, a set $s \in [V]^{k-1}$ such that $t \cap s = \emptyset \wedge x \in t \rightarrow \{x\} \cup s \in U^{(k)}\{P\}$. Especially, $x \in t \rightarrow [\{x\} \cup s]^{\langle k,2\rangle} \subseteq \dot{E}_k$ holds for each $k \geq 2$, where E_k is as in the first proof of 4.1. Put, for $k \geq 2$, $X_k = \{s \hat{\succeq} k; t \cap s = \emptyset \wedge (\forall x \in t)([\{x\} \cup s]^{\langle FN,2\rangle} \subseteq \dot{E}_k\}$. We have $\emptyset \neq X_{k+1} \subseteq X_k$ and each X_k is revealed. Thus there exists a set $u \in \bigcap_{k\geq 2} X_k$. We have $u \in [V]^\infty$ and $t \cap u = \emptyset$. To finish our proof it suffices to prove: $x \in t \rightarrow (\forall i)[\{x\} \cup u]^{\langle i,2\rangle} \subseteq \dot{E}\{P\}$. Let $i \in FN$. We have for each $k > i$, $k \geq 2 : x \in t \rightarrow [\{x\} \cup u]^{\langle i,2\rangle} \subseteq \dot{E}_k$, i.e. $x \in t \rightarrow [\{x\} \cup u]^{\langle i,2\rangle} \subseteq \bigcap_{k\geq 2} \dot{E}\{P\}$. ■

Corollary. *Let* $t \in [V]^n \wedge n \geq 2$. *Assume that* $(\forall x, y \in t)(\langle x, y\rangle \in E\{P\})$. *Then there exists an infinite set* u *such that* $t \cap u = \emptyset$ *and* $(\forall x \in t)(\{x\} \cup u$ *is a set of* $\{P\}$*-indiscernibles).*

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